Algebraic Geometry (WS 2025)

PD Dr. Jürgen Müller, **Lecture 21** (10.12.2025)

(21.1) Sheaves again. Letting $\mathcal{F}$ and $\mathcal{G}$ be presheaves on a topological space $\mathcal{V}$ with values in a category $\mathcal{A}$, a **morphism of presheaves** is a natural transformation $\Phi\colon \mathcal{F} \Rightarrow \mathcal{G}$, where $\mathcal{F}$ and $\mathcal{G}$ are called **isomorphic** if they are naturally isomorphic; in this case we again write $\mathcal{F} \cong \mathcal{G}$.

Thus a morphism of presheaves Φ assigns a morphism $\Phi_{\mathcal{U}}\colon \mathcal{F}(\mathcal{U}) \to \mathcal{G}(\mathcal{U})$ to all open subsets $\mathcal{U} \subseteq \mathcal{V}$, such that for all $\mathcal{U}' \subseteq \mathcal{U} \subseteq \mathcal{V}$ open we have

$$\rho^{\mathcal{U}}_{\mathcal{U}',\mathcal{F}} \cdot \Phi_{\mathcal{U}'} = \Phi_{\mathcal{U}} \cdot \rho^{\mathcal{U}}_{\mathcal{U}',\mathcal{G}}\colon \mathcal{F}(\mathcal{U}) \to \mathcal{G}(\mathcal{U}').$$

We get the **functor category** $\mathbf{PSh}(\mathcal{V}) = \mathbf{PSh}(\mathcal{V}, \mathcal{A})$, whose objects are the presheaves on $\mathcal{V}$ with values in $\mathcal{A}$, having the morphism of presheaves as morphisms; it is immediately checked that it is a category indeed.

Moreover, we get the **full** subcategory $\mathbf{Sh}(\mathcal{V}) = \mathbf{Sh}(\mathcal{V}, \mathcal{A}) \subseteq \mathbf{PSh}(\mathcal{V}, \mathcal{A})$, whose objects are only the sheaves on $\mathcal{V}$ with values in $\mathcal{A}$, but still having the morphisms of presheaves as morphisms.

(21.2) Regular functions on affine varieties. We keep the above notation, let $\mathbf{V} \subseteq L^n$ be closed, and let $U \subseteq \mathbf{V}$ be open.

A function $\varphi\colon U \to L$ is called **regular** at a point $v \in U$, if there are $f, g \in K[\mathbf{V}]$, such that $v \in D_g \subseteq U$ and $\varphi(u) = \frac{f(u)}{g(u)}$, for all $u \in D_g$; note that we may assume $D_g \subseteq U$. Moreover, φ is called **regular** on U, if it is regular at any point of U.

If $K[\mathbf{V}]$ is a domain, that is $\mathbf{V}$ is irreducible, then in the above definition we may just write $\frac{f}{g} \in K(\mathbf{V}) := \mathrm{Q}(K[\mathbf{V}])$, where $K(\mathbf{V})$ is also called the field of **rational functions** on $\mathbf{V}$ (which are actually only defined on certain open subsets of $\mathbf{V}$). In general, lifting elements of $K[\mathbf{V}]$ to $A = K[\mathcal{X}]$, we may write $\frac{f}{g} \in K(\mathcal{X}) = \mathrm{Q}(A)$, subject to the condition $D_g \cap \mathbf{V} \subseteq U$.

It is immediate that the set $\mathcal{O}_{\mathbf{V}}(U)$ of regular functions on U is a K-algebra; recall that $D_g \cap D_{g'} = D_{gg'}$ for $g, g' \in K[\mathbf{V}]$. Then it is immediate that associating the K-algebra $\mathcal{O}_{\mathbf{V}}(U)$ to any open subset $U \subseteq \mathbf{V}$, together with restriction of functions $\rho^U_{U'}\colon \mathcal{O}_{\mathbf{V}}(U) \to \mathcal{O}_{\mathbf{V}}(U')\colon f \mapsto f|_{U'}$ for any pair $U' \subseteq U \subseteq \mathbf{V}$ of open subsets, and letting $\mathcal{O}_{\mathbf{V}}(\emptyset) := \{0\}$, defines a presheaf $\mathcal{O}_{\mathbf{V}}$ of K-algebras on $\mathbf{V}$. Hence we get the **restricted** presheaf $\mathcal{O}_U := (\mathcal{O}_{\mathbf{V}})|_U$ on any open subset $U \subseteq \mathbf{V}$, by letting $\mathcal{O}_U(U') := \mathcal{O}_{\mathbf{V}}(U')$ for all $U' \subseteq U$ open.

