

Algebraic Geometry (WS 2025)

PD Dr. Jürgen Müller, **Lecture 9** (05.11.2025)

(9.1) Corollary. Let $\varphi: \mathbf{V} \rightarrow \mathbf{W}$ be regular. Then φ is an isomorphism (of varieties) if and only if φ^* is an isomorphism (of K -algebras), in which case we have $(\varphi^*)^{-1} = (\varphi^{-1})^*$.

Proof. If $\varphi: \mathbf{V} \rightarrow \mathbf{W}$ is an isomorphism with inverse $\psi: \mathbf{W} \rightarrow \mathbf{V}$, then by functoriality we have $\varphi^*\psi^* = (\psi\varphi)^* = (\text{id}_{\mathbf{W}})^* = \text{id}_{K[\mathbf{W}]}$ and $\psi^*\varphi^* = (\varphi\psi)^* = (\text{id}_{\mathbf{V}})^* = \text{id}_{K[\mathbf{V}]}$, showing that φ^* is an isomorphism such that $(\varphi^*)^{-1} = (\varphi^{-1})^*$.

Conversely, if $\varphi^*: K[\mathbf{W}] \rightarrow K[\mathbf{V}]$ is an isomorphism, then by surjectivity there is a regular map $\psi: \mathbf{W} \rightarrow \mathbf{V}$ such that $\psi^* = (\varphi^*)^{-1}: K[\mathbf{V}] \rightarrow K[\mathbf{W}]$. Hence we get $(\varphi\psi)^* = \psi^*\varphi^* = \text{id}_{K[\mathbf{V}]} = (\text{id}_{\mathbf{V}})^*$ and $(\psi\varphi)^* = \varphi^*\psi^* = \text{id}_{K[\mathbf{W}]} = (\text{id}_{\mathbf{W}})^*$, by injectivity implying $\varphi\psi = \text{id}_{\mathbf{V}}$ and $\psi\varphi = \text{id}_{\mathbf{W}}$; thus φ is an isomorphism. $\sharp$

Being an isomorphism of varieties is a property strictly stronger than being a homeomorphism, let alone being bijective, as the following example shows:

(9.2) Example: Frobenius morphisms, cont.. Let q be a prime power, let $\mathbb{F}_q \subseteq \mathbf{F}$ be an algebraic closure, and let $\Phi = \Phi_q: \mathbf{F} \rightarrow \mathbf{F}: x \mapsto x^q$ be the associated Frobenius morphism. We have already seen that Φ is bijective, but not an isomorphism of varieties.

Moreover, Φ fixes all closed subsets of $\mathbf{F}$: Let $\emptyset \neq \mathbf{V} \subset \mathbf{F}$ be closed, thus there is $f \in A$ such that $\mathbf{V} = \mathbf{V}_{\mathbf{F}}(f)$, hence we have $\Phi^{-1}(\mathbf{V}) = \{x^{\frac{1}{q}} \in \mathbf{F}; x \in \mathbf{V}\} = \mathbf{V}_{\mathbf{F}}(\Phi^*(f)) = \mathbf{V}_{\mathbf{F}}(f) = \mathbf{V}$. Thus Φ is continuous (which also follows from regularity), and is a **closed map**, that is it maps closed sets to closed sets, which implies that Φ^{-1} is continuous as well, so that Φ is a homeomorphism.

(9.3) Proposition: Properties of morphisms. Keeping the above notation, let $\varphi: \mathbf{V} \rightarrow \mathbf{W}$ be regular. Then we have:

- a) φ is continuous (with respect to the Zariski topology).
- b) φ^* is injective if and only if φ is **dominant**, that is $\varphi(\mathbf{V}) \subseteq \mathbf{W}$ is dense.
- c) φ^* is surjective if and only if φ is a **closed immersion**, that is $\varphi(\mathbf{V}) \subseteq \mathbf{W}$ is closed such that φ induces an isomorphism $\varphi': \mathbf{V} \rightarrow \varphi(\mathbf{V})$.

Proof. a) Let $\mathbf{U} \subseteq \mathbf{W}$ be closed, and let $I := \mathbf{I}_{\mathbf{W}}(\mathbf{U}) \trianglelefteq K[\mathbf{W}]$. Then $\mathbf{V}_{\mathbf{W}}(I) = \mathbf{U}$ yields $\varphi^{-1}(\mathbf{U}) = \{v \in \mathbf{V}; \varphi(v) \in \mathbf{U}\} = \{v \in \mathbf{V}; f(\varphi(v)) = 0 \text{ for all } f \in I\} = \{v \in \mathbf{V}; (\varphi^*(f))(v) = 0 \text{ for all } f \in I\} = \{v \in \mathbf{V}; (\varphi^*(I))(v) = 0\} = \mathbf{V}_{\mathbf{V}}(\varphi^*(I))$, which is closed in $\mathbf{V}$ indeed.

b) Let φ be dominant, and let $f \in \ker(\varphi^*) \trianglelefteq K[\mathbf{W}]$. Then we have $f(\varphi(v)) = (\varphi^*(f))(v) = 0$, for $v \in \mathbf{V}$, implying that $f|_{\varphi(\mathbf{V})} = 0$. Since $\{0\} \subseteq L$ is

(K -)closed and f is continuous, we infer that $f^{-1}(\{0\}) \subseteq \mathbf{W}$ is closed, hence $\varphi(\mathbf{V}) \subseteq f^{-1}(\{0\})$ entails $\mathbf{W} = \overline{\varphi(\mathbf{V})} \subseteq f^{-1}(\{0\}) \subseteq \mathbf{W}$, thus $f = 0 \in K[\mathbf{W}]$.

