Algebraic Geometry (WS 2025)

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(8.1) Algebraic-geometric correspondence. Keeping the notation introduced earlier, let $\mathbf{V} \subseteq L^n$ be closed. Then we have the following **algebraic-geometric correspondence**:

For any subset $\mathcal{S} \subseteq K[\mathbf{V}]$ we get the (closed) subset

$$\mathbf{V}_{\mathbf{V}}(\mathcal{S}) := \{v \in \mathbf{V}; f(v) = 0 \text{ for all } f \in \mathcal{S}\} = \mathbf{V}_L(\mathcal{S} + \mathbf{I}_K(\mathbf{V})) \subseteq \mathbf{V}.$$

Conversely, for any subset $W \subseteq \mathbf{V}$ we get the (radical) vanishing ideal

$$\mathbf{I}_{\mathbf{V}}(W) := \{f \in K[\mathbf{V}]; f(w) = 0 \text{ for all } w \in W\} = \mathbf{I}_K(W)/\mathbf{I}_K(\mathbf{V}) \trianglelefteq K[\mathbf{V}].$$

Then we have $\overline{W} = \mathbf{V}_{\mathbf{V}}(\mathbf{I}_{\mathbf{V}}(W)) \subseteq \mathbf{V}$. For any closed subset $\mathbf{W} \subseteq \mathbf{V}$ we have

$$K[\mathbf{W}] = A/\mathbf{I}_K(\mathbf{W}) \cong (A/\mathbf{I}_K(\mathbf{V}))/(\mathbf{I}_K(\mathbf{W})/\mathbf{I}_K(\mathbf{V})) = K[\mathbf{V}]/\mathbf{I}_{\mathbf{V}}(\mathbf{W}),$$

where the natural epimorphism $K[\mathbf{V}] \to K[\mathbf{W}]$ with kernel $\mathbf{I}_{\mathbf{V}}(\mathbf{W})$ is given by restricting regular functions on $\mathbf{V}$ to $\mathbf{W}$. Moreover, $\mathbf{W}$ is irreducible if and only if $K[\mathbf{W}]$ is a domain, or equivalently if and only if $\mathbf{I}_{\mathbf{V}}(\mathbf{W}) \trianglelefteq K[\mathbf{V}]$ is prime.

If $\mathbf{V} \subseteq L^n$ is closed, the operator $\mathbf{I}_{\mathbf{V}}$ induces an inclusion-reversing (with respect to set-theoretic inclusion) injective correspondence

$$\{\mathbf{W} \subseteq \mathbf{V} \ K\text{-closed}\} \to \{I \trianglelefteq K[\mathbf{V}]; I = \sqrt{I} \text{ radical}\},$$

whose inverse on the image of $\mathbf{I}_{\mathbf{V}}$ is the operator $\mathbf{V}_{\mathbf{V}}$. If L is algebraically closed, then $\mathbf{I}_{\mathbf{V}}$ is surjective, and for $I \trianglelefteq K[\mathbf{V}]$ we have $\mathbf{I}_{\mathbf{V}}(\mathbf{V}_{\mathbf{V}}(I)) = \sqrt{I} \trianglelefteq K[\mathbf{V}]$.

(8.2) Affine algebras. A reduced finitely generated K-algebra is called an **affine K-algebra**. Actually, any affine K-algebra R is the coordinate algebra of a K-closed set: Let L be an algebraic closure of K. Since R is finitely generated, by $n \in \mathbb{N}_0$ elements say, we have $R \cong A/I$, where since R is reduced we have $I = \sqrt{I}$; hence letting $\mathbf{V} := \mathbf{V}_L(I) \subseteq L^n$ we get $K[\mathbf{V}] = A/\mathbf{I}_K(\mathbf{V}) = A/\mathbf{I}_K(\mathbf{V}_L(I)) = A/\sqrt{I} = A/I \cong R$.

(8.3) Example. i) If L is infinite, then we have $\mathbf{I}_K(L^n) = \{0\} \triangleleft A = K[\mathcal{X}]$, thus $K[L^n] = A/\{0\} \cong A$; since A is a domain, L^n is irreducible.

ii) If $K = L$, for $v = [x_1, \ldots, x_n] \in K^n$ letting $I := \langle X_1 - x_1, \ldots, X_n - x_n \rangle \trianglelefteq A$, we have $\mathbf{V}_K(I) = \{v\}$, showing that all singleton sets are closed and irreducible. Moreover, polynomial division shows that $\dim_K(A/I) \leq 1$. Hence from $I \subseteq \mathbf{I}_K(v) \triangleleft A$ we get $\mathbf{I}_K(v) = I \triangleleft A$, being maximal, and $K[\{v\}] = A/I \cong K$.

iii) If moreover $K = L$ is finite, then all subsets of K^n are closed, the irreducible ones being the singleton subsets. Thus in this case the Zariski topology coincides with the discrete topology; in particular, K^n is reducible for $n \geq 1$.

(8.4) Regular maps. a) We keep the above notation, and additionally let $B := K[\mathcal{Y}]$, where $\mathcal{Y} := \{Y_1, \ldots, Y_m\}$ are (further) indeterminates for some $m \in \mathbb{N}_0$. Moreover, let $\mathbf{V} \subseteq L^n$ and $\mathbf{W} \subseteq L^m$ be closed. A map $\varphi\colon \mathbf{V} \to \mathbf{W}$ is called **regular** or a **(K-)morphism (of affine varieties)**, if there are 'polynomials' $f_1, \ldots, f_m \in K[\mathbf{V}]$ such that $\varphi(v) = [f_1(v), \ldots, f_m(v)] \in \mathbf{W}$, for all $v \in \mathbf{V}$.

