

Algebraic Geometry (WS 2025)

PD Dr. Jürgen Müller, Lecture 7 (29.10.2025)

(7.1) Irreducible spaces. Let $\mathcal{V}$ be a topological space.

Proposition. A subset $\mathcal{W} \subseteq \mathcal{V}$ is irreducible if and only if $\overline{\mathcal{W}} \subseteq \mathcal{V}$ is.

Proof. Let first $\mathcal{U} \subseteq \mathcal{V}$ be open such $\mathcal{U} \cap \mathcal{W} = \emptyset$. Then $\mathcal{V} \setminus \mathcal{U}$ is closed and contains $\mathcal{W}$, so that we have $\mathcal{W} \subseteq \overline{\mathcal{W}} \subseteq \mathcal{V} \setminus \mathcal{U}$, thus $\mathcal{U} \cap \overline{\mathcal{W}} = \emptyset$ as well. (Conversely, if $\mathcal{U} \cap \overline{\mathcal{W}} = \emptyset$ then we trivially have $\mathcal{U} \cap \mathcal{W} = \emptyset$ as well.)

i) Now let $\mathcal{W}$ be irreducible, and let $\mathcal{U}', \mathcal{U}'' \subseteq \mathcal{V}$ be open such that $\mathcal{U}' \cap \overline{\mathcal{W}} \neq \emptyset \neq \mathcal{U}'' \cap \overline{\mathcal{W}}$. Then we have $\mathcal{U}' \cap \mathcal{W} \neq \emptyset \neq \mathcal{U}'' \cap \mathcal{W}$ as well, so that by irreducibility of $\mathcal{W}$ we have $\emptyset \neq \mathcal{U}' \cap \mathcal{U}'' \cap \mathcal{W} \subseteq \mathcal{U}' \cap \mathcal{U}'' \cap \overline{\mathcal{W}}$. Thus $\overline{\mathcal{W}}$ is irreducible.

ii) Let conversely $\overline{\mathcal{W}}$ be irreducible, and let $\mathcal{U}', \mathcal{U}'' \subseteq \mathcal{V}$ be open such that $\mathcal{U}' \cap \mathcal{W} \neq \emptyset \neq \mathcal{U}'' \cap \mathcal{W}$. Then we trivially have $\mathcal{U}' \cap \overline{\mathcal{W}} \neq \emptyset \neq \mathcal{U}'' \cap \overline{\mathcal{W}}$ as well, so that by irreducibility of $\overline{\mathcal{W}}$ we have $\mathcal{U}' \cap \mathcal{U}'' \cap \overline{\mathcal{W}} \neq \emptyset$. This in turn implies $\mathcal{U}' \cap \mathcal{U}'' \cap \mathcal{W} \neq \emptyset$ as well. Thus $\mathcal{W}$ is irreducible. $\sharp$

(7.2) Noetherian spaces. A topological space $\mathcal{V}$ is called **Noetherian**, if any strictly descending chain of closed subsets of $\mathcal{V}$ terminates, or equivalently if any non-empty subset of closed subsets of $\mathcal{V}$ has a minimal element.

By taking complements, $\mathcal{V}$ is Noetherian if and only if any strictly ascending chain of open subsets of $\mathcal{V}$ terminates, or equivalently if any non-empty subset of open subsets of $\mathcal{V}$ has a maximal element.

Theorem: Irreducible components. Let $\mathcal{V} \neq \emptyset$ be a Noetherian topological space. Then $\mathcal{V}$ can be written as an **irredundant** finite union $\mathcal{V} = \bigcup_{i=1}^r \mathcal{V}_i$, for some $r \in \mathbb{N}$, of irreducible closed subsets, that is $\mathcal{V}_i \not\subseteq \bigcup_{j \neq i} \mathcal{V}_j$ for all i . Moreover, the subsets $\mathcal{V}_1, \dots, \mathcal{V}_r$ are precisely the (finitely many) maximal irreducible (closed) subsets of $\mathcal{V}$, being called its **irreducible components**.

Proof. Assume first that $\mathcal{V}$ cannot be written as a finite union of irreducible closed subsets. Hence $\mathcal{V}$ is reducible, so that it is a union $\mathcal{V} = \mathcal{W}_1 \cup \mathcal{W}'_1$ of proper closed subsets, where we may assume that $\mathcal{W}_1$ is reducible again. Iterating this process yields an infinite strictly descending chain $\mathcal{V} \supset \mathcal{W}_1 \supset \mathcal{W}_2 \supset \dots$ of closed subsets, a contradiction. Hence we may write $\mathcal{V} = \bigcup_{i=1}^r \mathcal{V}_i$, for some $r \in \mathbb{N}$, as an irredundant union of irreducible closed subsets.