Actually, $\mathcal{O}_U$ is a sheaf, being called the **sheaf of regular functions** on U, or the **structure sheaf** of U: For any open subset $W \subseteq U$, and any open covering $\{W_i \subseteq W; i \in \mathcal{I}\}$, where $\mathcal{I}$ is an index set, any function $f\colon W \to L$ is uniquely defined by its restrictions $f|_{W_i}$. Conversely, since f is regular if and only if for any $v \in W$ there is a principal open subset $v \in D_g \subseteq W$ such that $f|_{D_g} = \frac{h}{g}$ for some $g, h \in K[\mathbf{V}]$, where the principal open subsets are a basis of the Zariski topology, prescribing compatible regular functions $f_i : W_i \to L$ defines a regular function f such that $f|_{W_i} = f_i$.

Then $\mathbf{V}$, together with the sheaf $\mathcal{O}_\mathbf{V}$, is called an **affine (K-)variety**; and U, together with the sheaf $\mathcal{O}_U$, is called a **quasi-affine (K-)variety**. Note that the Zariski topology is built into the structure sheaf anyway. If L is algebraically closed, will show later that $\Gamma(\mathcal{O}_\mathbf{V}) = K[\mathbf{V}]$, and that conversely $\mathcal{O}_\mathbf{V}$ is determined by $K[\mathbf{V}]$, so that the present definition of an affine variety coincides with the earlier one. But first we turn to the projective case:

(21.3) Regular functions on projective varieties. We keep the above notation, let $\mathbf{V} \subseteq \mathbf{P}$ be closed, and let $U \subseteq \mathbf{V}$ be open. Recall that the homogeneous coordinate algebra $K[\mathbf{V}]$ does not induce functions on $\mathbf{P}$, due to the use of homogeneous coordinates. But $K[\mathbf{V}]$ is a graded K-algebra:

Let $0 \neq f, g \in K[\mathbf{V}]$ such that $d := \deg(f) = \deg(g)$. Then for $[x_0 : \ldots : x_n] \in D_g \subseteq \mathbf{V}$ we have $\frac{f(\lambda \cdot [x_0,\ldots,x_n])}{g(\lambda \cdot [x_0,\ldots,x_n])} = \frac{\lambda^d \cdot f(x_0,\ldots,x_n)}{\lambda^d \cdot g(x_0,\ldots,x_n)} = \frac{f(x_0,\ldots,x_n)}{g(x_0,\ldots,x_n}$, for all $\lambda \in L^*$. Hence this indeed defines a function on D_g. Having observed this:

A function $\varphi : U \to L$ is called **regular** at a point $v \in U$, if there are $f, g \in K[\mathbf{V}]$, where $g \neq 0$, and $f = 0$ or $\deg(f) = \deg(g)$, such that $v \in D_g \subseteq U$ and $\varphi(u) = \frac{f(u)}{g(u)}$, or all $u \in D_g$; note that we may indeed assume that $D_g \subseteq U$. Moreover, φ is called **regular** on U, if it is regular at any point of U.

If $K[\mathbf{V}]$ is a domain, that is $\mathbf{V}$ is irreducible, then letting $K(\mathbf{V}) := \mathrm{Q}(K[\mathbf{V}])$ we have the **graded field of fractions**

$$K(\mathbf{V})_0 := \{0\} \,\dot\cup\, \{\frac{f}{g} \in K(\mathbf{V}); 0 \neq f, g \in K[\mathbf{V}] \text{ homogeneous}, \deg(f) = \deg(g)\},$$

which is immediately seen to be a field indeed. Then in the above definition we may just write $\frac{f}{g} \in K(\mathbf{V})_0$, so that the latter is also called the field of **rational functions** on $\mathbf{V}$ (which are actually only defined on certain open subsets of $\mathbf{V}$). In general, lifting elements of $K[\mathbf{V}]$ to $A^\sharp = K[\mathcal{X}^\sharp]$, we may write $\frac{f}{g} \in K(\mathcal{X}^\sharp)_0$, subject to the condition $D_g \cap \mathbf{V} \subseteq U$.