Let $\mathrm{Mor}_K(\mathbf{V}, \mathbf{W})$ be the set of all regular maps from $\mathbf{V}$ to $\mathbf{W}$. In particular, for $\mathbf{W} = L$ we have $\mathrm{Mor}_K(\mathbf{V}, L) = K[\mathbf{V}]$ as sets. Moreover, $\varphi \in \mathrm{Mor}_K(\mathbf{V}, \mathbf{W})$ is called an **isomorphism (of affine varieties)**, if it is bijective and its inverse is regular again; in particular, $\mathrm{id}_{\mathbf{V}} \in \mathrm{Mor}_K(\mathbf{V}, \mathbf{V})$ is an isomorphism.

b) Let $\mathrm{Hom}(K[\mathbf{W}], K[\mathbf{V}])$ be the set of all K-algebra homomorphisms from $K[\mathbf{W}]$ to $K[\mathbf{V}]$; recall that an algebra homomorphism is an isomorphism if and only if it is bijective. Letting $\varphi\colon \mathbf{V} \to \mathbf{W}$ be regular, by pre-composition we get the **dual** K-algebra homomorphism or **comorphism** of coordinate K-algebras

$$\varphi^*\colon K[\mathbf{W}] \to K[\mathbf{V}]\colon g \mapsto g \circ \varphi = g(f_1, \ldots, f_m).$$

We only have to show that φ^* is well-defined: For $g \in K[\mathbf{W}]$ and $v \in \mathbf{V}$ we get $g(\varphi(v)) = g(f_1(v), \ldots, f_m(v)) = g(f_1, \ldots, f_m)(v)$, where $g(f_1, \ldots, f_m) \in K[\mathbf{V}]$; note that this independent of the choice of representatives for $g \in K[\mathbf{W}]$ and the $f_j \in K[\mathbf{V}]$ in $K[\mathcal{Y}]$ and $K[\mathcal{X}]$, respectively.

(8.5) Functoriality. Keeping the above notation, we vary the regular map:

Let $\psi \in \mathrm{Mor}_K(\mathbf{U}, \mathbf{V})$, where $\mathbf{U}$ is an affine variety. Then it is immediate (by concatenation of 'polynomial' functions) that $\psi\varphi \in \mathrm{Mor}_K(\mathbf{U}, \mathbf{W})$. For $g \in K[\mathbf{W}]$ we have $g \circ (\varphi \circ \psi) = (g \circ \varphi) \circ \psi$, so that we get $(\psi\varphi)^* = \varphi^*\psi^*$. Moreover, we have $(\mathrm{id}_{\mathbf{W}})^* = \mathrm{id}_{K[\mathbf{W}]}$. In other words, the collection of maps $?^*\colon \varphi \mapsto \varphi^*$ is **contravariantly functorial**.

Theorem. The map $\mathrm{Mor}_K(\mathbf{V}, \mathbf{W}) \to \mathrm{Hom}(K[\mathbf{W}], K[\mathbf{V}])\colon \varphi \mapsto \varphi^*$ is bijective.

Proof. Let $\varphi\colon \mathbf{V} \to \mathbf{W}$ be regular, given by $f_1, \ldots, f_m \in K[\mathbf{V}]$. Then for $Y_j \in K[\mathbf{W}]$, where $j \in \{1, \ldots, m\}$, we have $\varphi^*(Y_j) = f_j$, implying injectivity.

Let $\alpha\colon K[\mathbf{W}] \to K[\mathbf{V}]$ be a homomorphism of K-algebras. For $j \in \{1, \ldots, m\}$ let $f_j := \alpha(Y_j + \mathbf{I}_K(\mathbf{W})) \in K[\mathbf{V}]$, and let $\varphi\colon \mathbf{V} \to L^m\colon v \mapsto [f_1(v), \ldots, f_m(v)]$ be the associated regular map. Then we have $\varphi^*\colon B \to K[\mathbf{V}]\colon Y_j \mapsto f_j$. Thus by construction φ^* factors through $B/\mathbf{I}_K(\mathbf{W}) = K[\mathbf{W}]$. Hence for $f \in \mathbf{I}_K(\mathbf{W})$ we have $\varphi^*(f) = 0$, which for $v \in \mathbf{V}$ implies $f(\varphi(v)) = (\varphi^*(f))(v) = 0$. This says that $\varphi(v) \in \mathbf{V}_L(\mathbf{I}_K(\mathbf{W})) = \mathbf{W}$. Thus we have $\varphi(\mathbf{V}) \subseteq \mathbf{W}$. Hence, slightly abusing notation, we have a regular map $\varphi\colon \mathbf{V} \to \mathbf{W}$, having comorphism $\varphi^*\colon K[\mathbf{W}] \to K[\mathbf{V}]\colon Y_j \mapsto f_j$. This shows $\varphi^* = \alpha$, implying surjectivity. $\sharp$

Being an isomorphism of varieties is a property strictly stronger than being a homeomorphism, let alone being bijective, as the following example shows (once we have shown that the map $\varphi \mapsto \varphi^*$ reflects and respects isomorphisms):

Example. Let q be a prime power, let $\mathbb{F}_q \subseteq \mathbf{F}$ be an algebraic closure, and let $A := \mathbb{F}_q[X] = \mathbb{F}_q[\mathbf{F}]$. The **Frobenius map** $\Phi = \Phi_q \colon \mathbf{F} \to \mathbf{F} \colon x \mapsto x^q$ is an $\mathbb{F}_q$-morphism, whose comorphism is the $\mathbb{F}_q$-algebra homomorphism $\Phi^* \colon A \to A \colon X \mapsto X^q$. Then we have $\Phi^*(f) = f(X^q) = f^q$, for $f \in A$, hence Φ^* is injective, but not surjective. Thus Φ^* is not an isomorphism, so Φ neither is. $\sharp$
