Association Schemes —
Group Theory Without Groups

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Association schemes are a central notion in algebraic combinatorics. They provide a unified approach to various questions from design theory, coding theory, algebraic graph theory, discrete geometry, group theory and representation theory. This is facilitated by viewing ideas from the theory of groups and their representations from a combinatorial perspective, thus leading to a more general picture. The aims of this lecture are to introduce association schemes and the related basic structural notions and to present newer developments, in particular as far as their representation theory is concerned.

Contents

0 Introduction

The theory of association schemes has its origin in the design of statistical experiments [4], in the construction of so-called **balanced incomplete block designs** [1, Ch.1.2]; In [8], association schemes were first recognised and fully used as the basic underlying structure of **coding theory** and **design theory** [1], giving birth to **algebraic combinatorics** as a mathematical discipline [3]. Besides that, association schemes appear e. g. in the theory of **distance-regular graphs** [5], in group theory related to **coherent configurations** [14, 15, 16] and to **Schur rings** [21], in representation theory related to **centraliser algebras** [19], and are interesting mathematical objects themselves deserving serious study, in particular as far as their representation theory is concerned [17].

1 Association schemes

(1.1) Association schemes. Let $X \neq \emptyset$ be a finite set and $n := |X| \in \mathbb{N}$. For $d \in \mathbb{N}_0$ let $X^2 := X \times X = \coprod_{i=0}^{d} R_i$ be a disjoint union of **relations** $R_i \neq \emptyset$, where $R_0 := \{[x,x]; x \in X\}$ is the **identity relation**, having the following **pairing** and **regularity properties**:

i) For all $i \in \{0, \dots, d\}$ there is $i^* \in \{0, \dots, d\}$ such that for the **transposed relation** we have $R_i^{\mathrm{tr}} := \{[y,x] \in X^2; [x,y] \in R_i\} = R_{i^*}$. In particular we have $i^{**} = i$ and $0^* = 0$; let $\mathcal{I} := \{i \in \{0, \dots, d\}; i^* = i\}$.

ii) For all $i,j,k \in \{0, \dots, d\}$ there is an **intersection number** $p_{ij}^k \in \mathbb{N}_0$ such that for all $[x,z] \in R_k$ we have $|\{y \in X; [x,y] \in R_i, [y,z] \in R_j\}| = p_{ij}^k$, independent of the particular choice of $[x,z] \in R_k$.

Then $\mathcal{X} := [X, \{R_0, \dots, R_d\}]$ is called a **(non-commutative) association scheme** or **homogeneous coherent configuration** [16] on X, of **order** n, and of **class** d or **rank** $d+1$. Elements $x, y \in X$ such that $[x,y] \in R_i$, for $i \in \{0, \dots, d\}$, are called **i-th associates**. Let the **valency** of R_i be defined as $n_i := p_{ii^*}^0$; if $n_i \leq 1$ for all $i \in \{0, \dots, d\}$ then $\mathcal{X}$ is called **thin**.

If $p_{ij}^k = p_{ji}^k$ for all $i,j,k \in \{0, \dots, d\}$, then $\mathcal{X}$ is called **commutative**. If $\mathcal{I} = \{0, \dots, d\}$ then $\mathcal{X}$ is called **symmetric** or of **Bose-Mesner type**; in this case by (1.2) we have $p_{ij}^k = p_{j^*i^*}^{k^*} = p_{ji}^k$, thus $\mathcal{X}$ is commutative.

E. g. for $|X| \geq 2$, letting $R_0 := \{[x,x]; x \in X\}$ and $R_1 := X^2 \setminus R_0$, we get the **trivial scheme** $[X, \{R_0, R_1\}]$, which is the unique association scheme on X of class 1. It is symmetric, and we have $n_0 = 1$ and $n_1 = n - 1$, as well as $P_0 := [p_{i0}^k]_{i,k} = I_2$ and $P_1 := [p_{i1}^k]_{i,k} = \begin{bmatrix} . & 1 \\ n-1 & n-2 \end{bmatrix}$.

(1.2) Proposition. Let $\mathcal{X}$ be an association scheme and $i,j,k \in \{0, \dots, d\}$.
a) We have $n_i = |\{y \in X; [x,y] \in R_i\}| \in \mathbb{N}$, for any $x \in X$. In particular we have $n_0 = 1$ and $\sum_{i=0}^{d} n_i = n$, as well as $n_i = n_{i^*}$.

b) We have $p_{0j}^k = \delta_{jk}$ and $p_{i0}^k = \delta_{ik}$ and $p_{ij}^0 = n_i\delta_{ij^*}$, as well as $p_{ij}^k = p_{j^*i^*}^{k^*}$. Moreover, we have $\sum_{j=0}^d p_{ij}^k = n_i$ and $\sum_{i=0}^d p_{ij}^k = n_j$, as well as the **triangle formula** $n_k p_{ij}^k = n_j p_{i^*k}^j = n_i p_{kj^*}^i$.

c) For $l, s, t \in \{0, \ldots, d\}$ we have the **quadrangle formula** $\sum_{k=0}^d p_{ij}^k p_{sk}^t = \sum_{l=0}^d p_{si}^l p_{lj}^t$, and if $p_{ij}^k \neq 0$ we have the inequality $p_{st}^k \leq \sum_{l=0}^d \min\{p_{sl}^i, p_{l^*t}^j\}$.

Proof. a) For any $[x,z] \in R_0$, that is for any $x = z \in X$, we have $n_i = p_{ii^*}^0 = |\{y \in X; [x,y] \in R_i, [y,z] \in R_{i^*}\}| = |\{y \in X; [x,y], [z,y] \in R_i\}| = |\{y \in X; [x,y] \in R_i\}|$, thus since $R_i \neq \emptyset$ we conclude $n_i \neq 0$. Moreover, we have $n_0 = |\{y \in X; [x,y] \in R_0\}| = |\{x\}| = 1$, and from $X = \coprod_{i=0}^d \{y \in X; [x,y] \in R_i\}$, for any $x \in X$, we get $\sum_{i=0}^d n_i = \sum_{i=0}^d |\{y \in X; [x,y] \in R_i\}| = |X| = n$. Finally, from $R_i = \coprod_{x \in X}\{y \in X; [x,y] \in R_i\}$ we get $nn_i = |R_i| = |R_{i^*}| = nn_{i^*}$.

b) For any $[x,z] \in R_k$ we have $p_{0j}^k = |\{y \in X; [x,y] \in R_0, [y,z] \in R_j\}| = |\{y \in X; y = x, [x,z] \in R_j\}|$, hence $p_{0j}^k = |\{x\}| = 1$ if $j = k$, and $p_{0j}^k = |\emptyset| = 0$ if $j \neq k$. Similarly we have $p_{i0}^k = |\{y \in X; [x,y] \in R_i, [y,z] \in R_0\}| = |\{y \in X; y = z, [x,z] \in R_i\}|$, hence $p_{i0}^k = |\{z\}| = 1$ if $i = k$, and $p_{i0}^k = |\emptyset| = 0$ if $i \neq k$.

For any $[x,z] \in R_0$, that is for any $x = z \in X$, we have $p_{ij}^0 = |\{y \in X; [x,y] \in R_i, [y,x] \in R_j\}| = |\{y \in X; [x,y] \in R_i \cap R_{j^*}\}|$, hence $p_{ij}^0 = 0$ if $i \neq j^*$. For any $[x,z] \in R_k$, that is $[z,x] \in R_{k^*}$, we have $p_{ij}^k = |\{y \in X; [x,y] \in R_i, [y,z] \in R_j\}| = |\{y \in X; [y,x] \in R_{i^*}, [z,y] \in R_{j^*}\}| = p_{j^*i^*}^{k^*}$.

For any $[x,z] \in R_k$ we have $\sum_{j=0}^d p_{ij}^k = \sum_{j=0}^d |\{y \in X; [x,y] \in R_i, [y,z] \in R_j\}| = |\{y \in X; [x,y] \in R_i\}| = n_i$ and $\sum_{i=0}^d p_{ij}^k = \sum_{i=0}^d |\{y \in X; [x,y] \in R_i, [y,z] \in R_j\}| = |\{y \in X; [y,z] \in R_j\}| = n_j$.

For any $x \in X$ we have $n_k p_{ij}^k = |\{[y,z] \in X^2; [x,y] \in R_i, [y,z] \in R_j, [x,z] \in R_k\}|$, thus we get $nn_k p_{ij}^k = |\{[x,y,z] \in X^3; [x,y] \in R_i, [y,z] \in R_j, [x,z] \in R_k\}| = |\{[x,y,z] \in X^3; [y,x] \in R_{i^*}, [x,z] \in R_k, [y,z] \in R_j\}| = nn_j p_{i^*k}^j$ and $nn_k p_{ij}^k = |\{[x,y,z] \in X^3; [x,z] \in R_k, [z,y] \in R_{j^*}, [x,y] \in R_i\}| = nn_i p_{kj^*}^i$.

c) For $[w,z] \in R_t$ we get $\sum_{k=0}^d p_{ij}^k p_{sk}^t = \sum_{k=0}^d |\{[x,y] \in X^2; [w,x] \in R_s, [x,z] \in R_k, [x,y] \in R_i, [y,z] \in R_j\}| = |\{[x,y] \in X^2; [w,x] \in R_s, [x,y] \in R_i, [y,z] \in R_j\}| = \sum_{l=0}^d |\{[x,y] \in X^2; [w,y] \in R_l, [y,z] \in R_j, [w,x] \in R_s, [x,y] \in R_i\}| = \sum_{l=0}^d p_{si}^l p_{lj}^t$.

For any $[x,y,z] \in X^3$ such that $[x,y] \in R_i$, $[y,z] \in R_j$, and $[x,z] \in R_k$, which by assumption exists, we have $p_{st}^k = |\{w \in X; [x,w] \in R_s, [w,z] \in R_t\}| = \sum_{l=0}^d |\{w \in X; [x,w] \in R_s, [w,z] \in R_t, [w,y] \in R_l\}|$, hence from $\{w \in X; [x,w] \in R_s, [w,z] \in R_t, [w,y] \in R_l\} \subseteq \{w \in X; [x,w] \in R_s, [w,y] \in R_l\}$ and $\{w \in X; [x,w] \in R_s, [w,z] \in R_t, [w,y] \in R_l\} \subseteq \{w \in X; [w,z] \in R_t, [w,y] \in R_l\}$ we deduce $p_{st}^k \leq \sum_{l=0}^d \min\{p_{sl}^i, p_{l^*t}^j\}$. $\sharp$

(1.3) Schurian schemes. Let $X \neq \emptyset$ be a finite set and $n := |X| \in \mathbb{N}$, and let $G \leq \mathcal{S}_n$ be a transitive permutation group on X. Then G acts diagonally on X^2, yielding the disjoint union $X^2 = \coprod_{i=0}^{d} \mathcal{O}_i$ of G-orbits, called the associated **orbitals**, where $d+1 \in \mathbb{N}$ is called the **rank** of the permutation action. The **Schurian scheme** $\mathcal{X} := [X, \{\mathcal{O}_0, \ldots, \mathcal{O}_d\}]$ is an association scheme of class d:

Since $x_0^G = X$ for any $x_0 \in X$, the **diagonal** $\mathcal{O}_0 := \{[x,x] \in X^2; x \in X\} = \{[x_0^g, x_0^g] \in X^2; g \in G\}$ indeed is an orbital. For any orbital $\mathcal{O} = \{[x^g, y^g] \in X^2; g \in G\}$, where $x, y \in X$, we have the **paired** orbital $\mathcal{O}^* = \{[y,x] \in X^2; [x,y] \in \mathcal{O}\}$. For $i,j,k \in \{0,\ldots,d\}$ let $[x,z],[x',z'] \in \mathcal{O}_k$. Hence there is $g \in G$ such that $[x',z'] = [x^g, z^g]$, thus we have $\{y \in X; [x,y] \in \mathcal{O}_i, [y,z] \in \mathcal{O}_j\}^g = \{y' \in X; [x, y'^{g^{-1}}] \in \mathcal{O}_i, [y'^{g^{-1}}, z] \in \mathcal{O}_j\} = \{y' \in X; [x^g, y'] \in \mathcal{O}_i, [y', z^g] \in \mathcal{O}_j\} = \{y' \in X; [x', y'] \in \mathcal{O}_i, [y', z'] \in \mathcal{O}_j\}$, hence $|\{y \in X; [x,y] \in \mathcal{O}_i, [y,z] \in \mathcal{O}_j\}| = |\{y' \in X; [x',y'] \in \mathcal{O}_i, [y',z'] \in \mathcal{O}_j\}|$, implying regularity. $\sharp$

The association scheme $\mathcal{X}$ is symmetric if and only if for all $x, y \in X$ there is an orbital $\mathcal{O}$ such that $[x,y],[y,x] \in \mathcal{O}$, that is there is $g \in G$ such that $[y,x] = [x^g, y^g]$, that is $x^g = y$ and $y^g = x$, that is G is **generously transitive**.

For any $x_0 \in X$ let $H := \mathrm{Stab}_G(x_0) \leq G$; hence $H\backslash G \to X \colon Hg \mapsto x_0^g$ is an isomorphism of G-sets, and we have $n = |X| = [G \colon H]$. Let $X = \coprod_{i=0}^{r} X_i$ be the disjoint union of H-orbits, for some $r \in \mathbb{N}_0$, called the associated H**-suborbits**. Letting $x_i \in X_i$ and $g_i \in G$ such that $x_i = x_0^{g_i}$ as well as $H_i := \mathrm{Stab}_H(x_i) = H \cap H^{g_i} \leq H$, for all $i \in \{1,\ldots,r\}$, and for completeness $g_0 := 1$ and $H_0 := H$, we get an isomorphism of H-sets $H_i\backslash H \to X_i \colon H_i h \mapsto x_i^h$.

