Algebraic Geometry (WS 2025)

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(4.1) Radical ideals. Keeping the notation introduced earlier, let $I \trianglelefteq A$ be an ideal. Then the **radical** of I be given as

$$\sqrt{I} = \mathrm{rad}(I) := \{f \in A; f^k \in I \text{ for some } k \in \mathbb{N}\};$$

in other words, $\sqrt{I}$ consists of the elements of A which are **nilpotent** modulo the ideal I. It is immediate that for $f, g \in \sqrt{I}$ we have $f \cdot A \subseteq \sqrt{I}$; and from $(f+g)^k = \sum_{l=1}^{k} \binom{k}{l} f^l g^{k-l}$ we infer that $\sqrt{I}$ is additively closed; hence $\sqrt{I} \trianglelefteq A$ is an ideal again.

We have $\sqrt{I} = \langle 1 \rangle$ if and only if $I = \langle 1 \rangle$. Moreover, we have $I \subseteq \sqrt{I} = \sqrt{\sqrt{I}}$, and if $\sqrt{I} = I$ then I is called a **radical ideal**. In particular, if $P \lhd A$ is prime, then A/P is a domain, which hence does not have non-zero nilpotent elements, implying that $\sqrt{P} = P$, that is P is radical.

Now, for any subset $V \subseteq L^n$ we have $\mathbf{I}_K(V) = \sqrt{\mathbf{I}_K(V)}$, and for any ideal $I \trianglelefteq A$ we have $\mathbf{V}_L(\sqrt{I}) = \mathbf{V}_L(I)$: Since L does not contain any non-zero nilpotent elements, from $f^k \in \mathbf{I}_K(V)$, for some $f \in A$ and $k \in \mathbb{N}$, we get $f \in \mathbf{I}_K(V)$ already. Similarly, if $v \in \mathbf{V}_L(I)$, then for $f \in \sqrt{I}$ we have $f^k(v) = 0$, for some $k \in \mathbb{N}$, implying $f(v) = 0$, hence $v \in \mathbf{V}_L(\sqrt{I})$.

(4.2) Algebraic sets and their ideals. a) Keeping the above notation, we consider the interplay between the operators $\mathbf{V}_L$ and $\mathbf{I}_K$: By definition, for $V \subseteq L^n$ we have $V \subseteq \mathbf{V}_L(\mathbf{I}_K(V))$, and for $I \trianglelefteq A$ we have $I \subseteq \mathbf{I}_K(\mathbf{V}_L(I))$.

For three-fold compositions this yields: The inclusion $I \subseteq \mathbf{I}_K(\mathbf{V}_L(I))$ implies $\mathbf{V}_L(\mathbf{I}_K(\mathbf{V}_L(I))) \subseteq \mathbf{V}_L(I)$, so that the inclusion $\mathbf{V}_L(I) \subseteq \mathbf{V}_L(\mathbf{I}_K(\mathbf{V}_L(I)))$ implies equality; that is we have

$$\mathbf{V}_L(\mathbf{I}_K(\mathbf{V}_L(I))) = \mathbf{V}_L(I), \quad \text{for any } I \trianglelefteq A.$$

Similarly, the inclusion $V \subseteq \mathbf{V}_L(\mathbf{I}_K(V))$ implies $\mathbf{I}_K(\mathbf{V}_L(\mathbf{I}_K(V))) \subseteq \mathbf{I}_K(V)$, so that the inclusion $\mathbf{I}_K(V) \subseteq \mathbf{I}_K(\mathbf{V}_L(\mathbf{I}_K(V)))$ implies equality; that is we have

$$\mathbf{I}_K(\mathbf{V}_L(\mathbf{I}_K(V))) = \mathbf{I}_K(V), \quad \text{for any } V \subseteq L^n.$$

b) Let $\mathbf{V}, \mathbf{W} \subseteq L^n$ be algebraic. Then we have $\mathbf{V} \subseteq \mathbf{W}$ if and only if $\mathbf{I}_K(\mathbf{W}) \subseteq \mathbf{I}_K(\mathbf{V})$, and $\mathbf{V} = \mathbf{W}$ if and only if $\mathbf{I}_K(\mathbf{W}) = \mathbf{I}_K(\mathbf{V})$: From $\mathbf{I}_K(\mathbf{W}) \subseteq \mathbf{I}_K(\mathbf{V})$ we get $\mathbf{V} = \mathbf{V}_L(\mathbf{I}_K(\mathbf{V})) \subseteq \mathbf{V}_L(\mathbf{I}_K(\mathbf{W})) = \mathbf{W}$; similarly, equality $\mathbf{I}_K(\mathbf{W}) = \mathbf{I}_K(\mathbf{V})$ implies $\mathbf{V} = \mathbf{V}_L(\mathbf{I}_K(\mathbf{V})) = \mathbf{V}_L(\mathbf{I}_K(\mathbf{W})) = \mathbf{W}$.

