Algebraic Geometry (WS 2025)

PD Dr. Jürgen Müller, **Lecture 6** (28.10.2025)

(6.1) Nullstellensatz (geometric form)
Keeping the notation introduced earlier, let L be algebraically closed, and let $I \trianglelefteq A$. Then we have $\mathbf{I}_K(\mathbf{V}_L(I)) = \sqrt{I} \trianglelefteq A$.

Proof: 'Rabinowitsch Trick'. We have already seen that

$$I \subseteq \sqrt{I} \subseteq \mathbf{I}_K(\mathbf{V}_L(I)) = \sqrt{\mathbf{I}_K(\mathbf{V}_L(I))} \trianglelefteq A.$$

Now let $0 \neq f \in \mathbf{I}_K(\mathbf{V}_L(I))$, let T be an additional indeterminate, and let $J := \langle I, fT-1 \rangle \trianglelefteq A[T] = K[\mathcal{X}, T]$; in other words, we have $T = \frac{1}{f} \in A[T]/J$.

We show that $J = A[T]$: Let $[x_1, \ldots, x_n, t] \in \mathbf{V}_L(J) \subseteq L^{n+1}$, then $[x_1, \ldots, x_n] \in \mathbf{V}_L(I)$, thus $f(x_1, \ldots, x_n) = 0$, implying that $0 = (fT-1)(x_1, \ldots, x_n, t) = f(x_1, \ldots, x_n) \cdot t - 1 = -1$, a contradiction; hence $\mathbf{V}_L(J) = \emptyset$, thus by the strong form of Hilbert's Nullstellensatz we infer that $J = A[T]$ indeed.

Hence there are $g, g_1, \ldots, g_r \in A[T]$ and $f_1, \ldots, f_r \in I$, for some $r \in \mathbb{N}_0$, such that $1 = g \cdot (fT-1) + \sum_{i=1}^r g_i f_i \in A[T]$. Now let $\varphi \colon A[T] \to K(\mathcal{X})$ be the K-algebra homomorphism defined by $X_i \mapsto X_i$ and $T \mapsto \frac{1}{f}$. Then we have $\varphi(fT-1) = 0$, so that $\sum_{i=1}^r \varphi(g_i) f_i = 1 \in K(\mathcal{X})$. We may assume that $\varphi(g_i) = \frac{g_i'}{f^k} \in K(\mathcal{X})$, for all i, where $g_i' \in A$ and $k \in \mathbb{N}$ is large enough. This yields $f^k = \sum_{i=1}^r g_i' f_i \in K(\mathcal{X})$, showing that $f^k \in I$, thus $f \in \sqrt{I}$. ♯

(6.2) Topological spaces. We recall some notions from general topology: A collection of subsets of a set $\mathcal{V}$, being called **open**, is called a **topology** on $\mathcal{V}$, provided the following properties hold: Both $\emptyset$ and $\mathcal{V}$ are open; if $\mathcal{U}, \mathcal{U}' \subseteq \mathcal{V}$ are open, then $\mathcal{U} \cap \mathcal{U}' \subseteq \mathcal{V}$ is open; and if $\mathcal{U}_i \subseteq \mathcal{V}$ are open, for $i \in \mathcal{I}$, where $\mathcal{I}$ is a (possibly infinite) index set, then $\bigcup_{i \in \mathcal{I}} \mathcal{U}_i \subseteq \mathcal{V}$ is open.

Then $\mathcal{V}$ together with a topology on it is called a **topological space**. For example, the collection $\{\emptyset, \mathcal{V}\}$ is a topology, being called the **trivial topology**; and the set of all subsets of $\mathcal{V}$ is a topology, being called the **discrete topology**.

A subset $\mathcal{W} \subseteq \mathcal{V}$ is called **closed** if its complement $\mathcal{V} \setminus \mathcal{W} \subseteq \mathcal{V}$ is open. Hence by taking complements a topology is equivalently given by a collection of closed subsets, having the following properties: Both $\emptyset$ and $\mathcal{V}$ are closed; if $\mathcal{W}, \mathcal{W}' \subseteq \mathcal{V}$ are closed, then $\mathcal{W} \cup \mathcal{W}' \subseteq \mathcal{V}$ is closed; and if $\mathcal{W}_i \subseteq \mathcal{V}$ are closed, for $i \in \mathcal{I}$, where $\mathcal{I}$ is a (possibly infinite) index set, then $\bigcap_{i \in \mathcal{I}} \mathcal{W}_i \subseteq \mathcal{V}$ is closed.