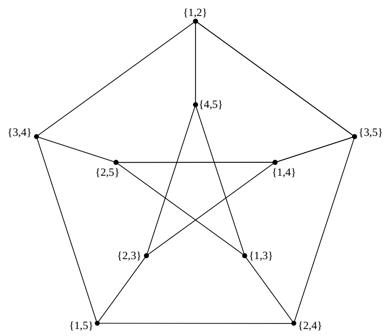
Since $x_0^G = X$, for any orbital $\mathcal{O}$ there is $x \in X$ such that $[x_0, x] \in \mathcal{O}$. For $x, y \in X$ we have $[x_0,x],[x_0,y] \in \mathcal{O}$ if and only if x and y are in one and the same H-suborbit: If $y = x^h$, for some $h \in H$, then $[x_0,y] = [x_0,x^h] = [x_0,x]^h$; if conversely $[x_0,x],[x_0,y] \in \mathcal{O}$ then there is $g \in G$ such that $[x_0,y]^g = [x_0,x]$, implying that $g \in \mathrm{Stab}_G(x_0) = H$ and $y = x^g$. Hence there is a bijection between the H-suborbits and the orbitals, implying that $r = d$, and we may assume that $X_i = \{x \in X; [x_0,x] \in \mathcal{O}_i\}$, for all $i \in \{0,\ldots,d\}$.

The valencies coincide with the subgroup indices $n_i = |\{x \in X; [x_0,x] \in \mathcal{O}_i\}| = |X_i| = [H \colon H_i]$, and $p_{ij}^k = |\{x \in X_i; [x, x_0^{g_k}] \in \mathcal{O}_j\}| = |\{x \in X_i; [x_0, x^{g_k^{-1}}] \in \mathcal{O}_{j^*}\}| = |X_i \cap X_{j^*}^{g_k}|$, for all $i,j,k \in \{0,\ldots,d\}$; to straightforwardly determine the matrix $P_j = [p_{ij}^k]_{ik}$ we better use $p_{ij}^k = \frac{n_j}{n_k} \cdot p_{i^*k}^j = \frac{n_j}{n_k} \cdot |X_{i^*} \cap X_{k^*}^{g_j}|$.

E. g. $X := G$ becomes a G-set by right multiplication action; hence we have $n = |G|$. The orbitals are given by $\mathcal{O}_g := \{[x, x^g] \in X^2; x \in X\}$, thus we have $n_g = 1$, for all $g \in G$. Hence the **regular** Schurian scheme $[X, \{\mathcal{O}_g; g \in G\}]$ is thin; it is symmetric if and only if $\exp(G) := \mathrm{lcm}\{|g| \in \mathbb{N}; g \in G\} \leq 2$.

(1.4) Example: Johnson schemes. Let $V \neq \emptyset$ be a finite set and $v := |V| \in \mathbb{N}$, and let $k \in \mathbb{N}_0$ such that $k \leq \frac{v}{2}$. Let X be the set of all k-element subsets of V, hence $n := |X| = \binom{v}{k}$. For $i \in \{0,\ldots,k\}$ let $R_i := \{[x,y] \in X^2; |x \cap y| = k-i\}$. Hence we have $X^2 = \coprod_{i=0}^{k} R_i$, where $R_0 = \{[x,x] \in X^2; x \in X\}$ and $R_i^{\mathrm{tr}} = R_i$

Table 1: Petersen graph.



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for all $i \in \{0, \ldots, k\}$. Regularity is fulfilled as well, by specifying an appropriate group action, hence the **Johnson scheme** $\mathcal{J}(v, k) := [X, \{R_0, \ldots, R_k\}]$ is a symmetric Schurian association scheme of class k:

The symmetric group $\mathcal{S}_v$ acts k-transitively on V, that is $\mathcal{S}_v$ acts transitively on X. For all $x, y \in X$ and $\pi \in \mathcal{S}_v$ we have $|x \cap y| = |x^\pi \cap y^\pi|$, hence the R_i are unions of orbitals. Conversely, let $x, y, x', y' \in X$ such that $|x \cap y| = |x' \cap y'|$. Then we have $|x \setminus (x \cap y)| = |x' \setminus (x' \cap y')|$ and $|y \setminus (x \cap y)| = |y' \setminus (x' \cap y')|$, and let $\pi \in \mathcal{S}_v$ such that $(x \cap y)^\pi = x' \cap y'$ and $x \setminus (x \cap y)^\pi = x' \setminus (x' \cap y')$ and $y \setminus (x \cap y)^\pi = y' \setminus (x' \cap y')$. Then we have $x^\pi = x'$ and $y^\pi = y'$, implying that the R_i are precisely the orbitals. ♯

For $x \in X$ we have $H := \mathrm{Stab}_{\mathcal{S}_v}(x) \cong \mathcal{S}_{v-k} \times \mathcal{S}_k$, and for $i \in \{0, \ldots, k\}$ and $y \in X$ such that $[x, y] \in R_i$, that is $|x \cap y| = k - i$, we have $\mathrm{Stab}_H(y) \cong (\mathcal{S}_{v-k-i} \times \mathcal{S}_i) \times (\mathcal{S}_{k-i} \times \mathcal{S}_i)$, hence $n_i = [H \colon \mathrm{Stab}_H(y)] = \binom{v-k}{i}\binom{k}{i}$; thus from $n = \sum_{i=0}^{k} n_i$ we recover the identity $\binom{v}{k} = \sum_{i=0}^{k} \binom{v-k}{i}\binom{k}{i}$.

E. g. the relation R_1 of $\mathcal{J}(4, 2)$ can be depicted as the regular octahedron, R_2 becoming the space diagonals, hence $n_0 = 1$, $n_1 = 4$ and $n_2 = 1$, and $P_0 := [p_{i0}^k]_{i,k} = I_3$, $P_1 := [p_{i1}^k]_{i,k} = \begin{bmatrix} . & 1 & . \\ 4 & 2 & 4 \\ . & 1 & . \end{bmatrix}$, $P_2 := [p_{i2}^k]_{i,k} = \begin{bmatrix} . & . & 1 \\ . & 1 & . \\ 1 & . & . \end{bmatrix}$.

E. g. the relation R_2 of $\mathcal{J}(5, 2)$ can be depicted as the Petersen graph, Table 1. Hence we have $n_0 = 1$, $n_1 = 6$ and $n_2 = 3$, as well as $P_0 := [p_{i0}^k]_{i,k} = I_3$ and $P_1 := [p_{i1}^k]_{i,k} = \begin{bmatrix} . & 1 & . \\ 6 & 3 & 4 \\ . & 2 & 2 \end{bmatrix}$ and $P_2 := [p_{i2}^k]_{i,k} = \begin{bmatrix} . & . & 1 \\ . & 2 & 2 \\ 3 & 1 & . \end{bmatrix}$.

(1.5) Example: Hamming schemes. Let F be a finite set and $q := |F| \geq 2$, and for $n \in \mathbb{N}$ let $X := F^n$, hence $|X| = q^n$. For $x = [x_1, \ldots, x_n] \in F^n$ and

$y = [y_1, \ldots, y_n] \in F^n$ let $d(x, y) := |\{i \in \{1, \ldots, n\}; x_i \neq y_i\}| \in \mathbb{N}_0$ be the associated **Hamming distance**. This defines a **metric** on F^n: The Hamming distance is **positive definite**, that is we have $d(x, y) = 0$ if and only if $x = y$, **symmetric**, that is we have $d(x, y) = d(y, x)$, and the **triangle inequality** holds: From $\{i \in \{1, \ldots, n\}; x_i \neq z_i\} = \{i \in \{1, \ldots, n\}; y_i = x_i \neq z_i\} \,\dot\cup\, \{i \in \{1, \ldots, n\}; y_i \neq x_i \neq z_i\} \subseteq \{i \in \{1, \ldots, n\}; y_i \neq z_i\} \cup \{i \in \{1, \ldots, n\}; x_i \neq y_i\}$ we get $d(x, z) \leq d(x, y) + d(y, z)$, for all $x, y, z \in F^n$.

For $i \in \{0, \ldots, n\}$ let $R_i := \{[x, y] \in X^2; d(x, y) = i\}$. Hence we have $X^2 = \coprod_{i=0}^n R_i$, where $R_0 = \{[x, x] \in X^2; x \in X\}$ and $R_i^{\text{tr}} = R_i$ for all $i \in \{0, \ldots, n\}$. Regularity is fulfilled as well, by specifying an appropriate group action, hence the **Hamming scheme** $\mathcal{H}(q, n) := [X, \{R_0, \ldots, R_n\}]$ is a symmetric Schurian association scheme of class n:

We consider the **wreath product** $\mathcal{S}_q \wr \mathcal{S}_n := \mathcal{S}_n \ltimes \mathcal{S}_q^n$, the semidirect product of $\mathcal{S}_n$ with the direct product $\mathcal{S}_q^n := \mathcal{S}_q \times \cdots \times \mathcal{S}_q$ with n factors, where $\mathcal{S}_n$ acts on $\mathcal{S}_q^n$ by permuting the direct factors. The group $\mathcal{S}_q$ acts naturally on $F := \{1, \ldots, q\}$, and the natural action of $\mathcal{S}_n$ on $\{1, \ldots, n\}$ induces an action on X, yielding a transitive action of $\mathcal{S}_q \wr \mathcal{S}_n$, given by $[x_1, \ldots, x_n] \mapsto [x_{1^{\tau^{-1}}}^{\sigma_1}, \ldots, x_{n^{\tau^{-1}}}^{\sigma_n}]$, for all $x \in X$ and $\pi := [\tau; \sigma_1, \ldots, \sigma_n] \in \mathcal{S}_q \wr \mathcal{S}_n$.

For all $x, y \in X$ and $\pi \in \mathcal{S}_q \wr \mathcal{S}_n$ we have $d(x, y) = d(x^\pi, y^\pi)$, thus $\mathcal{S}_q \wr \mathcal{S}_n$ induces isometries of X with respect to the Hamming distance, and the R_i are unions of orbitals; actually, $\mathcal{S}_q \wr \mathcal{S}_n$ is the group of all such isometries of X. Conversely, let $x, y, x', y' \in X$ such that $d(x, y) = d(x', y')$. Then let $\mathcal{J} := \{i \in \{1, \ldots, n\}; x_i \neq y_i\}$ and $\mathcal{J}' := \{i \in \{1, \ldots, n\}; x'_i \neq y'_i\}$, and let $\tau \in \mathcal{S}_n$ such that $\mathcal{J}^\tau = \mathcal{J}'$. Moreover, for $i \in \mathcal{J}'$ let $\sigma_i \in \mathcal{S}_q$ such that $x_{i^{\tau^{-1}}}^{\sigma_i} = x'_i$ and $y_{i^{\tau^{-1}}}^{\sigma_i} = y'_i$, while for $i \notin \mathcal{J}'$ let $\sigma_i \in \mathcal{S}_q$ such that $x_{i^{\tau^{-1}}}^{\sigma_i} = x'_i$. Then for $\pi := [\tau; \sigma_1, \ldots, \sigma_n] \in \mathcal{S}_q \wr \mathcal{S}_n$ we have $x^\pi = x'$ and $y^\pi = y'$, implying that the R_i are precisely the orbitals. $\sharp$

For $x \in X$ we have $H := \text{Stab}_{\mathcal{S}_q \wr \mathcal{S}_n}(x) \cong \mathcal{S}_{q-1} \wr \mathcal{S}_n$, and for $i \in \{0, \ldots, n\}$ and $y \in X$ such that $[x, y] \in R_i$, that is $d(x, y) = i$, we have $\text{Stab}_H(y) \cong (\mathcal{S}_{q-2} \wr \mathcal{S}_i) \times (\mathcal{S}_{q-1} \wr \mathcal{S}_{n-i})$, hence $n_i = [H : \text{Stab}_H(y)] = \frac{(q-1)!^n \cdot n!}{(q-2)!^i \cdot i! \cdot (q-1)!^{n-i} \cdot (n-i)!} = (q-1)^i \cdot \binom{n}{i}$; thus from $|X| = \sum_{i=0}^k n_i$ we recover the identity $q^n = \sum_{i=0}^n (q-1)^i \cdot \binom{n}{i}$.

E. g. the relation R_1 of $\mathcal{H}(2, 2)$ can be depicted as the regular quadrangle, the lower left hand vertex being located in the origin; then R_2 is depicted as the diagonals. Hence we have $n_0 = 1$, $n_1 = 2$ and $n_2 = 1$, as well as $P_0 := [p_{i0}^k]_{i,k} = I_3$ and $P_1 := [p_{i1}^k]_{i,k} = \begin{bmatrix} . & 1 & . \\ 2 & . & 2 \\ . & 1 & . \end{bmatrix}$ and $P_2 := [p_{i2}^k]_{i,k} = \begin{bmatrix} . & . & 1 \\ . & 1 & . \\ 1 & . & . \end{bmatrix}$.

E. g. the relation R_1 of $\mathcal{H}(2, 3)$ can be depicted as the regular cube, the lower left front vertex being located in the origin; then R_2 and R_3 are depicted as the face and space diagonals, respectively. Hence we have $n_0 = 1$, $n_1 = 3$, $n_2 = 3$

and $n_3 = 1$, as well as $P_0 := [p_{i0}^k]_{i,k} = I_4$ and $P_1 := [p_{i1}^k]_{i,k} = \begin{bmatrix} . & 1 & . & . \\ 3 & . & 2 & . \\ . & 2 & . & 3 \\ . & . & 1 & . \end{bmatrix}$ and $P_2 := [p_{i2}^k]_{i,k} = \begin{bmatrix} . & . & 1 & . \\ . & 2 & . & 3 \\ 3 & . & 2 & . \\ . & 1 & . & . \end{bmatrix}$ and $P_3 := [p_{i3}^k]_{i,k} = \begin{bmatrix} . & . & . & 1 \\ . & . & 1 & . \\ . & 1 & . & . \\ 1 & . & . & . \end{bmatrix}$.