Conversely, let φ^* be injective, and let $\mathbf{U} := \overline{\varphi(\mathbf{V})} \subseteq \mathbf{W}$. We show that $\mathbf{I}_{\mathbf{W}}(\mathbf{U}) = \{0\}$, implying $\mathbf{U} = \mathbf{V}_{\mathbf{W}}(\mathbf{I}_{\mathbf{W}}(\mathbf{U})) = \mathbf{W}$: Let $f \in \mathbf{I}_{\mathbf{W}}(\mathbf{U})$, then we have $(\varphi^*(f))(v) = f(\varphi(v)) = 0$, for $v \in \mathbf{V}$, implying $\varphi^*(f) = 0$, thus $f = 0$.

c) Here is a much more straightforward proof than the one presented in the lecture, avoiding Hilbert's Nullstellensatz as well:

Let φ be a closed immersion. Then $\mathbf{U} := \varphi(\mathbf{V}) \subseteq \mathbf{W}$ is closed such that $\varphi = \varphi' \cdot \iota_{\mathbf{U}}^{\mathbf{W}}$, where $\varphi': \mathbf{V} \rightarrow \mathbf{U}$ is an isomorphism, and $\iota_{\mathbf{U}}^{\mathbf{W}}: \mathbf{U} \rightarrow \mathbf{W}$ is the natural inclusion morphism. Moreover, $(\iota_{\mathbf{U}}^{\mathbf{W}})^* = \rho_{\mathbf{U}}^{\mathbf{W}}: K[\mathbf{W}] \rightarrow K[\mathbf{U}]$ is the restriction epimorphism. Then we have $\varphi^* = (\iota_{\mathbf{U}}^{\mathbf{W}})^* \cdot (\varphi')^* = \rho_{\mathbf{U}}^{\mathbf{W}} \cdot (\varphi')^*$. Since $\rho_{\mathbf{U}}^{\mathbf{W}}$ is surjective, and $(\varphi')^*: K[\mathbf{U}] \rightarrow K[\mathbf{V}]$ is bijective, φ^* is surjective.

Conversely, let φ^* be surjective. Then let $\mathbf{U} := \overline{\varphi(\mathbf{V})} \subseteq \mathbf{W}$, and let $\varphi': \mathbf{V} \rightarrow \mathbf{U}$ such that $\varphi = \varphi' \cdot \iota_{\mathbf{U}}^{\mathbf{W}}$. Then we have $\varphi^* = (\iota_{\mathbf{U}}^{\mathbf{W}})^* \cdot (\varphi')^* = \rho_{\mathbf{U}}^{\mathbf{W}} \cdot (\varphi')^*$. Since φ^* is surjective, so is $(\varphi')^*$. Moreover, φ' is dominant, thus $(\varphi')^*$ is injective. Thus $(\varphi')^*$ is an isomorphism, hence so is φ' , in particular $\varphi(\mathbf{V}) = \mathbf{U}$ is closed. $\#$

The image of a morphism $\varphi: \mathbf{V} \rightarrow \mathbf{W}$ needs neither be closed nor open in $\mathbf{W}$, as the example below shows. (The structure of $\varphi(\mathbf{V})$ will be discussed later.)

Example. Let $K = L = \mathbb{C}$, let $\mathbf{V} = \mathbb{C}^2$, let $A := \mathbb{C}[X, Y] \cong \mathbb{C}[\mathbf{V}]$, and let $\varphi: \mathbf{V} \rightarrow \mathbf{V}: [x, y] \mapsto [xy, y]$, being regular with $f_x = XY \in A$ and $f_y = Y \in A$. The associated comorphism is the $\mathbb{C}$ -algebra homomorphism given by $\varphi^*: A \rightarrow A: X \mapsto XY, Y \mapsto Y$. Hence $\varphi^*(\sum_{i,j \geq 0} a_{ij} X^i Y^j) = \sum_{i,j \geq 0} a_{ij} X^i Y^{i+j}$ shows that φ^* is injective, but $X \notin \varphi^*(A)$ shows that φ^* is not surjective.

We have $\varphi(\mathbf{V}) = \{[0, 0]\} \dot{\cup} (\mathbb{C} \times \mathbb{C}^*) \subseteq \mathbf{V}$, that is $\mathbf{V} \setminus \varphi(\mathbf{V}) = \mathbb{C}^* \times \{0\}$. We show that $\varphi(\mathbf{V}) \subseteq \mathbf{V}$ is neither closed nor open: The injectivity of φ^* implies that $\varphi(\mathbf{V}) \subset \mathbf{V}$ is dense, thus $\varphi(\mathbf{V})$ is not closed. Assume that $\varphi(\mathbf{V})$ is open, then $\mathbf{V} \setminus \varphi(\mathbf{V})$ is closed; since $\{[0, 0]\} \subseteq \varphi(\mathbf{V})$ is closed as well, there is $f \in A$ such that $f(x, 0) = 0$ for all $x \neq 0$, but $f(0, 0) \neq 0$, a contradiction. $\#$