We now consider (arbitrary) intersections and (finite) unions of algebraic sets: Firstly, for algebraic sets $\mathbf{V}_i \subseteq L^n$, for $i \in \mathcal{I}$, where $\mathcal{I}$ is a (possibly infinite) index set, we have $\mathbf{V}_L(\sum_{i \in \mathcal{I}} \mathbf{I}_K(\mathbf{V}_i)) = \bigcap_{i \in \mathcal{I}} \mathbf{V}_L(\mathbf{I}_K(\mathbf{V}_i)) = \bigcap_{i \in \mathcal{I}} \mathbf{V}_i$. In particular, an arbitrary intersection of algebraic sets is algebraic again. Secondly:

Proposition. We have $\mathbf{V}\cup\mathbf{W} = \mathbf{V}_L(\mathbf{I}_K(\mathbf{V})\cap\mathbf{I}_K(\mathbf{W})) = \mathbf{V}_L(\mathbf{I}_K(\mathbf{V})\cdot\mathbf{I}_K(\mathbf{W}))$. In particular, a finite union of algebraic sets is algebraic again.

Proof. We have $\mathbf{I}_K(\mathbf{V})\cdot\mathbf{I}_K(\mathbf{W}) \subseteq \mathbf{I}_K(\mathbf{V})\cap\mathbf{I}_K(\mathbf{W}) \subseteq \mathbf{I}_K(\mathbf{V}\cup\mathbf{W})$, implying

$$\mathbf{V}\cup\mathbf{W} \subseteq \mathbf{V}_L(\mathbf{I}_K(\mathbf{V}\cup\mathbf{W})) \subseteq \mathbf{V}_L(\mathbf{I}_K(\mathbf{V})\cap\mathbf{I}_K(\mathbf{W})) \subseteq \mathbf{V}_L(\mathbf{I}_K(\mathbf{V})\cdot\mathbf{I}_K(\mathbf{W})).$$

Conversely, let $v \in L^n\setminus(\mathbf{V}\cup\mathbf{W})$. Since $\mathbf{V} = \mathbf{V}_L(\mathbf{I}_K(\mathbf{V}))$ and $\mathbf{W} = \mathbf{V}_L(\mathbf{I}_K(\mathbf{W}))$, there are $f \in \mathbf{I}_K(\mathbf{V})$ and $g \in \mathbf{I}_K(\mathbf{W})$ such that $f(v) \neq 0 \neq g(v)$. Hence we have $fg \in \mathbf{I}_K(\mathbf{V})\cdot\mathbf{I}_K(\mathbf{W})$ such that $(fg)(v) \neq 0$, thus $v \notin \mathbf{V}_L(\mathbf{I}_K(\mathbf{V})\cdot\mathbf{I}_K(\mathbf{W}))$. $\sharp$

Thus the smallest algebraic set containing V is given as the intersection $\overline{V} := \bigcap\{\mathbf{W} \subseteq L^n \text{ algebraic}; V \subseteq \mathbf{W}\}$. Hence we have $V \subseteq \mathbf{V}_L(\mathbf{I}_K(V)) \subseteq \overline{V}$. Since conversely for any algebraic set $V \subseteq \mathbf{V}_L(I)$ we already have $\overline{V} = \mathbf{V}_L(\mathbf{I}_K(V)) \subseteq \mathbf{V}_L(\mathbf{I}_K(\mathbf{V}_L(I))) = \mathbf{V}_L(I)$, we get $\overline{\overline{V}} = \mathbf{V}_L(\mathbf{I}_K(V))$.

For subsets $V \subseteq W$ we have $\overline{V} \subseteq \overline{W}$, and V is algebraic if and only if $V = \overline{V}$.

(4.3) Algebraic-geometric correspondence. Keeping the above notation, we conclude that the operator $\mathbf{I}_K$ induces an inclusion-reversing (with respect to set-theoretic inclusion) injective correspondence

$$\mathbf{I}_K\colon \{\mathbf{V} \subseteq L^n \text{ affine } K\text{-algebraic}\} \to \{I \trianglelefteq A; I = \sqrt{I} \text{ radical}\},$$

whose inverse on $\mathrm{im}(\mathbf{I}_K)$ is given by the operator $\mathbf{V}_L$.

In general, $\mathbf{I}_K$ is not surjective: For example, let $K = L = \mathbb{R}$ and $n = 1$. Then $f := X^2+1 \in \mathbb{R}[X]$ is irreducible, hence $\langle f\rangle \lhd \mathbb{R}[X]$ is prime, thus radical. But $\mathbf{V}_\mathbb{R}(f) = \emptyset$, so that $\mathbf{I}_\mathbb{R}(\mathbf{V}_\mathbb{R}(f)) = \mathbf{I}_\mathbb{R}(\emptyset) = \langle 1\rangle$, implying that $\langle f\rangle \notin \mathrm{im}(\mathbf{I}_\mathbb{R})$. $\sharp$

But it will follow from (the geometric form of) Hilbert's Nullstellensatz, that if L is algebraically closed then $\mathbf{I}_K$ is surjective as well. In this case, the bijective correspondence given by $\mathbf{I}_K$ and $\mathbf{V}_L$ provides an **algebraic-geometric dictionary**, allowing us to translate between geometric properties of algebraic sets and algebraic properties of ideals.