2 Adjacency algebras

(2.1) Adjacency matrices. Let $\mathcal{X}$ be an association scheme. For all $i \in \{0, \ldots, d\}$ let the i-th **adjacency matrix** $A_i := [a_{ixy}]_{x,y \in X} \in \{0,1\}^{n \times n} \subseteq \mathbb{Z}^{n \times n}$ be defined as $a_{ixy} := 1$ if and only if $[x,y] \in R_i$. Then the defining properties of $\mathcal{X}$ yield the following: We have $A_0 = I_n$ and $\sum_{i=0}^d A_i = J_n$, where $J_n := [1]_{x,y \in X} \in \mathbb{Z}^{n \times n}$ is the **all-1 matrix**. For all $i \in \{0, \ldots, d\}$ we have $A_i^{\mathrm{tr}} = A_{i^*}$, and thus $\mathcal{X}$ is a symmetric association scheme if and only if all the A_i are symmetric matrices.

By the definition of matrix products, we for all $i, j, k \in \{0, \ldots, d\}$ have $A_i A_j = \sum_{k=0}^d p_{ij}^k A_k \in \mathbb{Z}^{n \times n}$. Thus $\mathcal{A} := \langle A_0, \ldots, A_d \rangle_{\mathbb{Z}} \subseteq \mathbb{Z}^{n \times n}$ is a $\mathbb{Z}$-algebra, called the **adjacency algebra** or **Bose-Mesner algebra** associated with $\mathcal{X}$. The algebra $\mathcal{A}$ is $\mathbb{Z}$-free such that $\mathrm{rk}_{\mathbb{Z}}(\mathcal{A}) = d+1$, and $\{A_0, \ldots, A_d\}$ is a $\mathbb{Z}$-basis, called the **Schur basis**. Moreover, $\mathcal{X}$ is commutative if and only if $\mathcal{A}$ is commutative.

E. g. for the regular Schurian scheme associated to a finite group G we have $\mathcal{A} \cong \mathbb{Z}G$, the associated **group algebra**. For the Hamming scheme $\mathcal{H}(2,2)$, letting $X := \{[0,0],[0,1],[1,0],[1,1]\}$, we get $A_0 = I_4$ and $A_1 = \begin{bmatrix} . & 1 & 1 & . \\ 1 & . & . & 1 \\ 1 & . & . & 1 \\ . & 1 & 1 & . \end{bmatrix}$ and $A_2 = \begin{bmatrix} . & . & . & 1 \\ . & . & 1 & . \\ . & 1 & . & . \\ 1 & . & . & . \end{bmatrix}$. For the Johnson scheme $\mathcal{J}(4,2)$, letting $X := \{\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\}\}$, we finally get $A_0 = I_6$ and $A_1 = \begin{bmatrix} . & 1 & 1 & 1 & 1 & . \\ 1 & . & 1 & 1 & . & 1 \\ 1 & 1 & . & . & 1 & 1 \\ 1 & 1 & . & . & 1 & 1 \\ 1 & . & 1 & 1 & . & 1 \\ . & 1 & 1 & 1 & 1 & . \end{bmatrix}$ and $A_2 = \begin{bmatrix} . & . & . & . & . & 1 \\ . & . & . & . & 1 & . \\ . & . & . & 1 & . & . \\ . & . & 1 & . & . & . \\ . & 1 & . & . & . & . \\ 1 & . & . & . & . & . \end{bmatrix}$.

(2.2) Representations. Let R be a commutative ring. Then $\mathcal{A}_R := \mathcal{A} \otimes_{\mathbb{Z}} R \subseteq R^{n \times n}$ is an R-free R-algebra such that $\mathrm{rk}_R(\mathcal{A}_R) = d+1$, and $\{A_0, \ldots, A_d\} \subseteq \mathcal{A} \subseteq \mathcal{A}_R$ is an R-basis, where we identify $A \in \mathbb{Z}^{n \times n}$ with $A \otimes 1_R \in R^{n \times n}$ for all

$A \in \mathcal{A}$. We have $\mathcal{A}_{\mathbb{Z}} \cong \mathcal{A}$, and $\mathcal{A}_R$ is commutative if and only if $\mathcal{A}$ is.

Let V be an $\mathcal{A}_R$-**lattice**, that is an R-free $\mathcal{A}_R$-module, of **degree** $r := \mathrm{rk}_R(V) \in \mathbb{N}_0$, and let the homomorphism of R-algebras $\delta\colon \mathcal{A}_R \to \mathrm{End}_R(V) \cong R^{r\times r}$ be the associated representation. Then the R-linear map $\varphi_\delta\colon \mathcal{A}_R \to R\colon A \mapsto \mathrm{tr}(\delta(A))$ is called the **character** of $\mathcal{A}_R$ **afforded by** V, where φ_δ is independent of the particular choice of the R-algebra isomorphism to $R^{r\times r}$, that is the particular choice of an R-basis of V, and $\varphi_\delta(A_0) = r$.

Let F be a field. Then up to isomorphism there are only finitely many irreducible $\mathcal{A}_F$-modules $S_0, \ldots, S_t$, for some $t \in \mathbb{N}_0$; for $i \in \{0, \ldots, t\}$ let $d_i := \dim_F(S_i) \in \mathbb{N}$ be the associated **degree**. Let $\mathrm{Irr}(\mathcal{A}_F) := \{\varphi_0, \ldots, \varphi_t\}$ be the characters afforded by the irreducible $\mathcal{A}_F$-modules; hence we have $d_i = \varphi_i(A_0) \in F$, and if $\mathcal{A}_F$ is **split**, that is F is a splitting field of $\mathcal{A}_F$, then by **Wedderburn's Theorem** $\mathrm{Irr}(\mathcal{A}_F)$ is F-linearly independent. The matrix $\Phi(\mathcal{A}_F) := [\varphi_i(A_j)]_{ij} \in F^{(t+1)\times(d+1)}$ is called the F-**character table** of $\mathcal{A}_F$; we may identify $\mathrm{Irr}(\mathcal{A}_F)$ with the rows of $\Phi(\mathcal{A}_F)$.

If V is an $\mathcal{A}_F$-module then by the Jordan-Hölder Theorem the multiplicity $[V\colon S_i] \in \mathbb{N}_0$ of S_i in an $\mathcal{A}_F$-module composition series of V is independent of the particular choice of the composition series, and we have $\varphi_V = \sum_{i=0}^{t}[V\colon S_i]\cdot\varphi_i$. If V is a **faithful** $\mathcal{A}_F$-module, that is δ_V is injective, we have $[V\colon S_i] > 0$ for all $i \in \{0, \ldots, t\}$: Letting $e \in A$ be an S_i-primitive idempotent, we since $\delta_V(e) \neq 0$ have $[V\colon S_i]\cdot\dim_F(\mathrm{End}_{A_F}(S_i)) = \dim_F(\mathrm{Hom}_{A_F}(eA_F, V)) = \dim_F(Ve) \neq 0$.

E. g. $V_{\mathrm{nat}} := R^n$ is an $\mathcal{A}_R$-lattice, called the **natural** or **standard module**, the associated representation being $\delta_{\mathrm{nat}} = \mathrm{id}_{\mathcal{A}}$, which is faithful. For its character we have $\nu(A_i) = \mathrm{tr}(A_i) = n\delta_{0i}$, hence $\nu(A_iA_j) = \sum_{k=0}^{d} p_{ij}^k \mathrm{tr}(A_k) = np_{ij}^0 = nn_i\delta_{ij^*}$, for all $i, j \in \{0, \ldots, d\}$. If F is a field, the integer $m_i := [V_{\mathrm{nat}}\colon S_i] \in \mathbb{N}$ is called the **multiplicity** associated with φ_i, for all $i \in \{0, \ldots, t\}$.

E. g. $V := R$ becomes an $\mathcal{A}_R$-lattice, called the **trivial** or **index module**, by letting $\varphi_0\colon \mathcal{A}_R \to \mathrm{End}_R(R) \cong R\colon A_i \mapsto n_i$, for all $i \in \{0, \ldots, d\}$: By (1.2) we have $\varphi_0(A_iA_j) = \varphi_0(\sum_{k=0}^{d} p_{ij}^k A_k) = \sum_{k=0}^{d} n_k p_{ij}^k = \sum_{k=0}^{d} n_i p_{kj^*}^i = n_i \cdot \sum_{k=0}^{d} p_{kj^*}^i = n_in_j$, for all $i, j \in \{0, \ldots, d\}$. If F is a field then $S_0 := F$ is an irreducible $\mathcal{A}_F$-module.

(2.3) Intersection matrices. Let $\mathcal{X}$ be an association scheme, and let R be a commutative ring. Then $\mathcal{A}_R$ is an $\mathcal{A}_R$-lattice with R-basis $\{A_0, \ldots, A_d\}$, and for the associated **(right) regular representation** $\delta_\rho\colon \mathcal{A}_R \mapsto \mathrm{End}_R(\mathcal{A}_R) \cong R^{(d+1)\times(d+1)}\colon A_j \mapsto (\mathcal{A}_R \to \mathcal{A}_R\colon A \mapsto AA_j)$ the multiplication rule yields $\delta_\rho\colon A_j \mapsto P_j$, for all $j \in \{0, \ldots, d\}$, where the j-th **intersection matrix** or **collapsed adjacency matrix** is defined as $P_j := [p_{ij}^k]_{ik} \in \mathbb{N}_0^{(d+1)\times(d+1)} \subseteq \mathbb{Z}^{(d+1)\times(d+1)} \subseteq R^{(d+1)\times(d+1)}$.

From $A_0A = A \neq 0$ for all $0 \neq A \in \mathcal{A}_R$ we conclude that δ_ρ is faithful. Hence letting $\mathcal{P} := \langle P_0, \ldots, P_d\rangle_{\mathbb{Z}} \subseteq \mathbb{Z}^{(d+1)\times(d+1)}$ be the **intersection algebra** associated with $\mathcal{X}$, and $\mathcal{P}_R := \mathcal{P} \otimes_{\mathbb{Z}} R \subseteq R^{(d+1)\times(d+1)}$, we get an isomorphism of R-algebras $\delta_\rho\colon \mathcal{A}_R \to \mathcal{P}_R\colon A_j \to P_j$, for all $j \in \{0,\ldots,d\}$. For its character we have $\rho(A_j) = \sum_{i=0}^d p_{ij}^i$ for all $j \in \{0,\ldots,d\}$.

E. g. for the Hamming scheme $\mathcal{H}(2,2)$ we have $\delta_\rho\colon A_j \mapsto P_j$, for all $j \in \{0,1,2\}$, where A_j is as given in (2.1) and P_j is as given in (1.5). For the Johnson scheme $\mathcal{J}(4,2)$ we have $\delta_\rho\colon A_j \mapsto P_j$, for all $j \in \{0,1,2\}$, where A_j is as given in (2.1) and P_j is as given in (1.4).

(2.4) Theorem. Let F be a field such that $p := \mathrm{char}(F) \geq 0$, let $\mathcal{X}$ be an association scheme, and let $\mathcal{J} := \{j \in \{0,\ldots,d\}; p \mid nn_j\}$.
a) For the **Jacobson radical** of $\mathcal{A}_F$ we have $\mathrm{rad}(\mathcal{A}_F) \subseteq \langle A_j; j \in \mathcal{J}\rangle_F$. In particular, if $p \not| \, n \cdot \prod_{i=0}^d n_i$ then $\mathcal{A}_F$ is **semisimple**, that is the regular module $\mathcal{A}_F$ is **completely reducible**, that is the direct sum of irreducible submodules.
b) We have $\langle J_n\rangle_F \trianglelefteq \mathcal{A}_F$, and $\langle J_n\rangle_F \subseteq \mathrm{rad}(\mathcal{A}_F)$ if and only if $p \mid n$. In particular, if $p \mid n$ then $\mathcal{A}_F$ is not semisimple.

Proof. We have $\mathrm{rad}(\mathcal{A}_F) := \bigcap\{U < \mathcal{A}_F \text{ maximal}\} = \bigcap_{i=0}^t \mathrm{ann}_{\mathcal{A}_F}(S_i) \lhd \mathcal{A}_F$, where $\mathrm{ann}_{\mathcal{A}_F}(S_i) := \{A \in \mathcal{A}_F; S_i A = \{0\}\} \lhd \mathcal{A}_F$ is the **annihilator** of S_i. Then $\mathrm{rad}(\mathcal{A}_F) \lhd \mathcal{A}_F$ is the largest **nilpotent** ideal, that is we have $\mathrm{rad}(\mathcal{A}_F)^k = \{0\}$ for some $k \in \mathbb{N}$, and $\mathcal{A}_F$ is semisimple if and only if $\mathrm{rad}(\mathcal{A}_F) = \{0\}$.
a) Let $A := \sum_{k=0}^d \alpha_k A_k \in \mathrm{rad}(\mathcal{A}_F)$. Considering an $\mathcal{A}_F$-module composition series of the natural module F^n, we from $AA_j \in \mathrm{rad}(\mathcal{A}_F) \subseteq \mathrm{ann}_{\mathcal{A}_F}(S_i)$, for all $i \in \{0,\ldots,t\}$ and $j \in \{0,\ldots,d\}$, get that $0 = \nu(AA_j) = \sum_{k=0}^d \alpha_k n n_k \delta_{kj^*} = \alpha_{j^*} n n_{j^*}$. Hence for any $j \in \{0,\ldots,d\}$ such that $\alpha_j \neq 0$ we have $p \mid nn_j$, implying that $A \in \langle A_j; j \in \mathcal{J}\rangle_F$. The assumption $p \not| \, n \cdot \prod_{i=0}^d n_i$ is equivalent to $\mathcal{J} = \emptyset$, which implies $\mathrm{rad}(\mathcal{A}_F) = \{0\}$.
b) We have $A_i J_n = n_i J_n$ and $J_n A_i = n_{i^*} J_n = n_i J_n$, for all $i \in \{0,\ldots,d\}$, hence $\langle J_n\rangle_F \trianglelefteq \mathcal{A}_F$. Moreover, we have $J_n^2 = nJ_n$, hence $\langle J_n\rangle_F$ is nilpotent if and only if $p \mid n$. Hence, if $p \mid n$ then $\{0\} \neq \langle J_n\rangle_F \subseteq \mathrm{rad}(\mathcal{A}_F)$. ♯

(2.5) Corollary: Maschke's Theorem. If $\mathcal{X}$ is thin, then $\mathcal{A}_F$ is semisimple if and only if $p \not| \, n$.