Then any irreducible closed subset $\mathcal{V}' \subseteq \mathcal{V}$ is contained in (at least) one of the $\mathcal{V}_i$: We have $\mathcal{V}' = \mathcal{V}' \cap (\bigcup_{i=1}^r \mathcal{V}_i) = \bigcup_{i=1}^r (\mathcal{V}' \cap \mathcal{V}_i)$, where the sets $\mathcal{V}' \cap \mathcal{V}_i \subseteq \mathcal{V}'$ are closed; since $\mathcal{V}'$ is irreducible, we have $\mathcal{V}' \cap \mathcal{V}_i = \mathcal{V}'$, for some i , thus $\mathcal{V}' \subseteq \mathcal{V}_i$. Hence the $\mathcal{V}_i$ are precisely the maximal irreducible closed subsets of $\mathcal{V}$. $\sharp$

(7.3) Zariski topology. We again place ourselves in the algebraic-geometric setting built up earlier. We have seen that we have quite a rich supply of algebraic sets: Both $\emptyset$ and L^n are algebraic, and both arbitrary intersections and finite unions of algebraic sets are algebraic again. Thus the set of algebraic subsets of L^n can be considered as the closed subsets of a topology on L^n , being called its **(K-)Zariski topology**. Thus henceforth algebraic sets will just be called **(K-)closed**.

The Zariski topology is Noetherian: Let $L^n \supseteq \mathbf{V}_1 \supset \mathbf{V}_2 \supset \cdots \supset \emptyset$ be an infinite strictly descending chain of closed subsets; then the chain $\{0\} \subseteq \mathbf{I}_K(L^n) \subseteq \mathbf{I}_K(\mathbf{V}_1) \subset \mathbf{I}_K(\mathbf{V}_2) \subset \cdots \trianglelefteq A$ of ideals is strictly increasing, a contradiction. $\#$

In particular, the closure of any subset $V \subseteq L^n$, that is the smallest closed set containing V , is given as $\overline{V} = \mathbf{V}_L(\mathbf{I}_K(V))$, justifying the notation already used earlier. Moreover, any closed set is the finite union of its irreducible components. Thus the study of closed sets can often be reduced to the case of irreducible ones. We characterize irreducible closed sets algebraically:

Proposition. A closed set $\mathbf{V} \subseteq L^n$ is irreducible if and only if $\mathbf{I}_K(\mathbf{V})$ is prime.

Proof. i) Let first $\mathbf{V}$ be irreducible. Then $\mathbf{V} \neq \emptyset$ implies that $I := \mathbf{I}_K(\mathbf{V}) \triangleleft A$ is proper. In order to show that I is prime, let $f, g \in A$ such that $fg \in I$. Then we have $\mathbf{V} = \mathbf{V}_L(I) \subseteq \mathbf{V}_L(fg) = \mathbf{V}_L(f) \cup \mathbf{V}_L(g)$, implying that $\mathbf{V} = (\mathbf{V}_L(f) \cap \mathbf{V}) \cup (\mathbf{V}_L(g) \cap \mathbf{V})$. Since $\mathbf{V}$ is irreducible, we have $\mathbf{V}_L(f) \cap \mathbf{V} = \mathbf{V}$, say, that is $\mathbf{V} \subseteq \mathbf{V}_L(f)$, or equivalently $f \in I$.

ii) Let now I be prime. Then $I \triangleleft A$ being proper implies $\mathbf{V} \neq \emptyset$. In order to show that $\mathbf{V}$ is irreducible, assume to the contrary that $\mathbf{V} = \mathbf{V}' \cup \mathbf{V}''$, where $\mathbf{V}', \mathbf{V}'' \subset \mathbf{V}$ are closed and proper. Letting $I' := \mathbf{I}_K(\mathbf{V}') \trianglelefteq A$ and $I'' := \mathbf{I}_K(\mathbf{V}'') \trianglelefteq A$, we have $I = \mathbf{I}_K(\mathbf{V}) = \mathbf{I}_K(\mathbf{V}' \cup \mathbf{V}'') = \mathbf{I}_K(\mathbf{V}') \cap \mathbf{I}_K(\mathbf{V}'') = I' \cap I''$, where both $I \subset I'$ and $I \subset I''$. Hence there are $f \in I' \setminus I$ and $g \in I'' \setminus I$. But now we have $fg \in I' \cdot I'' \subseteq I' \cap I'' = I$, a contradiction. $\#$

(7.4) Reduced rings. Let R be a ring. Then the ideal

$$\text{nil}(R) := \sqrt{\{0\}} = \{f \in R; f^k = 0 \text{ for some } k \in \mathbb{N}\} \trianglelefteq R,$$

consisting of the **nilpotent** elements of R , is called the **nilradical** of R . Moreover, the ring R is called **reduced** if $\text{nil}(R) = \{0\}$.