Given a subset $\mathcal{U} \subseteq \mathcal{V}$, its **closure** in $\mathcal{V}$ is defined as the closed subset $\overline{\mathcal{U}} := \bigcap \{\mathcal{W} \subseteq \mathcal{V} \text{ closed}; \mathcal{U} \subseteq \mathcal{W}\} \subseteq \mathcal{V}$; note that the latter set of sets contains $\mathcal{V}$ as an element, thus is non-empty, so that the intersection is well-defined. In other words, $\overline{\mathcal{U}} \subseteq \mathcal{V}$ is the smallest closed subset (with respect to set-theoretic inclusion) containing $\mathcal{U}$. The subset $\mathcal{U} \subseteq \mathcal{V}$ is called **dense** if $\overline{\mathcal{U}} = \mathcal{V}$.

Any subset $\mathcal{V}' \subseteq \mathcal{V}$ carries the **induced topology**, whose open subsets are given as $\mathcal{U} \cap \mathcal{V}' \subseteq \mathcal{V}'$, where $\mathcal{U} \subseteq \mathcal{V}$ is open; likewise, its closed subsets are given as $\mathcal{W} \cap \mathcal{V}' \subseteq \mathcal{V}'$, where $\mathcal{W} \subseteq \mathcal{V}$ is closed.

(6.3) Irreducible spaces. A topological space $\mathcal{V} \neq \emptyset$ is called **reducible**, if $\mathcal{V} = \mathcal{V}' \cup \mathcal{V}''$ is a union of proper closed subsets $\mathcal{V}', \mathcal{V}'' \subset \mathcal{V}$; note that we do not require $\mathcal{V}'$ and $\mathcal{V}''$ to be disjoint. If $\mathcal{V} \neq \emptyset$ is not reducible, that is whenever $\mathcal{V} = \mathcal{V}' \cup \mathcal{V}''$ is a union of closed subsets we necessarily have $\mathcal{V}' = \mathcal{V}$ or $\mathcal{V}'' = \mathcal{V}$, then $\mathcal{V}$ is called **irreducible**. A subset $\emptyset \neq \mathcal{W} \subseteq \mathcal{V}$ is called **(ir)reducible** if it is so with respect to the induced topology.

We present various characterisations of irreducible topological spaces:

i) By taking complements, it follows that $\mathcal{V}$ is irreducible if and only if whenever $\mathcal{U}', \mathcal{U}'' \subseteq \mathcal{V}$ are open such that $\mathcal{U}' \cap \mathcal{U}'' = \emptyset$, then we have $\mathcal{U}' = \emptyset$ or $\mathcal{U}'' = \emptyset$.

ii) This can be rephrased as follows: $\mathcal{V}$ is irreducible if and only if whenever $\emptyset \neq \mathcal{U}', \mathcal{U}'' \subseteq \mathcal{V}$ are open, then we have $\mathcal{U}' \cap \mathcal{U}'' \neq \emptyset$.

In particular, any irreducible topological space is **connected**, that is cannot be written as the disjoint union of two non-empty open (hence closed) subsets. (But the converse does not hold in general.)

iii) Thus $\mathcal{V}$ is irreducible if and only if any non-empty open subset of $\mathcal{V}$ is dense:

Let $\mathcal{V}$ be irreducible, and let $\emptyset \neq \mathcal{U} \subseteq \mathcal{V}$ be open; then $\mathcal{V} \setminus \overline{\mathcal{U}} \subseteq \mathcal{V}$ is open, and we have $\mathcal{U} \cap (\mathcal{V} \setminus \overline{\mathcal{U}}) = \emptyset$; thus we have $\mathcal{V} \setminus \overline{\mathcal{U}} = \emptyset$, that is $\overline{\mathcal{U}} = \mathcal{V}$.

Conversely, let $\mathcal{V}$ be such that any non-empty open subset is dense, and assume there are $\emptyset \neq \mathcal{U}', \mathcal{U}'' \subseteq \mathcal{V}$ open such that $\mathcal{U}' \cap \mathcal{U}'' = \emptyset$; then $\mathcal{V} \setminus \mathcal{U}'' \subset \mathcal{V}$ is closed, so that $\mathcal{U}' \subseteq \mathcal{V} \setminus \mathcal{U}''$ implies $\mathcal{U}' \subseteq \overline{\mathcal{U}'} \subseteq \mathcal{V} \setminus \mathcal{U}'' \subset \mathcal{V}$, a contradiction. $\sharp$