(2.6) Corollary. Let $p \not| \, n$. Then $\mathcal{A}_F \cong \langle J_n\rangle_F \oplus \mathcal{A}'_F$ as F-algebras, where $\langle J_n\rangle_F$ affords the trivial character; and the multiplicity of the trivial module as a constituent of the natural $\mathcal{A}_F$-module is $m_0 = 1$.

Proof. The first assertion follows from $\langle J_n\rangle_F \cap \mathrm{rad}(\mathcal{A}_F) = \{0\}$. As for the second assertion, we from $J_n^2 = nJ_n$ and $0 \neq J_n \neq nI_n$ get the minimum polynomial $\mu_{J_n} = T(T-n) \in F[T]$. Hence we have eigenspace decomposition $F^n = E_n(J_n) \oplus E_0(J_n)$ with respect to the eigenvalues n and 0, respectively, where $\dim_F(\mathrm{im}(J_n)) = 1$ yields $\dim_F(E_0(J_n)) = n-1$, thus $\dim_F(E_n(J_n)) = 1$. Since $J_n \in Z(\mathcal{A}_F)$ we conclude that $E_0(J_n)$ and $E_n(J_n)$ are $\mathcal{A}_F$-submodules. Since $\varphi_0(J_n) = \sum_{j=0}^d \varphi_0(A_j) = \sum_{j=0}^d n_j = n$, and J_n acts on $E_0(J_n)$ by the zero map, $E_0(J_n)$ does not have the trivial module as a constituent. ♯

(2.7) Example: Johnson scheme $\mathcal{J}(7,2)$. We have $n = \binom{7}{2} = 21$ as well as $n_0 = 1$ and $n_1 = \binom{5}{1}\binom{2}{1} = 10$ and $n_2 = \binom{5}{2}\binom{2}{2} = 10$. Thus $\mathcal{A}_F$, where F is a field, is semisimple if $\mathrm{char}(F) \notin \{2,3,5,7\}$, it is not semisimple if $\mathrm{char}(F) \in \{3,7\}$, while (2.4) does not assert anything if $\mathrm{char}(F) \in \{2,5\}$; actually $\mathcal{A}_F$ is semisimple if $\mathrm{char}(F) = 2$ and it is not semisimple if $\mathrm{char}(F) = 5$:

We have $P_0 = I_3$, $P_1 = \begin{bmatrix} . & 1 & . \\ 10 & 5 & 4 \\ . & 4 & 6 \end{bmatrix}$, $P_2 = \begin{bmatrix} . & . & 1 \\ . & 4 & 6 \\ 10 & 6 & 3 \end{bmatrix}$, hence $P_1P_2 = P_2P_1$, thus $\mathcal{J}(7,2)$ is commutative. For the associated minimum polynomials we get $\mu_{A_1} = \mu_{P_1} = (T-10)(T-3)(T+2) \in \mathbb{Z}[T]$ and $\mu_{A_2} = \mu_{P_2} = (T-10)(T+4)(T-1) \in \mathbb{Z}[T]$. Hence P_1 and P_2 have simultaneous Jordan normal forms over any field F, all irreducible representations of $\mathcal{A}_F$ have degree 1, implying that $\mathcal{A}_F$ is split.

If $\mathrm{char}(F) \notin \{2,3,5,7\}$, then both P_1 and P_2 have three pairwise distinct eigenvalues in F, thus are diagonalisable, where simultaneous eigenvectors are $\begin{bmatrix} 1 & 1 & 1 \\ -10 & -3 & 1 \\ 10 & -2 & 1 \end{bmatrix}$, the eigenvalues being ordered as above. Thus $\mathcal{A}_F \cong S_0 \oplus S_1 \oplus S_2$, and the character table $\Phi(\mathcal{A}_F)$ is given as:

	m_i	d_i	A_0	A_1	A_2
φ_0	1	1	1	10	10
φ_1	6	1	1	3	-4
φ_2	14	1	1	-2	1

The multiplicities are computed as follows: If $\mathrm{char}(F) = 0$ then the multiplicities can be just read off from the decomposition $\nu = \varphi_0 + 6\varphi_1 + 14\varphi_2$. Hence for the associated characteristic polynomials we have $\chi_{A_1} = (T-10)(T-3)^6(T+2)^{14} \in \mathbb{Z}[T]$ and $\chi_{A_2} = (T-10)(T+4)^6(T-1)^{14} \in \mathbb{Z}[T]$. This yields the characteristic polynomials of A_1 and A_2 over any field F, and thus the associated multiplicities. For the remaining cases we have:

i) If $\mathrm{char}(F) = 7$, then $\mu_{P_1} = (T-3)^2(T-5) \in F[T]$ and $\mu_{P_2} = (T-3)^2(T-1) \in F[T]$. Vectors inducing simultaneous Jordan normal forms are $\left[\begin{array}{ccc} 2 & 1 & . \\ 1 & 1 & 1 \\ \hline 3 & 5 & 1 \end{array}\right]$.

Thus $\mathcal{A}_F \cong \begin{bmatrix} S_0 \\ S_0 \end{bmatrix} \oplus S_1$, and $\Phi(\mathcal{A}_F)$ is given as:

	m_i	d_i	A_0	A_1	A_2
φ_0	7	1	1	3	3
φ_1	14	1	1	5	1

ii) If $\mathrm{char}(F) = 3$, then $\mu_{P_1} = (T-1)^2T \in F[T]$ and $\mu_{P_2} = (T-1)(T-2) \in F[T]$. Vectors inducing simultaneous Jordan normal forms are $\left[\begin{array}{ccc} . & 1 & . \\ 1 & 1 & 1 \\ \hline 2 & . & 1 \end{array}\right]$. Thus $\mathcal{A}_F \cong \begin{bmatrix} S_0 \\ S_0 \end{bmatrix} \oplus S_1$, and $\Phi(\mathcal{A}_F)$ is given as:

	m_i	d_i	A_0	A_1	A_2
φ_0	15	1	1	1	1
φ_1	6	1	1	0	2

iii) If $\mathrm{char}(F) = 5$, then $\mu_{P_1} = T(T-3)^2 \in F[T]$ and $\mu_{P_2} = T(T-1)^2 \in F[T]$. Vectors inducing simultaneous Jordan normal forms are $\left[\begin{array}{ccc} 1 & 1 & 1 \\ \hline . & 1 & . \\ . & 3 & 1 \end{array}\right]$. Thus $\mathcal{A}_F \cong S_0 \oplus \begin{bmatrix} S_1 \\ S_1 \end{bmatrix}$, and $\Phi(\mathcal{A}_F)$ is given as:

	m_i	d_i	A_0	A_1	A_2
φ_0	1	1	1	0	0
φ_1	20	1	1	3	1

iv) If $\mathrm{char}(F) = 2$, then $\mu_{P_1} = T(T-1) \in F[T]$ and $\mu_{P_2} = T(T-1) \in F[T]$. Simultaneous eigenvectors are $\begin{bmatrix} 1 & 1 & 1 \\ . & 1 & . \\ . & . & 1 \end{bmatrix}$. Thus $\mathcal{A}_F \cong S_0 \oplus S_1 \oplus S_2$, and $\Phi(\mathcal{A}_F)$ is given as:

	m_i	d_i	A_0	A_1	A_2
φ_0	1	1	1	0	0
φ_1	6	1	1	1	0
φ_2	14	1	1	0	1

3 Symmetric algebras

In this section we place ourselves in a more general setting.

(3.1) Symmetric algebras. a) Let R be a commutative ring, let $\mathcal{A}$ be an R-free R-algebra such that $n := \mathrm{rk}_R(\mathcal{A})$, let λ be a symmetrising form, and let $\{A_1, \ldots, A_n\} \subseteq \mathcal{A}$ and $\{A_1^*, \ldots, A_n^*\} \subseteq \mathcal{A}$ be mutually dual R-bases of $\mathcal{A}$.

Comparing the right and left regular representations of $\mathcal{A}$, for $A \in \mathcal{A}$ let $A_i A = \sum_{k=1}^n p_i^k(A) A_k$ and $AA_i^* = \sum_{k=1}^n {}^*p_k^i(A) A_k^*$, where $p_i^k(A), {}^*p_k^i(A) \in R$. Then we have ${}^*p_k^i(A) = \lambda(AA_i^* \cdot A_k) = \lambda(A_k A \cdot A_i^*) = p_k^i(A)$, for all $i, k \in \{1, \ldots, n\}$. Hence for the associated representing matrices with respect to the given R-basis of $\mathcal{A}$ and the associated dual basis, respectively, we have $P(A) := [p_i^k(A)]_{ik} = [{}^*p_k^i(A)]_{ki} =: {}^*P(A) \in R^{n \times n}$.

Let V and V' be $\mathcal{A}$-modules with associated representations δ and δ', respectively. For $M \in \mathrm{Hom}_R(V, V')$ let $M^+ := \sum_{i=1}^n \delta(A_i^*) M \delta'(A_i) \in \mathrm{Hom}_R(V, V')$. Then $\delta(A) M^+ = \sum_{i=1}^n \delta(AA_i^*) M \delta'(A_i) = \sum_{i=1}^n \sum_{k=1}^n {}^*p_k^i(A) \delta(A_k^*) M \delta'(A_i) = \sum_{i=1}^n \sum_{k=1}^n p_k^i(A) \delta(A_k^*) M \delta'(A_i) = \sum_{k=1}^n \delta(A_k^*) M \delta'(A_k A) = M^+ \delta'(A)$ for $A \in \mathcal{A}$ implies $M^+ \in \mathrm{Hom}_{\mathcal{A}}(V, V')$.

Moreover, M^+ is independent of the particular choice of an R-basis of $\mathcal{A}$: Let $\{A_1', \ldots, A_n'\} \subseteq \mathcal{A}$ be an R-basis of $\mathcal{A}$, and let $\{A_1'^*, \ldots, A_n'^*\} \subseteq \mathcal{A}$ be the associated dual basis. For the base change matrices $B = [b_{ij}]_{ij} := {}_{\{A_i'\}}\mathrm{id}_{\{A_j\}} \in R^{n \times n}$ and $B^* = [b_{ij}^*]_{ij} := {}_{\{A_i'^*\}}\mathrm{id}_{\{A_j^*\}} \in R^{n \times n}$ we have $BB^{*\mathrm{tr}} = I_n$, that is $\sum_{i=1}^n b_{ij} b_{ik}^* = \delta_{jk}$, for all $j, k \in \{1, \ldots, n\}$. This implies $\sum_{i=1}^n \delta(A_i'^*) M \delta'(A_i') = \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n b_{ij}^* b_{ik} \delta(A_j^*) M \delta'(A_k) = \sum_{j=1}^n \delta(A_j^*) M \delta'(A_j) = M^+$.

b) Let now $R := F$ be a field such that $\mathcal{A}$ is split, and let S and S' be irreducible $\mathcal{A}$-modules with associated representations δ and δ', respectively, such that either $\delta \not\cong \delta'$ or $\delta = \delta'$. Letting $E_{qr} := [\delta_{qk}\delta_{rl}]_{kl} \in F^{d\times d}$ be a **matrix unit**, for $q \in \{1,\ldots,d\}$ and $r \in \{1,\ldots,d'\}$ where $d := \dim_F(S)$ and $d' := \dim_F(S')$, we have $E_{qr}^+ := \sum_{i=1}^n \delta(A_i^*)E_{qr}\delta'(A_i) \in \mathrm{Hom}_{\mathcal{A}}(S,S')$. Hence by Schur's Lemma there are $c_{S,qr} \in F$ such that $E_{qr}^+ = \delta_{S,S'}c_{S,qr} \cdot \mathrm{diag}[1,\ldots,1] \in F^{d\times d'}$. In terms of coordinate functions, where $p \in \{1,\ldots,d\}$ and $s \in \{1,\ldots,d'\}$, we infer $\sum_{i=1}^n \delta(A_i^*)_{pq}\delta'(A_i)_{rs} = \delta_{S,S'}\delta_{ps}c_{S,qr} \in F$. Interchanging the roles of $\{A_1,\ldots,A_n\}$ and $\{A_1^*,\ldots,A_n^*\}$ yields $c_{S,ps}^* \in F$ such that $\sum_{i=1}^n \delta(A_i^*)_{pq}\delta(A_i)_{rs} = \delta_{qr}c_{S,ps}^* \in F$. Thus from $\delta_{ps}c_{S,qr} = \delta_{qr}c_{S,ps}^*$ we infer that there is a **Schur element** $c_S \in F$ such that $c_{S,qr} = \delta_{qr}c_S$, and we have the **Frobenius-Schur relations** $\sum_{i=1}^n \delta(A_i^*)_{pq}\delta'(A_i)_{rs} = \delta_{S,S'}\delta_{ps}\delta_{qr}c_S \in F$.