From $\sqrt{\sqrt{\{0\}}} = \sqrt{\{0\}}$ we get $\text{nil}(R/\text{nil}(R)) = \{0\}$, that is $R_{\text{red}} := R/\text{nil}(R)$ is reduced, being called the **reduced ring** associated with R . Then, for any ideal $I \trianglelefteq R$ we have $\text{nil}(R/I) = \sqrt{I}/I$, that is $(R/I)_{\text{red}} \cong R/\sqrt{I}$. In particular, R/I is reduced if and only if $I = \sqrt{I}$ is radical.

(7.5) Coordinate algebras. a) Keeping the notation introduced earlier, let $\mathbf{V} \subseteq L^n$ be closed. Then the vanishing ideal $\mathbf{I}_K(\mathbf{V}) \trianglelefteq A$ is radical, thus the

(finitely generated) **coordinate algebra** $K[\mathbf{V}] := A/\mathbf{I}_K(\mathbf{V})$ of $\mathbf{V}$ is reduced. In general, any reduced finitely generated K -algebra is called an **affine K -algebra**.

The closed set $\mathbf{V} \subseteq L^n$, equipped with the (induced) Zariski topology, together with its affine coordinate algebra $K[\mathbf{V}]$ is called an **affine (K -)variety**. Note that this definition depends on the given embedding $\mathbf{V} \subseteq L^n$ (which will be remedied by defining **abstract** affine varieties later). The coordinate algebra plays a decisive role within the structure of $\mathbf{V}$ as an affine variety:

Any $f \in K[\mathbf{V}]$ induces a (polynomial) **regular function** $f^\bullet : \mathbf{V} \rightarrow L : v \mapsto f(v)$, where $f(v)$ is given by choosing any representative of f in A modulo $\mathbf{I}_K(\mathbf{V})$. (We will show later that regular functions are actually continuous with respect to the Zariski topology.)

This gives rise to the homomorphism of K -algebras $K[\mathbf{V}] \rightarrow \text{Maps}(\mathbf{V}, L) : f \mapsto f^\bullet$. Since for $g, h \in A$ we have $g^\bullet = h^\bullet$ if and only if $g - h \in \mathbf{I}_K(\mathbf{V})$, we conclude that the latter map is injective. Hence $K[\mathbf{V}]$ can be identified with the algebra of functions it induces, so that $K[\mathbf{V}]$ is also called the **algebra of regular functions** on $\mathbf{V}$. In particular, for $X_i \in K[\mathbf{V}]$ we get the **coordinate function** $(X_i)^\bullet$ mapping $[x_1, \dots, x_n] \in \mathbf{V}$ to its i -th coordinate $x_i \in L$.

We have the following **algebraic-geometric correspondence**: For any subset $\mathcal{S} \subseteq K[\mathbf{V}]$ we get the (closed) subset

$$\mathbf{V}_{\mathbf{V}}(\mathcal{S}) := \{v \in \mathbf{V}; f(v) = 0 \text{ for all } f \in \mathcal{S}\} = \mathbf{V}_L(\mathcal{S} + \mathbf{I}_K(\mathbf{V})) \subseteq \mathbf{V}.$$

Conversely, for any subset $W \subseteq \mathbf{V}$ we get the (radical) vanishing ideal

$$\mathbf{I}_{\mathbf{V}}(W) := \{f \in K[\mathbf{V}]; f(w) = 0 \text{ for all } w \in W\} = \mathbf{I}_K(W)/\mathbf{I}_K(\mathbf{V}) \trianglelefteq K[\mathbf{V}].$$

Then we have $\overline{W} = \mathbf{V}_{\mathbf{V}}(\mathbf{I}_{\mathbf{V}}(W)) \subseteq \mathbf{V}$. For any closed subset $\mathbf{W} \subseteq \mathbf{V}$ we have

$$K[\mathbf{W}] = A/\mathbf{I}_K(\mathbf{W}) \cong (A/\mathbf{I}_K(\mathbf{V})) / (\mathbf{I}_K(\mathbf{W})/\mathbf{I}_K(\mathbf{V})) = K[\mathbf{V}]/\mathbf{I}_{\mathbf{V}}(\mathbf{W}),$$

where the natural epimorphism $K[\mathbf{V}] \rightarrow K[\mathbf{W}]$ with kernel $\mathbf{I}_{\mathbf{V}}(\mathbf{W})$ is given by restricting regular functions on $\mathbf{V}$ to $\mathbf{W}$. Moreover, $\mathbf{W}$ is irreducible if and only if $K[\mathbf{W}]$ is a domain, or equivalently if and only if $\mathbf{I}_{\mathbf{V}}(\mathbf{W}) \trianglelefteq K[\mathbf{V}]$ is prime.