Algebraic Geometry (WS 2025)

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All rings and algebras occurring in the sequel will be commutative, associative, and unital, unless otherwise specified.

Affine varieties

(3.1) Hilbert's Basis Theorem. Recall that a ring R is called **Noetherian** if any ideal of R is finitely generated. This is equivalent to saying that R fulfills the **ascending chain condition (A.C.C.)** on ideals, that is any strictly ascending chain of ideals of R terminates. Moreover, it is equivalent to the **maximum condition** on ideals, that is any set of ideals R contains a maximal element.

This implies that any generating set of an ideal of R contains a finite generating set. For example, any principal ideal ring, and thus any field is Noetherian.

Theorem: Hilbert [1890]. Let R be a Noetherian ring, and let X be an indeterminate. Then the polynomial ring $R[X]$ is Noetherian as well.

Proof. Assume to the contrary that there is an ideal $I \trianglelefteq R[X]$ which is not finitely generated. Then there is a sequence $[f_1, f_2, \ldots] \subseteq I \subseteq R[X]$ such that $f_i \in I \setminus \langle f_1, \ldots, f_{i-1} \rangle$, being chosen such that its **degree** $d_i := \deg(f_i) \geq 0$ is minimal, for $i \geq 1$; note that $I \setminus \langle\rangle = I \setminus \{0\}$, and that $d_i \leq d_{i+1}$.

Let $a_i := \mathrm{lc}(f_i) \in R \setminus \{0\}$ be the **leading coefficient** of f_i, so that f_i has **leading monomial** $\mathrm{lm}(f_i) := X^{d_i}$ and **leading term** $\mathrm{lt}(f_i) := a_i X^{d_i}$, for $i \geq 1$. (These notions appearing here is reminiscent of Gordan's proof [1900]; the present formulation seems to bo back to Sarges [1976]).

Now let $J := \langle a_i; i \geq 1 \rangle \trianglelefteq R$. Then, since R is Noetherian, there is $n \geq 0$ such that $J = \langle a_1, \ldots, a_n \rangle$. Thus we have $a_{n+1} = \sum_{i=1}^n a_i b_i$, for some $b_1, \ldots, b_n \in R$. Hence $g := \sum_{i=1}^n b_i f_i X^{d_{n+1}-d_i} \in \langle f_1, \ldots, f_n \rangle$ has degree $\deg(g) = d_{n+1}$ and leading coefficient $\mathrm{lc}(g) = \sum_{i=1}^n a_i b_i = a_{n+1}$, which implies that $f_{n+1} - g \in I \setminus \langle f_1, \ldots, f_n \rangle$ has degree $\deg(f_{n+1} - g) < d_{n+1}$, a contradiction. $\sharp$

Corollary. If R is Noetherian, then so is any finitely generated R-algebra.

Proof. Any such algebra can be written as $R[\mathcal{X}]/I$, where $\mathcal{X}$ is a finite set of indeterminates, and $I \trianglelefteq R[\mathcal{X}]$. Now, by Hilbert's Basis Theorem and induction, $R[\mathcal{X}]$ is Noetherian, and so is $R[\mathcal{X}]/I$ by the Homomorphism Theorem. $\sharp$

(3.2) Algebraic sets. Let $K \subseteq L$ be a field extension, let $\mathcal{X} := \{X_1, \ldots, X_n\}$ be indeterminates, where $n \in \mathbb{N}_0$, and let $A := K[\mathcal{X}]$ be the associated polynomial K-algebra; recall that $\mathbf{A}^n(L) = L^n$ is also called the n-dimensional **affine space** over L. Then, letting $\mathcal{S} \subseteq A$, the set

$$\mathbf{V}_L(\mathcal{S}) := \{[x_1, \ldots, x_n] \in L^n; f(x_1, \ldots, x_n) = 0 \text{ for all } f \in \mathcal{S}\}$$

is called the **(affine) algebraic (K-)set** given by the **defining set** $\mathcal{S}$; then K and L are called its **field of definition** and its **field of coordinates**, respectively. If $R \subseteq L$ is a subring, then $\mathbf{V}_R(\mathcal{S}) := \mathbf{V}_L(\mathcal{S}) \cap R^n$ is called the set of **R-rational points** of $\mathbf{V}_L(\mathcal{S})$.

We have $\mathbf{V}_L(\mathcal{S}) = \mathbf{V}_L(\langle \mathcal{S} \rangle_A)$. Thus by Hilbert's Basis Theorem there are $f_1, \ldots, f_r \in \mathcal{S}$, for some $r \in \mathbb{N}_0$, such that $\mathbf{V}_L(\mathcal{S}) = \mathbf{V}_L(f_1, \ldots, f_r)$. Hence any algebraic set is defined by an ideal, or alternatively by finitely many equations.

For ideals $I, J \trianglelefteq A$ such that $I \subseteq J$ we have $\mathbf{V}_L(J) \subseteq \mathbf{V}_L(I)$. Given ideals $I_i \trianglelefteq A$, for $i \in \mathcal{I}$, where $\mathcal{I}$ is a (possibly infinite) index set, we have $\bigcap_{i \in \mathcal{I}} \mathbf{V}_L(I_i) = \mathbf{V}_L(\sum_{i \in \mathcal{I}} I_i)$, but in general we only have $\bigcup_{i \in \mathcal{I}} \mathbf{V}_L(I_i) \subseteq \mathbf{V}_L(\bigcap_{i \in \mathcal{I}} I_i)$.

For example, we have $\mathbf{V}_L(1) = \emptyset$ and $\mathbf{V}_L(0) = L^n$. Moreover, for $K = L$ and $n = 1$, we have $A = L[X]$, so that the associated algebraic sets are L itself and its finite subsets; this shows that the infinite union of algebraic sets need not be algebraic again.

(3.3) Vanishing ideals. Keeping the above notation, letting $V \subseteq L^n$ be any subset, the set

$$\mathbf{I}_K(V) := \{f \in A; f(x_1, \ldots, x_n) = 0 \text{ for all } [x_1, \ldots, x_n] \in V\} \trianglelefteq A$$

is called the **vanishing ideal** of V; it is immediate that $\mathbf{I}_K(V)$ is an ideal.

For $V \subseteq W \subseteq L^n$ we have $\mathbf{I}_K(W) \subseteq \mathbf{I}_K(V)$. Given $V_i \subseteq L^n$, for $i \in \mathcal{I}$, where $\mathcal{I}$ is a (possibly infinite) index set, we have $\bigcap_{i \in \mathcal{I}} \mathbf{I}_K(V_i) = \mathbf{I}_K(\bigcup_{i \in \mathcal{I}} V_i)$, but in general (even for finite sums) we only have $\sum_{i \in \mathcal{I}} \mathbf{I}_K(V_i) \subseteq \mathbf{I}_K(\bigcap_{i \in \mathcal{I}} V_i)$.

For example, we have $\mathbf{I}_K(\emptyset) = \langle 1 \rangle$, and if L is infinite then we indeed have $\mathbf{I}_K(L^n) = \{0\}$. Moreover, for $K = L$ infinite and $n = 1$, if $V_1, V_2 \subseteq L$ are infinite such that $V_1 \cap V_2$ is finite, then we have $\mathbf{I}_L(V_1) = \{0\} = \mathbf{I}_L(V_2)$, but $\mathbf{I}_L(V_1 \cap V_2) \neq \{0\}$, so that we have $\mathbf{I}_L(V_1) + \mathbf{I}_L(V_2) \neq \mathbf{I}_L(V_1 \cap V_2)$.