This yields the **orthogonality relations** for irreducible characters: Let φ and φ' be the characters afforded by S and S', respectively. Then we have $\sum_{i=1}^n \varphi(A_i^*)\varphi'(A_i) = \sum_{i=1}^n \sum_{j=1}^d \sum_{k=1}^{d'} \delta(A_i^*)_{jj}\delta'(A_i)_{kk} = \delta_{\varphi,\varphi'}c_\varphi \cdot \sum_{j=1}^d 1 = \delta_{\varphi,\varphi'}c_\varphi d_\varphi$, where $c_\varphi := c_S$ and $d_\varphi := d = \varphi(1_{\mathcal{A}}) \in F$ is the associated degree.

c) By the Gaschütz-Higman-Ikeda Theorem [6, Thm.IX.62.11], S is projective if and only if there is $M \in \mathrm{End}_F(S)$ such that $M^+ = \mathrm{id}_S \in \mathrm{End}_{\mathcal{A}}(S)$. Hence this is the case if and only if $c_S \neq 0 \in F$. In that case, S lies in a **block** $\mathcal{A}_S \cong \mathrm{End}_F(S) \cong F^{d\times d}$ of its own, and we have the λ-orthogonal decomposition $\mathcal{A} \cong \mathcal{A}_S \oplus \mathcal{A}'$ as F-algebras.

Let S be an irreducible $\mathcal{A}$-module such that $c_\varphi = c_S \neq 0 \in F$, and for $i,j \in \{1,\ldots,d\}$ let $e_{\varphi,ij} := \frac{1}{c_\varphi} \cdot \sum_{k=1}^n \delta(A_k^*)_{ji}A_k \in \mathcal{A}$. Hence for the natural representation $\rho_{\mathcal{A}'}\colon \mathcal{A} \to \mathrm{End}_F(\mathcal{A}')$ we have $\rho_{\mathcal{A}'}(e_{\varphi,ij}) \in \mathrm{Hom}_{\mathcal{A}}(S,\mathcal{A}') = \{0\}$, implying that $e_{\varphi,ij} \in \mathcal{A}_S$. The Frobenius-Schur relations yield $\delta(e_{\varphi,ij}) = \frac{1}{c_\varphi} \cdot \sum_{k=1}^n \delta(A_k^*)_{ji}\delta(A_k) = E_{ij} \in F^{d\times d}$. Since $\delta\colon \mathcal{A}_S \to \mathrm{End}_F(S)$ is faithful, we conclude that $\{e_{\varphi,ij}; i,j \in \{1,\ldots,d\}\} \subseteq \mathcal{A}_S$ is an F-basis, and that $e_{\varphi,11},\ldots,e_{\varphi,dd} \in \mathcal{A}_S$ are mutually **orthogonal primitive idempotents**, that is we have $e_{\varphi,ii}e_{\varphi,jj} = \delta_{ij}e_{\varphi,ii}$ for all $i,j \in \{1,\ldots,d\}$. Thus we have $\epsilon_\varphi := \sum_{i=1}^d e_{\varphi,ii} = \frac{1}{c_\varphi} \cdot \sum_{k=1}^n \varphi(A_k^*)A_k = 1_{\mathcal{A}_S} \in \mathcal{A}_S$, hence $\epsilon_\varphi \in \mathcal{A}$ is the **centrally primitive idempotent** associated with $\mathcal{A}_S$.

Since for the **commutator subspace** $[\mathcal{A}_S,\mathcal{A}_S] := \langle AB - BA; A,B \in \mathcal{A}_S\rangle_F$ we have $[\mathcal{A}_S,\mathcal{A}_S] = \ker(\mathrm{tr})$, where $\mathrm{tr}\colon \mathcal{A}_S \cong F^{d\times d} \to F$ is the usual matrix trace, we infer that $\langle\mathrm{tr}\rangle_F$ coincides with the F-vector space of all **trace forms** on $\mathcal{A}_S$, that is the F-linear forms having $[\mathcal{A}_S,\mathcal{A}_S]$ in their kernel. Since λ restricts to a symmetrising form on $\mathcal{A}_S$, there is $0 \neq c \in F$ such that $\lambda|_{\mathcal{A}_S} = c \cdot \mathrm{tr}$, and hence the dual basis associated with $\{E_{ij}; i,j \in \{1,\ldots,d\}\} \subseteq \mathcal{A}_S$ is given as $\{\frac{1}{c}E_{ji}; i,j \in \{1,\ldots,d\}\} \subseteq \mathcal{A}_S$. The Frobenius-Schur relations yield $\frac{1}{c} = \sum_{i=1}^d \sum_{j=1}^d \frac{1}{c}(E_{ji})_{kk}(E_{ij})_{kk} = c_\varphi$, for any $k \in \{1,\ldots,d\}$.

Since $\mathcal{A}$ is semisimple if and only if all irreducible $\mathcal{A}$-modules are projective, we conclude that $\mathcal{A}$ is semisimple if and only if $c_S \neq 0 \in F$ for all irreducible $\mathcal{A}$-modules S. In that case, we have the λ-orthogonal decomposition $\mathcal{A} = \bigoplus_{\varphi\in\mathrm{Irr}(\mathcal{A})} F^{d_\varphi\times d_\varphi}$, and $\{\epsilon_\varphi \in \mathcal{A}; \varphi \in \mathrm{Irr}(\mathcal{A})\}$ are mutually orthogonal centrally primitive idempotents such that $\sum_{\varphi\in\mathrm{Irr}(\mathcal{A})} \epsilon_\varphi = 1$, and $\lambda = \sum_{\varphi\in\mathrm{Irr}(\mathcal{A})} \frac{1}{c_\varphi}\varphi$.

We place ourselves into an even more general setting, until further notice.

(3.2) Decomposition theory. Let R be the ring of integers in an algebraic number field K, let $\wp \lhd R$ be a prime ideal, and let $R_\wp \subseteq K$ be the **localisation** of R with respect to $\wp$; hence $R_\wp$ is a **discrete valuation ring** in K. Moreover, let $\bar{\ }\colon R_\wp \to R_\wp/\wp R_\wp \cong R/\wp =: F$ be the natural map onto the finite residue class field F. Let $\mathcal{A}$ be an R-free R-algebra with R-basis $\{A_1, \ldots, A_n\} \subseteq \mathcal{A}$ where $n := \mathrm{rk}_R(\mathcal{A}) \in \mathbb{N}$, such that $\mathcal{A}_K := \mathcal{A} \otimes_R K$ is split.

a) Let V be an $\mathcal{A}_K$-module, with representation δ, and let $\{b_1, \ldots, b_d\} \subseteq V$ be a K-basis where $d := \dim_K(V) \in \mathbb{N}$. Then V is **realisable** over $R_\wp$: The subset $\widetilde{V} := \langle b_j A_i; j \in \{1, \ldots, d\}, i \in \{1, \ldots, n\}\rangle_{R_\wp} \subseteq V$ is an $\mathcal{A}_{R_\wp}$-submodule, and since $R_\wp$ is a principal ideal domain we conclude that $\widetilde{V}$ is an $R_\wp$-free $\mathcal{A}_{R_\wp}$-module, and since it contains a K-basis of V it hence is a **full** $\mathcal{A}_{R_\wp}$-lattice in V, that is we have $\widetilde{V}_K := \widetilde{V} \otimes_{R_\wp} K \cong V$.

Letting $\mathcal{A}_F := \mathcal{A} \otimes_R F$, we thus obtain the $\mathcal{A}_F$-module $\overline{\widetilde{V}} := \widetilde{V} \otimes_{R_\wp} F$ by **$\wp$-modular reduction**. Since F is perfect, we by the Brauer-Nesbitt Theorem [6, Thm.V.30.16, Thm.XII.82.1] get a **decomposition map** $D_\wp\colon \mathbb{Z}\mathrm{Irr}(\mathcal{A}_K) \to \mathbb{Z}\mathrm{Irr}(\mathcal{A}_F)$ between the **Grothendieck groups** of $\mathcal{A}_K$ and $\mathcal{A}_F$, that is the free abelian groups generated by the isomorphism types of irreducible $\mathcal{A}_K$-modules and $\mathcal{A}_F$-modules, respectively.

b) Let V be a projective indecomposable $\mathcal{A}_F$-module. Thus there is a primitive idempotent $e \in \mathcal{A}_F$ such that $V \cong e\mathcal{A}_F$ as $\mathcal{A}_F$-modules. Hence by **lifting of idempotents**, [7, Exc.6.16] for the case $\mathcal{A}_K$ semisimple and [20, Thm.3.4.1] for the general case, there is a primitive idempotent $\widehat{e} \in \mathcal{A}_{R_\wp} \subseteq \mathcal{A}_K$ such that $\overline{\widehat{e}} = e$. Thus the projective indecomposable $\mathcal{A}_{R_\wp}$-module $\widetilde{V} := \widehat{e}\mathcal{A}_{R_\wp}$ **lifts** V in the sense that $\overline{\widetilde{V}} \cong \overline{\widehat{e}\mathcal{A}_{R_\wp}} \cong e\mathcal{A}_F \cong V$ as $\mathcal{A}_F$-modules.

c) We show **Tits' Deformation Theorem**: Let $\mathcal{A}_F$ be semisimple. Since F is perfect, there is a finite field extension $F \subseteq F'$ such that $\mathcal{A}_{F'}$ is split semisimple, and there is a finite field extension $K \subseteq K'$ with ring of integers R' having a prime ideal $\wp' \lhd R'$ such that $\wp' \cap R = \wp$ and $R'/\wp' \cong F'$. Then for any $\psi \in \mathrm{Irr}(\mathcal{A}_{F'})$ the associated irreducible $\mathcal{A}_{F'}$-module being projective, there is $\widehat{\psi} \in \mathrm{Irr}(\mathcal{A}_{K'}) = \mathrm{Irr}(\mathcal{A}_K)$ such that $D_\wp(\widehat{\psi}) = \psi$, implying that ψ is realisable over F, that is $\mathcal{A}_F$ is split. By Wedderburn's Theorem we have $\sum_{\psi \in \mathrm{Irr}(\mathcal{A}_F)} d_\psi^2 = n$, and since the lifting map $\psi \to \widehat{\psi}$ is injective we conclude $\sum_{\widehat{\psi} \in \mathrm{Irr}(\mathcal{A}_K)} d_{\widehat{\psi}}^2 = n$, implying that $\mathcal{A}_K$ is semisimple as well. Moreover, the map $\psi \to \widehat{\psi}$ is also surjective, that is $D_\wp\colon \mathrm{Irr}(\mathcal{A}_K) \to \mathrm{Irr}(\mathcal{A}_F)$ is a bijection.

(3.3) Proposition. Let $\lambda\colon \mathcal{A} \to R$ be a trace form such that $\tau_\lambda\colon \mathcal{A}^2 \to R\colon [A, B] \mapsto \lambda(AB)$ has **discriminant** $0 \neq \Delta \in R$ with respect to $\{A_1, \ldots, A_n\}$. Then, letting $R_\Delta \subseteq K$ be the localisation of R with respect to $\{\Delta^k; k \in \mathbb{N}_0\} \subseteq R$, we for all $\varphi \in \mathrm{Irr}(\mathcal{A}_K)$ have $\varphi(A_i) \in R$ and $\varphi(A_i^*) \in R_\Delta$ as well as $c_\varphi \in R_\Delta$.

Proof. Let S be the irreducible $\mathcal{A}_K$-module affording φ, with associated representation δ. Since φ and c_φ are independent of the particular choice of a K-basis of S, we may assume that $\delta(\mathcal{A}_{R_\wp}) \subseteq R_\wp^{d\times d}$, where $d := \dim_K(S)$, implying that $\varphi(\mathcal{A}_{R_\wp}) \subseteq R_\wp$. Since R is a **Dedekind ring**, thus a **Krull ring** [18, Ch.IV.12], we have $R = \bigcap_{\wp \lhd R \text{ prime}} R_\wp \subseteq K$, thus we infer $\varphi(\mathcal{A}) \subseteq R$.

Moreover, $\mathcal{A}_{R_\Delta}$ is a symmetric R_Δ-algebra. Hence we have $A_i^* \in \mathcal{A}_{R_\Delta}$, for all $i \in \{1,\ldots,n\}$, thus $\varphi(A_i^*) \in R_\Delta$. For any prime ideal $\wp \lhd R$ such that $\Delta \notin \wp$, that is $\wp \cap \{\Delta^k; k \in \mathbb{N}_0\} = \emptyset$, we have $R_\Delta \subseteq R_\wp$. Hence we have $A_i^* \in \mathcal{A}_{R_\wp}$, for all $i \in \{1,\ldots,n\}$, and thus the Frobenius-Schur relations imply $c_\varphi = \sum_{i=1}^n \delta(A_i^*)_{jj}\delta(A_i)_{jj} \in R_\wp \subseteq K$, for any $j \in \{1,\ldots,d\}$. Hence we have $c_\varphi \in \bigcap_{\Delta\notin\wp\lhd R \text{ prime}} R_\wp = R_\Delta$. ♯

(3.4) Proposition. Let $\Delta \notin \wp \lhd R$ be a prime ideal such that $\mathcal{A}_F$ is split. Then $D_\wp$ induces a bijection $\{\varphi \in \mathrm{Irr}(\mathcal{A}_K); \overline{c}_\varphi \neq 0 \in F\} \to \{\psi \in \mathrm{Irr}(\mathcal{A}_F); c_\psi \neq 0 \in F\}$. In particular, $\mathcal{A}_F$ is semisimple if and only if $\overline{c}_\varphi \neq 0 \in F$ for all $\varphi \in \mathrm{Irr}(\mathcal{A}_K)$.

