# Algebraic Geometry (WS 2025)

PD Dr. Jürgen Müller, **Lecture 5** (22.10.2025)

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**(5.1) Hilbert's Nullstellensatz.** Keeping the notation introduced earlier, we prove the following fundamental 'Theorem of Zeroes':

**Theorem: Nullstellensatz (strong form)** [Hilbert, 1893]**.**
Let $L$ be algebraically closed, and let $I \lhd A$. Then we have $\mathbf{V}_L(I) \neq \emptyset$.

**Theorem: Nullstellensatz (field-theoretic form).**
Let the field $L$ be finitely generated as a $K$-algebra. Then $K \subseteq L$ is algebraic.

**(5.2) Proof of Hilbert's Nullstellensatz.** In order to prove this we proceed as follows: We first prove the equivalence of the strong and the field theoretic form. Then we prove two lemmas, from which the field theoretic form follows:

**Proof: Equivalence of strong and field theoretic form.**
**i)** Assume first that the strong form holds, and let $\overline{K}$ be an algebraic closure of $K$. Since $L$ is finitely generated as a $K$-algebra, by $n \in \mathbb{N}_0$ elements say, there is $P \lhd A$ maximal such that $L \cong A/P$ as $K$-algebras. Let $[x_1, \ldots, x_n] \in \mathbf{V}_{\overline{K}}(P)$, and let $\varphi\colon A \to \overline{K}$ be the (non-zero) homomorphism of $K$-algebras defined by $X_i \mapsto x_i$. Then for any $f \in P$ we have

$$\varphi(f) = \varphi(f(X_1, \ldots, X_n)) = f(\varphi(X_1), \ldots, \varphi(X_n)) = f(x_1, \ldots, x_n) = 0 \in \overline{K},$$

hence $\ker(\varphi) = P$. Thus there is an embedding $L \cong A/P \to \overline{K}$, implying that $K \subseteq L$ is an algebraic field extension.

**ii)** Assume now that the field-theoretic form holds. Let $I \subseteq P \lhd A$ be maximal. Then $A/P$ is a field extension of $K$, which is finitely generated as a $K$-algebra. Hence by assumption $K \subseteq (A/P)$ is algebraic. Since $L$ is algebraically closed, there is a homomorphism $\varphi\colon A \to L$ of $K$-algebras, inducing an embedding $\overline{\varphi}\colon A/P \to L$. Let $v := [\varphi(X_1), \ldots, \varphi(X_n)] \in L^n$. Then for any $f \in P$ we have

$$f(v) = f(\varphi(X_1), \ldots, \varphi(X_n)) = \varphi(f(X_1, \ldots, X_n)) = \varphi(f) = 0 \in L,$$

hence $v \in \mathbf{V}_L(P) \subseteq \mathbf{V}_L(I)$. Thus we have $\mathbf{V}_L(I) \neq \emptyset$. $\sharp$

**Lemma.** Let $R$ be a Noetherian ring, let $T$ be a finitely generated $R$-algebra, and let $S \subseteq T$ be an $R$-subalgebra, such that $T$ is a finitely generated $S$-module. Then $S$ is finitely generated as an $R$-algebra as well.

**Proof.** Let $T = R[\mathcal{G}]$, where $\mathcal{G}$ is finite, and let $T = \langle \mathcal{T} \rangle_S$ where $\mathcal{G} \subseteq \mathcal{T} = \{t_1, \ldots, t_r\}$, for some $r \in \mathbb{N}_0$. Then we have $t_i t_j = \sum_{k=1}^r t_k s_{ijk}$, for suitable $s_{ijk} \in S$. Let $S' := R[s_{ijk}; i, j, k \in \{1, \ldots, r\}] \subseteq S$. Then $S'$ is a finitely generated $R$-algebra, such that $T = R[\mathcal{G}] \subseteq \langle \mathcal{T} \rangle_{S'} \subseteq T$, implying that $T$ is a finitely generated $S'$-module.

Since $R$ is Noetherian, by Hilbert's Basis Theorem $S'$ is Noetherian as well. Thus $T$ is a Noetherian $S'$-module. Hence $S \subseteq T$ is a finitely generated $S'$-module as well. Thus, since $S'$ is a finitely generated $R$-algebra, $S$ is a finitely generated $R$-algebra as well. $\sharp$

**Lemma.** Let $R := K(\mathcal{X})$ be the associated rational function field, where $n \geq 1$. Then $R$ is not finitely generated as a $K$-algebra.

**Proof.** Assume to the contrary that $R = K[\frac{f_i}{g}; i \in \{1, \ldots, r\}]$, where $f_i \in A := K[\mathcal{X}]$ and $0 \neq g \in A$, for some $r \in \mathbb{N}_0$; note that we may indeed assume the elements of the generating set to have the same denominator. This implies that any element of $R$ can be written in the form $\frac{f}{g^k}$, for some $f \in A$ and $k \in \mathbb{N}_0$. Since $A$ is factorial, $g$ has only finitely many monic irreducible divisors.

Now, $A$ has infinitely many monic irreducible polynomials: If $K$ is infinite, there are $X - a \in K[X]$ for $a \in K$; if $K$ is finite, there is a monic irreducible polynomial of any degree $d \in \mathbb{N}$. Hence there is an irreducible polynomial $p \in A$ not dividing $g$. Then $\frac{1}{p} \in R$ is not of the required form, a contradiction. $\sharp$

**Proof: Field theoretic form** [Zariski, 1947; Artin–Tate, 1951]**.**
Assume that $K \subseteq L$ is not algebraic. Then, since $K \subseteq L$ is a finitely generated field extension, let $\mathcal{Y} := \{Y_1, \ldots, Y_r\}$ be a **transcendence basis** of $L$ over $K$, for some $r \in \mathbb{N}$; that is $K(\mathcal{Y})$ is a rational function field, such that $K(\mathcal{Y}) \subseteq L$ is algebraic. By finite generation, $L$ is a finitely generated $K(\mathcal{Y})$-vector space.

Now, since $L$ is a finitely generated $K$-algebra, the first lemma implies that $K(\mathcal{Y})$ is a finitely generated $K$-algebra, contradicting the second lemma. $\sharp$

**(5.3) Consequences from Hilbert's Nullstellensatz.**

**Corollary: Nullstellensatz (weak form).**
Let $K$ be algebraically closed, and let $P \lhd A$ be maximal. Then there is $v = [x_1, \ldots, x_n] \in K^n$ such that $P = \langle X_1 - x_1, \ldots, X_n - x_n \rangle \lhd A$.

**Proof.** By the strong form of Hilbert's Nullstellensatz, applied to $K = L$, the ideal $P$ has a zero $v = [x_1, \ldots, x_n] \in K^n$. Hence for any $f \in A \setminus P$ we have $f(v) \neq 0$: Assume that $f(v) = 0$, then $A = \langle P, f \rangle \subseteq \mathbf{I}_K(v) \lhd A$, a contradiction.

We conclude that $I := \langle X_1 - x_1, \ldots, X_n - x_n \rangle \subseteq P \lhd A$. Now polynomial division shows that $\dim_K(A/I) \leq 1$, thus we have $I = P \lhd A$. $\sharp$

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