It is immediate (as in the affine case) that the set $\mathcal{O}_\mathbf{V}(U)$ of regular functions on U is a K-algebra. Moreover, associating the K-algebra $\mathcal{O}_\mathbf{V}(U)$ to any open subset $U \subseteq \mathbf{V}$, together with restriction of functions between open subsets, and letting $\mathcal{O}_\mathbf{V}(\emptyset) := \{0\}$, defines a sheaf $\mathcal{O}_\mathbf{V}$ of K-algebras on $\mathbf{V}$, and by restriction a sheaf $\mathcal{O}_U := (\mathcal{O}_\mathbf{V})|_U$ of K-algebras on any open subset $U \subseteq \mathbf{V}$. Then $\mathcal{O}_U$ is called the **sheaf of regular functions** on U, or the **structure sheaf** of U.

Then $\mathbf{V}$, together with the sheaf $\mathcal{O}_\mathbf{V}$, is called a **projective (K-)variety**; and U, together with the sheaf $\mathcal{O}_U$, is called a **quasi-projective (K-)variety**. Note that the Zariski topology is built into the structure sheaf anyway. Unfortunately, If L is algebraically closed, it will turn out below that $\Gamma(\mathcal{O}_\mathbf{V}) \cong K$, so that $\Gamma(\mathcal{O}_\mathbf{V}) \ncong K[\mathbf{V}]$; thus the global sections are not too useful, and the homogeneous coordinate algebra cannot be recovered from the structure sheaf.

(21.4) Varieties. We keep the above notation. Then any affine, quasi-affine, projective or quasi-projective variety is simply called a **(K-)variety (over L)**. Thus, generally speaking, a variety is a topological space together with a sheaf of algebras of functions; later on this will be a special case of a **ringed space**.

If U and V are varieties, a continuous map $\varphi\colon U \to V$ is called a **morphism** (of varieties), if for any $W \subseteq V$ open and any $f \in \mathcal{O}_V(W)$ we have $\varphi^*(f) := f \circ \varphi \in \mathcal{O}_U(\varphi^{-1}(W))$. Thus the **comorphism** $\varphi^*\colon \mathcal{O}_V \Rightarrow \mathcal{O}_U$ induces natural homomorphisms of K-algebras $\varphi^*_W\colon \mathcal{O}_V(W) \to \mathcal{O}_U(\varphi^{-1}(W))$, which commute with the restriction of functions, that is for any $W' \subseteq W \subseteq V$ open we have

$$\varphi^*_{W'} \circ \rho^W_{W' \subseteq V} = \rho^{\varphi^{-1}(W)}_{\varphi^{-1}(W') \subseteq U} \circ \varphi^*_W.$$

In other words, the assignment $\varphi^*\colon \mathcal{O}_V \Rightarrow \mathcal{O}_U$ behaves like a natural transformation, that is morphism of sheaves, alone the sheaves in question are based on different topological spaces. Later on, the pair $[\varphi, \varphi^*]$ will be a special case of a **morphism of ringed spaces**.

In particular, id_U is a morphism, whose associated comorphism $(\mathrm{id}_U)^*\colon \mathcal{O}_U \Rightarrow \mathcal{O}_U$ induces the identity homomorphism on all sections of $\mathcal{O}_U$. Moreover, if W is a variety and $\psi\colon V \to W$ is a morphism, then $\varphi\psi\colon U \to W$ is a morphism again, where the associated comorphisms fulfill $(\varphi\psi)^* = \psi^*\varphi^*$. In other words, the assignment $?^*\colon \varphi \mapsto \varphi^*$ is **contravariantly functorial**.

A morphism $\varphi\colon U \to V$ is called an **isomorphism** (of varieties), if there is a morphism $\psi\colon V \to U$ such that $\varphi\psi = \mathrm{id}_U$ and $\psi\varphi = \mathrm{id}_V$. This is equivalent to saying that $\varphi\colon U \to V$ is a homeomorphism, such that the associated comorphisms fulfill $\psi^*\varphi^* = (\mathrm{id}_U)^*\colon \mathcal{O}_U \Rightarrow \mathcal{O}_U$ and $\varphi^*\psi^* = (\mathrm{id}_V)^*\colon \mathcal{O}_V \Rightarrow \mathcal{O}_V$, where the latter in turn is equivalent to saying that $\varphi^*_W\colon \mathcal{O}_V(W) \to \mathcal{O}_U(\psi(W))$ is an isomorphism with inverse $\psi^*_{\psi(W)}\colon \mathcal{O}_U(\psi(W)) \to \mathcal{O}_V(W)$, for any $W \subseteq V$ open.

If L is algebraically closed, we will show later that for affine varieties the present definition of morphisms and the earlier notion of regular maps coincide.