Proof. For all $\varphi \in \mathrm{Irr}(\mathcal{A}_K)$ we have $c_\varphi \in R_\wp$, and $\mathcal{A}_F$ is a symmetric F-algebra, hence the sets are well-defined. We show that the above map is well-defined:

Let $\varphi \in \mathrm{Irr}(\mathcal{A}_K)$ such that $\overline{c}_\varphi \neq 0 \in F$, hence we have $\frac{1}{c_\varphi} \in R_\wp$. Let $\delta\colon \mathcal{A}_K \to K^{d\times d}$ be the representation of degree d affording φ, where we may assume that $\delta(\mathcal{A}_{R_\wp}) \subseteq R_\wp^{d\times d}$. Since $\mathcal{A}_{R_\wp}$ is a symmetric $R_\wp$-algebra, letting $e_{\varphi,ij} = \frac{1}{c_\varphi} \cdot \sum_{k=1}^n \delta(A_k^*)_{ji} A_k \in \mathcal{A}_{R_\wp}$, for all $i,j \in \{1,\ldots,d\}$, the Frobenius-Schur relations yield $\delta(e_{\varphi,ij}) = E_{ij} \in \delta(\mathcal{A}_{R_\wp}) \subseteq R_\wp^{d\times d}$. Thus we have $\delta(\mathcal{A}_{R_\wp}) = R_\wp^{d\times d}$, implying $\overline{\delta}(\mathcal{A}_F) = F^{d\times d}$, that is the $\wp$-modular reduction is irreducible, and we have $c_{\overline{\delta}} = \overline{c}_\varphi \neq 0 \in F$.

As for injectivity, for the centrally primitive idempotent $\epsilon_\varphi \in \mathcal{A}_{R_\wp}$ associated with φ we have $\overline{\delta}(\overline{\epsilon}_\varphi) = \overline{\delta(\epsilon_\varphi)} = \overline{I_d} = I_d$, while for any $\varphi \neq \varphi' \in \mathrm{Irr}(\mathcal{A}_K)$ with associated representation δ' of degree d', where we also may assume that $\delta'(\mathcal{A}_{R_\wp}) \subseteq R_\wp^{d'\times d'}$, we have $\delta'(\epsilon_\varphi) = 0$ and thus $\overline{\delta}'(\overline{\epsilon}_\varphi) = 0 \in F^{d'\times d'}$. As for surjectivity, let $\psi \in \mathrm{Irr}(\mathcal{A}_F)$ such that $c_\psi \neq 0$, that is the irreducible $\mathcal{A}_F$-module affording ψ is projective, thus there is an irreducible $\mathcal{A}_K$-module lifting it.

Finally, $\mathcal{A}_F$ is semisimple if and only if $\sum_{\psi\in\mathrm{Irr}(\mathcal{A}_F), c_\psi\neq 0\in F} d_\psi^2 = n$, that is $\sum_{\varphi\in\mathrm{Irr}(\mathcal{A}_K), \overline{c}_\varphi\neq 0\in F} d_\varphi^2 = n$, that is $\overline{c}_\varphi \neq 0 \in F$ for all $\varphi \in \mathrm{Irr}(\mathcal{A}_K)$. ♯

(3.5) Theorem. Let $\mathcal{A}_K$ be semisimple.

a) Then for the **Frame number** we have $\mathcal{N} := \Delta \cdot \prod_{\varphi\in\mathrm{Irr}(\mathcal{A}_K)} c_\varphi^{d_\varphi^2} \in R$.
b) The ideal $\mathcal{N}R \trianglelefteq R$ is the square of an ideal in R.

Proof. We proceed towards in interpretation of $\mathcal{N}$: Given any trace form $\psi\colon \mathcal{A}_K \to K$, let $\tau_\psi\colon \mathcal{A}_K^2 \to K\colon [A,B] \mapsto \psi(AB)$ be the associated symmetric associative K-bilinear form. Given any K-basis $\mathcal{B} := \{B_1,\ldots,B_n\} \subseteq \mathcal{A}_K$, let

${}_{\mathcal{B}}(\tau_\psi)_{\mathcal{B}} := [\psi(B_iB_j)]_{ij} \in K^{n\times n}$ be the associated **Gram matrix**, hence the **discriminant** of τ_ψ with respect to $\mathcal{B}$ is given as $\det({}_{\mathcal{B}}(\tau_\psi)_{\mathcal{B}}) \in K$.

We have $\mathcal{A}_K \cong \bigoplus_{\varphi\in\mathrm{Irr}(\mathcal{A}_K)} K^{d_\varphi\times d_\varphi}$, hence let $\mathcal{E} := \coprod_{\varphi\in\mathrm{Irr}(\mathcal{A}_K)} \mathcal{E}_\varphi \subseteq \mathcal{A}_K$ be a **Wedderburn basis** of $\mathcal{A}_K$, that is $\mathcal{E}_\varphi := \{e_{\varphi,pq} \in \mathcal{A}_K; p,q \in \{1,\ldots,d_\varphi\}\}$ is the K-basis of the block associated with φ such that $\delta_\varphi(e_{\varphi,pq}) = E_{pq} \in K^{d_\varphi\times d_\varphi}$. This yields $\varphi(e_{\varphi,pq}e_{\varphi,rs}) = \delta_{ps}\delta_{qr}$ and $e_{\varphi,pq}e_{\varphi',rs} = 0 \in \mathcal{A}_K$, for all $\varphi \neq \varphi' \in \mathrm{Irr}(\mathcal{A}_K)$ and $p,q \in \{1,\ldots,d_\varphi\}$ and $r,s \in \{1,\ldots,d_{\varphi'}\}$. Thus for any trace form $\psi = \sum_{\varphi\in\mathrm{Irr}(\mathcal{A}_K)} \alpha_\varphi\varphi$, where $\alpha_\varphi \in K$, we get $\psi(e_{\varphi,pq}e_{\varphi',rs}) = \alpha_\varphi\delta_{\varphi,\varphi'}\delta_{ps}\delta_{qr}$, and hence the associated discriminant is $\det({}_{\mathcal{E}}(\tau_\psi)_{\mathcal{E}}) = \prod_{\varphi\in\mathrm{Irr}(\mathcal{A}_K)}((-1)^{\binom{d_\varphi}{2}} \cdot \prod_{p=1}^{d_\varphi}\prod_{q=1}^{d_\varphi}\alpha_\varphi) = \prod_{\varphi\in\mathrm{Irr}(\mathcal{A}_K)}(-1)^{\binom{d_\varphi}{2}}\alpha_\varphi^{d_\varphi^2}$.

For the symmetrising form $\lambda = \sum_{\varphi\in\mathrm{Irr}(\mathcal{A}_K)} \frac{1}{c_\varphi}\varphi$ we thus get $\det({}_{\mathcal{E}}(\tau_\lambda)_{\mathcal{E}}) = \prod_{\varphi\in\mathrm{Irr}(\mathcal{A}_K)}(-1)^{\binom{d_\varphi}{2}}(\frac{1}{c_\varphi})^{d_\varphi^2}$, while by assumption we have $\det({}_{\{A_i\}}(\tau_\lambda)_{\{A_i\}}) = \Delta$. Thus letting $C := {}_{\{A_i\}}\mathrm{id}_{\mathcal{E}} \in K^{n\times n}$ we from $\det({}_{\{A_i\}}(\tau_\lambda)_{\{A_i\}}) = \det({}_{\mathcal{E}}(\tau_\lambda)_{\mathcal{E}}) \cdot \det(C)^2$ infer $\det(C)^2 = \Delta \cdot \prod_{\varphi\in\mathrm{Irr}(\mathcal{A}_K)}(-1)^{\binom{d_\varphi}{2}}c_\varphi^{d_\varphi^2}$. For $\omega := \sum_{\varphi\in\mathrm{Irr}(\mathcal{A}_K)}\varphi$ we get $\det({}_{\mathcal{E}}(\tau_\omega)_{\mathcal{E}}) = \prod_{\varphi\in\mathrm{Irr}(\mathcal{A}_K)}(-1)^{\binom{d_\varphi}{2}}$, hence for the **reduced discriminant** we have $\det({}_{\{A_i\}}(\tau_\omega)_{\{A_i\}}) = \det({}_{\mathcal{E}}(\tau_\omega)_{\mathcal{E}}) \cdot \det(C)^2 = \Delta \cdot \prod_{\varphi\in\mathrm{Irr}(\mathcal{A}_K)} c_\varphi^{d_\varphi^2} = \mathcal{N}$.

a) Since $\omega(\mathcal{A}) \subseteq R$ we infer $\mathcal{N} = \det({}_{\{A_i\}}(\tau_\omega)_{\{A_i\}}) = \det([\omega(A_iA_j)]_{ij}) \in R$.

b) Letting $\wp \lhd R$ be a prime ideal we may assume that $\delta_\varphi(\mathcal{A}_{R_\wp}) \subseteq R_\wp^{d_\varphi\times d_\varphi}$. Since $\bigoplus_{\varphi\in\mathrm{Irr}(\mathcal{A}_K)}\delta_\varphi$ is faithful, we infer $A_i = \sum_{\varphi\in\mathrm{Irr}(\mathcal{A}_K)}\sum_{p=1}^{d_\varphi}\sum_{q=1}^{d_\varphi}\delta_\varphi(A_i)_{pq}e_{\varphi,pq} \in \mathcal{A}_K$ for all $i \in \{0,\ldots,d\}$. This shows that a Wedderburn basis can chosen such that $C = {}_{\{A_i\}}\mathrm{id}_{\mathcal{E}} \in R_\wp^{n\times n}$. Letting $\nu_\wp\colon K^* \to \mathbb{Z}$ be the discrete valuation of K associated with $\wp$ we infer $\nu_\wp(\mathcal{N}) = \nu_\wp(\mathcal{N} \cdot \prod_{\varphi\in\mathrm{Irr}(\mathcal{A}_K)}(-1)^{\binom{d_\varphi}{2}}) = \nu_\wp(\det(C)^2) = 2\nu_\wp(\det(C)) \in 2\mathbb{N}_0$. From $\mathcal{N}R = \prod_{k=1}^{s}\wp_k^{e_k} \unlhd R$, where $s, e_k \in \mathbb{N}_0$ and the $\wp_k \lhd R$ are pairwise different prime ideals, we thus get $e_k \in 2\mathbb{N}_0$. ♯

(3.6) Theorem: Fleischmann, 1993 [9]. Let $\mathcal{A}_K$ be semisimple. Then $\mathcal{A}_F$ is semisimple if and only if $\overline{\mathcal{N}} \neq 0 \in F$, that is $\mathcal{N} \notin \wp$.

Proof. Let $V := \bigoplus_{\varphi\in\mathrm{Irr}(\mathcal{A}_K)} S_\varphi$ with representation $\delta = \bigoplus_{\varphi\in\mathrm{Irr}(\mathcal{A}_K)}\delta_\varphi$, affording the character $\omega = \sum_{\varphi\in\mathrm{Irr}(\mathcal{A}_K)}\varphi$. We may assume that $\delta_\varphi(\mathcal{A}_{R_\wp}) \subseteq R_\wp^{d_\varphi\times d_\varphi}$ for all $\varphi \in \mathrm{Irr}(\mathcal{A}_K)$, hence the $\mathcal{A}_F$-module $\overline{V} = \bigoplus_{\varphi\in\mathrm{Irr}(\mathcal{A}_K)}\overline{S}_\varphi$ affords the character $\overline{\omega} = \sum_{\varphi\in\mathrm{Irr}(\mathcal{A}_K)}\overline{\varphi}$. Since $\overline{\mathcal{N}} \in F$ is the discriminant of $\tau_{\overline{\omega}}$ with respect to $\{A_1,\ldots,A_n\}$, we show that $\tau_{\overline{\omega}}$ is non-degenerate if and only if $\mathcal{A}_F$ is semisimple:

If $\mathcal{A}_F$ is semisimple, then by Tits' Deformation Theorem $\mathcal{A}_F$ is split and the decomposition map $D_\wp\colon \mathrm{Irr}(\mathcal{A}_K) \to \mathrm{Irr}(\mathcal{A}_F)\colon \varphi \mapsto \overline{\varphi}$ is a bijection, thus $\overline{\delta} = \bigoplus_{\varphi\in\mathrm{Irr}(\mathcal{A}_K)}\delta_{\overline{\varphi}}\colon \mathcal{A}_F \to \bigoplus_{\varphi\in\mathrm{Irr}(\mathcal{A}_K)} F^{d_\varphi\times d_\varphi}$ is an isomorphism of F-algebras, in particular $\overline{\delta}$ is faithful. Letting $A \in \mathrm{rad}(\tau_{\overline{\omega}}) \unlhd \mathcal{A}_F$, for any $\varphi \in \mathrm{Irr}(\mathcal{A}_K)$ and all $i,j \in \{1,\ldots,d_\varphi\}$, using the matrix units of the block of $\mathcal{A}_F$ associated with $\overline{\varphi}$, we have $\delta_{\overline{\varphi}}(A)_{ij} = \mathrm{tr}(E_{ki}^{\overline{\varphi}} A E_{jk}^{\overline{\varphi}}) = \overline{\omega}(E_{ki}^{\overline{\varphi}} A E_{jk}^{\overline{\varphi}}) = 0 \in F$, for any $k \in \{1, \ldots, d_\varphi\}$. Thus we have $\overline{\delta}(A) = 0$, implying $A = 0$.

If $\mathcal{A}_F$ is not semisimple, then let $0 \neq A \in \mathrm{rad}(\mathcal{A}_F) \lhd \mathcal{A}_F$. Hence $\delta_\psi(AB) = 0$ for all $\psi \in \mathrm{Irr}(\mathcal{A}_F)$ and $B \in \mathcal{A}_F$. Thus considering a composition series of $\overline{S}_\varphi$ shows that $\overline{\varphi}(AB) = 0 \in F$ for all $\varphi \in \mathrm{Irr}(\mathcal{A}_K)$ and $B \in \mathcal{A}_F$, hence $\overline{\omega}(AB) = 0 \in F$ for all $B \in \mathcal{A}_F$, that is $A \in \mathrm{rad}(\tau_{\overline{\omega}})$. ♯

(3.7) Proposition. Let $\mathcal{A}_K$ be semisimple, and let $K' \subseteq K$ be a subfield with ring of integers R', such that $\mathcal{A}_{R'} := \langle A_1, \ldots, A_n \rangle_{R'}$ is an R'-subalgebra and such that $\lambda \colon \mathcal{A}_{R'}^2 \to R'$. If the Schur elements $c_\varphi \in K$, for $\varphi \in \mathrm{Irr}(\mathcal{A}_K)$, are pairwise different, then the **character field** $K'(\varphi(\mathcal{A}_{K'}); \varphi \in \mathrm{Irr}(\mathcal{A}_K)) = K'(c_\varphi; \varphi \in \mathrm{Irr}(\mathcal{A}_K))$ is the unique minimal splitting field of $\mathcal{A}_{K'}$ in K.

Proof. From the Frobenius-Schur relations we infer that any splitting field of $\mathcal{A}_{K'}$ contains $L := K'(c_\varphi; \varphi \in \mathrm{Irr}(\mathcal{A}_K)) \subseteq K'(\varphi(\mathcal{A}_{K'}); \varphi \in \mathrm{Irr}(\mathcal{A}_K))$. We show that L is a splitting field:

We may assume that $K' \subseteq K$ is Galois. Let $\sigma \in \mathrm{Aut}_{K'}(K)$ and $\varphi \in \mathrm{Irr}(\mathcal{A}_K)$. Then from $\mathcal{A}_{K'}$ being a K'-subalgebra we infer that letting $A_i \mapsto \delta_\varphi(A_i)^\sigma$, for all $i \in \{1, \ldots, n\}$ and σ being applied entrywise, K-linear extension yields an irreducible representation of $\mathcal{A}_K$, affording the character $\varphi^\sigma(\sum_{i=1}^n \alpha_i A_i) := \sum_{i=1}^n \alpha_i \varphi(A_i)^\sigma$, where $\alpha_i \in K$. Since $A_i^* \in \mathcal{A}_{K'}$ for all $i \in \{1, \ldots, n\}$, the Frobenius-Schur relations imply $c_{\varphi^\sigma} = c_\varphi^\sigma \in K$. Thus for all $\sigma \in \mathrm{Aut}_L(K)$ we have $c_{\varphi^\sigma} = c_\varphi$, the Schur elements being pairwise different implying $\varphi^\sigma = \varphi$.

The field automorphism σ extends to a K'-algebra automorphism of $\mathcal{A}_K$ given by $(\sum_{i=1}^n \alpha_i A_i)^\sigma := \sum_{i=1}^n \alpha_i^\sigma A_i$. Thus for $A = \sum_{i=1}^n \alpha_i A_i \in \mathcal{A}_K$ we have $\varphi^\sigma(A^\sigma) = \varphi^\sigma(\sum_{i=1}^n \alpha_i^\sigma A_i) = \sum_{i=1}^n \alpha_i^\sigma \varphi(A_i)^\sigma = \varphi(A)^\sigma \in K$. In particular, σ permutes the set $\{\epsilon_\varphi; \varphi \in \mathrm{Irr}(\mathcal{A}_K)\}$ of centrally primitive idempotents of $\mathcal{A}_K$, and from $\varphi^\sigma(\epsilon_\varphi^\sigma) = \varphi(\epsilon_\varphi)^\sigma = 1^\sigma = 1$ we infer $\epsilon_\varphi^\sigma = \epsilon_{\varphi^\sigma}$. Thus for all $\sigma \in \mathrm{Aut}_L(K)$ we have $\epsilon_\varphi^\sigma = \epsilon_\varphi$, which since $\mathrm{Fix}_K(\mathrm{Aut}_L(K)) = L$ implies $\epsilon_\varphi \in \mathcal{A}_L \subseteq \mathcal{A}_K$ Hence $\mathcal{A}_L \cong \bigoplus_{\varphi \in \mathrm{Irr}(\mathcal{A}_K)} \epsilon_\varphi \mathcal{A}_L$ has at least as many irreducible representations as $\mathcal{A}_K$, hence equally many, thus is split. ♯

4 Structure of adjacency algebras

We return to our original setting of association schemes.

(4.1) Theorem. Let R be a commutative ring, and let $\mathcal{X}$ be an association scheme such that $\Delta := \prod_{i=0}^d n_i \in R^*$. Then $\mathcal{A}_R$ is a **symmetric** R-algebra with respect to the **symmetrising form** $\lambda \colon \mathcal{A}_R \to R \colon \sum_{k=0}^d \alpha_k A_k \mapsto \alpha_0$, where $\{A_i; i \in \{0, \ldots, d\}\}$ and $\{\frac{1}{n_i} \cdot A_{i^*}; i \in \{0, \ldots, d\}\}$ are mutually dual R-bases of $\mathcal{A}_R$; the latter is called the **dual Schur basis**.

Proof. Let $\tau_\lambda\colon \mathcal{A}_R^2 \to R\colon [A,B] \mapsto \lambda(AB)$ be the associated R-bilinear form. Hence we have $\tau_\lambda(AB,C) = \lambda(AB\cdot C) = \lambda(A\cdot BC) = \tau_\lambda(A,BC)$, for all $A,B,C \in \mathcal{A}_R$, that is τ_λ is associative. We show that τ_λ is symmetric, and that its discriminant is a unit in R: Let $A = \sum_{i=0}^d \alpha_i A_i$ and $B = \sum_{j=0}^d \beta_j A_j$, where $\alpha_i, \beta_j \in R$. Then we have $\lambda(AB) = \sum_{i=0}^d \sum_{j=0}^d \sum_{k=0}^d \alpha_i\beta_j p_{ij}^k \lambda(A_k) = \sum_{i=0}^d \alpha_i \beta_{i^*} n_i = \lambda(BA)$, and $\lambda(A_i A_{j^*}) = \sum_{k=0}^d p_{ij^*}^k \lambda(A_k) = n_i\delta_{ij} \in R^*$, for all $i,j \in \{0,\ldots,d\}$, hence the discriminant with respect to the Schur basis equals $(-1)^{\frac{d+1-|\mathcal{I}|}{2}} \cdot \Delta \in R^*$, where $\mathcal{I} := \{i \in \{0,\ldots,d\}; i^* = i\}$. ♯

(4.2) Theorem. Let $\mathcal{X}$ be an association scheme, and let F be a field such that $p := \mathrm{char}(F) \geq 0$ and $p \not| \, n\Delta$, and such that $\mathcal{A}_F$ is split. Then for any $\varphi \in \mathrm{Irr}(\mathcal{A}_F)$ we have $m_\varphi \neq 0 \in F$ and $c_\varphi = \frac{n}{m_\varphi} \in F$.

Proof. For the natural $\mathcal{A}_F$-module we have $F^n \cong \bigoplus_{\varphi\in\mathrm{Irr}(\mathcal{A}_F)} \bigoplus_{k=1}^{m_\varphi} S_\varphi$, where S_φ is the irreducible $\mathcal{A}_F$-module affording φ. For the centrally primitive idempotent $\epsilon_\varphi = \frac{1}{c_\varphi}\cdot\sum_{j=0}^d \frac{\varphi(A_{j^*})}{n_j} A_j \in \mathcal{A}_{F,\varphi}$ in the block associated with $\varphi \in \mathrm{Irr}(\mathcal{A}_F)$ and all $i \in \{0,\ldots,d\}$ we have $\epsilon_\varphi A_i \in \mathcal{A}_{F,\varphi}$. Hence for the natural character we get $\nu(\epsilon_\varphi A_i) = \sum_{\psi\in\mathrm{Irr}(\mathcal{A}_F)} m_\psi \psi(\epsilon_\varphi A_i) = m_\varphi\varphi(\epsilon_\varphi A_i) = m_\varphi \varphi(A_i) \in F$. Thus from $\nu(\epsilon_\varphi A_i) = \frac{1}{c_\varphi}\cdot\sum_{j=0}^d \frac{\varphi(A_{j^*})}{n_j}\nu(A_jA_i) = \frac{\varphi(A_i)}{c_\varphi n_i}\cdot n n_i = \frac{\varphi(A_i)n}{c_\varphi} \in F$, choosing $i \in \{0,\ldots,d\}$ such that $\varphi(A_i) \neq 0 \in F$, we infer $m_\varphi = \frac{n}{c_\varphi} \neq 0 \in F$. ♯

(4.3) Theorem: Hanaki, 2000 [12]. Let $\mathcal{X}$ be an association scheme.
a) Let K be a field such that $\mathrm{char}(K) = 0$ and $\mathcal{A}_K$ is split. Then for any $\varphi \in \mathrm{Irr}(\mathcal{A}_K)$ we have $c_\varphi \in \mathbb{Z}_\Delta \subseteq \mathbb{Q}$, and for the Frame number we have

$$\mathcal{N} = \frac{(-1)^{\frac{d+1-|\mathcal{I}|}{2}}\Delta\cdot n^{d+1}}{\prod_{\varphi\in\mathrm{Irr}(\mathcal{A}_K)} m_\varphi^{d_\varphi^2}} \in \mathbb{Z}.$$

b) Let F be a field such that $p := \mathrm{char}(F) \geq 0$. Then $\mathcal{A}_F$ is semisimple if and only if $p \not| \, \mathcal{N} \in \mathbb{Z}$.

Proof. a) Let $\overline{\mathbb{Q}}$ be the algebraic closure of $\mathbb{Q}$, hence $\mathcal{A}_{\overline{\mathbb{Q}}}$ is split. Thus we may assume that K is an algebraic number field. Let R be the ring of integers in K, then $\mathcal{A}_{R_\Delta}$ is a symmetric R_Δ-algebra, hence we infer that $c_\varphi \in R_\Delta \cap \mathbb{Q} = \mathbb{Z}_\Delta$. The discriminant of the symmetrising form with respect to the Schur basis being $(-1)^{\frac{d+1-|\mathcal{I}|}{2}}\Delta$, we from $d+1 = \dim_K(\mathcal{A}_K) = \sum_{\varphi\in\mathrm{Irr}(\mathcal{A}_K)} d_\varphi^2$ get $\mathcal{N} = (-1)^{\frac{d+1-|\mathcal{I}|}{2}}\Delta\cdot\prod_{\varphi\in\mathrm{Irr}(\mathcal{A}_K)}(\frac{n}{m_\varphi})^{d_\varphi^2} = \frac{(-1)^{\frac{d+1-|\mathcal{I}|}{2}}\Delta\cdot n^{d+1}}{\prod_{\varphi\in\mathrm{Irr}(\mathcal{A}_K)} m_\varphi^{d_\varphi^2}} \in \mathbb{Z}_\Delta \cap R = \mathbb{Z}$.
b) We may assume that $p \neq 0$. Then we have $\mathcal{A}_F \cong \mathcal{A}_{\mathbb{F}_p} \otimes_{\mathbb{F}_p} F$, and since $\mathbb{F}_p$ is perfect we get $\mathrm{rad}(\mathcal{A}_F) = \mathrm{rad}(\mathcal{A}_{\mathbb{F}_p}) \otimes_{\mathbb{F}_p} F$ [7, Thm.7.9]. Hence we may assume that F is a finite field such that $\mathcal{A}_F$ is split. Let K be an algebraic number field with ring of integers $R \subseteq K$, such that $\mathcal{A}_K$ is split and R has a

prime ideal $\wp \lhd R$ such that $R/\wp \cong F$. Then $\mathcal{A}_F$ is semisimple if and only if $\mathcal{N} \in \mathbb{Z} \setminus (\wp \cap \mathbb{Z}) = \mathbb{Z} \setminus p\mathbb{Z}$. ♯

(4.4) Theorem: Frame, 1941 [10]. Let $\mathcal{X}$ be an association scheme.
a) We have $n^2 \mid \mathcal{N} \in \mathbb{Z}$.
b) If $\mathcal{A}_\mathbb{Q}$ is split, then $|\mathcal{N}|$ is a square in $\mathbb{Z}$.
c) Let K be a field such that $\mathrm{char}(K) = 0$ and $\mathcal{A}_K$ is split. If the multiplicities m_φ, for $\varphi \in \mathrm{Irr}(\mathcal{A}_K)$, are pairwise different, then $A_\mathbb{Q}$ is split.

Proof. a) Let K be an algebraic number field such that $\mathcal{A}_K$ is split, let R be the ring of integers in K, and let $\wp \lhd R$ be a prime ideal. Then there is a Wedderburn basis $\mathcal{E} \subseteq \mathcal{A}_K$ such that $C := {}_{\{A_i\}}\mathrm{id}_\mathcal{E} \in R_\wp^{(d+1)\times(d+1)}$. We have $\sum_{i=0}^d A_i = J_n = n\epsilon_{\varphi_0} \in \mathcal{A}_K$, where $\epsilon_{\varphi_0} = e_{\varphi_0,11}$ is the centrally primitive idempotent associated with the trivial character φ_0. Thus we get $[1,\ldots,1] \cdot C = (J_n)_\mathcal{E} = [n, 0, \ldots, 0] \in R_\wp^{d+1}$. This implies $n \mid \det(C) \in R_\wp$ and hence $n^2 \mid \mathcal{N} \in R_\wp$. Thus we get $\frac{\mathcal{N}}{n^2} \in \mathbb{Q} \cap \bigcap_{\wp \lhd R \text{ prime}} R_\wp = \mathbb{Q} \cap R = \mathbb{Z}$. ♯

(4.5) Theorem: Wielandt, 1964 [21]; Higman, 1975 [14]. Let $\mathcal{X}$ be an association scheme, let $p \in \mathbb{Z}$ be a prime and $l \in \mathbb{N}$. Then we have $\sum_{\varphi\in\mathrm{Irr}(\mathcal{A}_\mathbb{C}), p^l \mid m_\varphi} d_\varphi^2 \leq |\{i \in \{0,\ldots,d\}; p^l \mid nn_i\}|$. In particular, if $\mathcal{X}$ is commutative then we have $|\{\varphi \in \mathrm{Irr}(\mathcal{A}_\mathbb{C}); p^l \mid m_\varphi\}| \leq |\{i \in \{0,\ldots,d\}; p^l \mid nn_i\}|$.

Proof. Let K be an algebraic number field such that $\mathcal{A}_K$ is split, let R be the ring of integers in K, and let $\wp \lhd R$ be a prime ideal such that $\wp \cap \mathbb{Z} = p\mathbb{Z}$; for the associated valuations we have $\nu_\wp|_\mathbb{Z} = e\nu_p$, where $e \in \mathbb{N}$ is the **ramification index**. We may assume that $\Psi := [\delta_\varphi(A_i)_{rs}; [\varphi, r, s] \in \mathcal{T}, i \in \{0,\ldots,d\}]_{\varphi,r,s;i} \in R_\wp^{(d+1)\times(d+1)}$, where $\mathcal{T} := \{[\varphi, r, s]; \varphi \in \mathrm{Irr}(\mathcal{A}_K), r, s \in \{1,\ldots,d_\varphi\}\}$ is ordered lexicographically; if $\mathcal{X}$ is commutative, that is $d_\varphi = 1$ for all $\varphi \in \mathrm{Irr}(\mathcal{A}_K)$, then $\Psi = \Phi(\mathcal{A}_K)$ coincides with the character table of $\mathcal{A}_K$.

Let $N := \mathrm{diag}[n_0,\ldots,n_d] \in \mathbb{Z}^{(d+1)\times(d+1)}$ and $M := \mathrm{diag}[m_\varphi I_{d_\varphi^2}; \varphi \in \mathrm{Irr}(\mathcal{A}_K)] \in \mathbb{Z}^{(d+1)\times(d+1)}$. The orthogonality relations read $(\Psi Q)N^{-1}(P\Psi)^{\mathrm{tr}} = nM^{-1} \in K^{(d+1)\times(d+1)}$, where $Q \in \mathbb{Z}^{(d+1)\times(d+1)}$ is the permutation matrix describing the involution $*\colon i \mapsto i^*$ on $\{0,\ldots,d\}$, and $P \in \mathbb{Z}^{(d+1)\times(d+1)}$ is the permutation matrix describing the appropriate reordering of $\mathcal{T}$. In particular $\Psi \in K^{(d+1)\times(d+1)}$ is invertible. Inverting yields $(P\Psi)^{-\mathrm{tr}}N(\Psi Q)^{-1} = n^{-1}M$ and thus $(P\Psi)^{\mathrm{tr}}M(\Psi Q) = nN \in R_\wp^{(d+1)\times(d+1)}$.

Since $R_\wp$ is a principal ideal domain, it has an elementary divisor theory with respect to its only prime ideal $\wp$, where elementary divisors are greatest common divisors of appropriate matrix minors. Writing $M = \mathrm{diag}[m_0,\ldots,m_d]$, we for $k \in \mathbb{N}_0$ have $|\{i \in \{0,\ldots,d\}; \nu_\wp(m_i) \geq k\}| \leq |\{i \in \{0,\ldots,d\}; \nu_\wp(nn_i) \geq k\}|$. ♯

(4.6) Remark. The above matrix calculation relates to the Frame number as follows: We have $\det(Q) = (-1)^{\frac{d+1-|\mathcal{I}|}{2}}$ and $\det(P) = \prod_{\varphi\in\mathrm{Irr}(\mathcal{A}_K)}(-1)^{\binom{d_\varphi}{2}}$, thus we get $(-1)^{\frac{d+1-|\mathcal{I}|}{2}}\det(\Psi)^2 \cdot \prod_{\varphi\in\mathrm{Irr}(\mathcal{A}_K)}(-1)^{\binom{d_\varphi}{2}} m_\varphi^{d_\varphi^2} = n^{d+1}\Delta$, thus $\mathcal{N} = \det(P\Psi^2)$. The associated Wedderburn basis $\mathcal{E} \subseteq \mathcal{A}_K$ is given as ${}_{\mathcal{E}}\mathrm{id}_{\{A_i\}} = n^{-1}MP(\Psi Q)N^{-1}$. Hence from $n^{-1}(\Psi Q)N^{-1} = M^{-1}(P\Psi)^{-\mathrm{tr}}$ and using $MP = PM$ and $P = P^{-1} = P^{\mathrm{tr}}$, we conclude ${}_{\mathcal{E}}\mathrm{id}_{\{A_i\}} = MPM^{-1}P^{-\mathrm{tr}}\Psi^{-\mathrm{tr}} = \Psi^{-\mathrm{tr}}$, implying $C := {}_{\{A_i\}}\mathrm{id}_{\mathcal{E}} = \Psi^{\mathrm{tr}}$.

(4.7) Theorem: Higman, 1975 [14]. Let $\mathcal{X}$ be an association scheme. Then $|\mathrm{Irr}(\mathcal{A}_\mathbb{C})| = 2$ implies $d = 1$. In particular, if $d \leq 4$ then $\mathcal{X}$ is commutative.

Proof. Let $\mathrm{Irr}(\mathcal{A}_\mathbb{C}) = \{\varphi_0, \varphi\}$, where φ_0 is the trivial character. Hence we have $d+1 = \dim_\mathbb{C}(\mathcal{A}_\mathbb{C}) = 1+d_\varphi^2$, thus $d_\varphi^2 = d$. For the natural character $\nu = \varphi_0 + m_\varphi\varphi$ we have $0 = \nu(A_i) = n_i + m_\varphi\varphi(A_i)$, for all $i \geq 1$, thus $\varphi(A_i) = \frac{-n_i}{m_\varphi} < 0$, and since $\varphi(A_i)$ is an integer we conclude $\varphi(A_i) \leq -1$. For $J_n = \sum_{i=0}^d A_i = n\epsilon_{\varphi_0}$ we have $0 = \varphi(J_n) = d_\varphi + \sum_{i=1}^d \varphi(A_i) \leq d_\varphi - d = d_\varphi - d_\varphi^2$, thus $d_\varphi^2 \leq d_\varphi$, implying $d_\varphi = 1$ and hence $d = d_\varphi^2 = 1$.

If $d \leq 3$, then from $\sum_{\varphi\in\mathrm{Irr}(\mathcal{A}_\mathbb{C})} d_\varphi^2 = d + 1 \leq 4$ and $d_{\varphi_0} = 1$ we conclude that $d_\varphi = 1$ for all $\varphi \in \mathrm{Irr}(\mathcal{A}_\mathbb{C})$, that is $\mathcal{A}_\mathbb{C}$ is commutative. Finally, if $d = 4$ then assume that $\mathcal{A}_\mathbb{C}$ is not commutative. Then we have $\mathrm{Irr}(\mathcal{A}_\mathbb{C}) = \{\varphi_0, \varphi\}$ where $d_\varphi = 2$, implying $d = 1$, a contradiction. $\sharp$

(4.8) Theorem: Bannai-Ito, 1984 [3]. Let $\mathcal{X}$ be an association scheme, such that all non-trivial characters in $\mathrm{Irr}(\mathcal{A}_\mathbb{C})$ have the same multiplicity m. Then $\mathcal{X}$ is commutative, and we have $n_1 = \cdots = n_d = m$ and $|\mathcal{N}| = n^{d+1}$.

Proof. We may assume that $d \geq 1$; we have $n_0 = m_{\varphi_0} = d_{\varphi_0} = 1$ anyway. Then $n = 1 + m \cdot \sum_{\varphi_0\neq\varphi\in\mathrm{Irr}(\mathcal{A}_\mathbb{C})} d_\varphi$ implies that n and m are coprime. Since $\sum_{\varphi_0\neq\varphi\in\mathrm{Irr}(\mathcal{A}_\mathbb{C})} d_\varphi^2 = d$ we infer that $|\mathcal{N}| = \frac{n^{d+1}\Delta}{m^d} \in \mathbb{Z}$, and thus $m^d \mid \Delta$, hence $m \leq \Delta^{\frac{1}{d}}$. The inequality between the geometric and arithmetic mean yields $\Delta^{\frac{1}{d}} = (\prod_{i=1}^d n_i)^{\frac{1}{d}} \leq \frac{1}{d} \cdot \sum_{i=1}^d n_i = \frac{n-1}{d} = \frac{m\cdot\sum_{\varphi_0\neq\varphi\in\mathrm{Irr}(\mathcal{A}_\mathbb{C})} d_\varphi}{d} \leq \frac{m\cdot\sum_{\varphi_0\neq\varphi\in\mathrm{Irr}(\mathcal{A}_\mathbb{C})} d_\varphi^2}{d} = m \leq \Delta^{\frac{1}{d}}$. Hence $\sum_{\varphi_0\neq\varphi\in\mathrm{Irr}(\mathcal{A}_\mathbb{C})} d_\varphi = \sum_{\varphi_0\neq\varphi\in\mathrm{Irr}(\mathcal{A}_\mathbb{C})} d_\varphi^2$ implies $d_\varphi = d_\varphi^2 = 1$ for all $\varphi_0 \neq \varphi \in \mathrm{Irr}(\mathcal{A}_\mathbb{C})$, that is $\mathcal{A}_\mathbb{C}$ is commutative, thus $|\mathrm{Irr}(\mathcal{A}_\mathbb{C})| = d + 1$. Moreover, $(\prod_{i=1}^d n_i)^{\frac{1}{d}} = \frac{1}{d} \cdot \sum_{i=1}^d n_i$ implies $n_1 = \cdots = n_d$, hence we have $1 + dn_1 = n = 1 + dm$, implying $n_1 = m$ and $\Delta = m^d$. $\sharp$

(4.9) Theorem: Hanaki, 2002 [11]; Hanaki-Uno, 2006 [13]. Let $\mathcal{X}$ be an association scheme such that $n = p^l$, where $p \in \mathbb{Z}$ is a prime and $l \in \mathbb{N}$.
a) Let F be a field such that $\mathrm{char}(F) = p$. Then $\mathcal{A}_F$ is a local algebra.
b) If $n = p$ then all non-trivial characters in $\mathrm{Irr}(\mathcal{A}_\mathbb{C})$ have the same multiplicity.

Proof. a) The adjacency algebra $\mathcal{A}_F$ is local if and only if the trivial representation is the only irreducible representation. Hence we may assume that F

is a finite field and $\mathcal{A}_F$ is split. Let K be an algebraic number field with ring of integers R, having a prime ideal $\wp \lhd R$ such that $R/\wp \cong F$, where we may assume that $\mathcal{A}_K$ is split. We show that $\mathcal{A}_F$ has a unique primitive idempotent:

Let first $e_\varphi \in \mathcal{A}_K$ be a primitive idempotent associated with $\varphi \in \mathrm{Irr}(\mathcal{A}_K)$, then from $\lambda = \sum_{\varphi\in\mathrm{Irr}(\mathcal{A}_K)} \frac{1}{c_\varphi}\varphi$ we get $\lambda(e_\varphi) = \frac{1}{c_\varphi} = \frac{m_\varphi}{n} \in K$. Let now $f \in \mathcal{A}_F$ be a primitive idempotent. Then there is a primitive idempotent $\widehat{f} \in \mathcal{A}_{R_\wp}$ such that $\overline{\widehat{f}} = f$. Let $\widehat{f} = \sum_{\varphi\in\mathrm{Irr}(\mathcal{A}_K)} \sum_{i=1}^{k_\varphi} e_{\varphi,i}$ be a decomposition into pairwise orthogonal primitive idempotents $e_{\varphi,i} \in \mathcal{A}_K$, where $e_{\varphi,i}$ is associated with $\varphi \in \mathrm{Irr}(\mathcal{A}_K)$ and $k_\varphi \in \{0,\dots,d_\varphi\}$. Hence we have $\lambda(\widehat{f}) = \frac{1}{p^l}\cdot \sum_{\varphi\in\mathrm{Irr}(\mathcal{A}_K)} k_\varphi m_\varphi \in R_\wp \cap \mathbb{Q} = \mathbb{Z}_{\langle p\rangle}$, implying that $n = p^l \mid \sum_{\varphi\in\mathrm{Irr}(\mathcal{A}_K)} k_\varphi m_\varphi \leq \sum_{\varphi\in\mathrm{Irr}(\mathcal{A}_K)} d_\varphi m_\varphi = n$, thus $k_\varphi = d_\varphi$ for all $\varphi \in \mathrm{Irr}(\mathcal{A}_K)$, hence $\widehat{f} = A_0$ and thus $f = \overline{\widehat{f}} = A_0$.

b) Let K be an algebraic number field with ring of integers R such that $\mathcal{A}_K$ is split, where we may assume that $\mathbb{Q} \subseteq K$ is Galois, and let $\wp \lhd R$ be a prime ideal such that $F := R/\wp$ is a field such that $\mathrm{char}(F) = p$. We show that all $\varphi_0 \neq \varphi \in \mathrm{Irr}(\mathcal{A}_K)$ are algebraically conjugate, then in particular all the Schur elements $c_\varphi = \frac{n}{m_\varphi} \in \mathbb{Q}$ are the same, and thus the multiplicities as well:

For $\varphi_0 \neq \varphi \in \mathrm{Irr}(\mathcal{A}_K)$ let $\mathcal{O} := \{\varphi^\sigma \in \mathrm{Irr}(\mathcal{A}_K); \sigma \in \mathrm{Aut}_\mathbb{Q}(K)\}$ and $\mathcal{O}' := \mathrm{Irr}(\mathcal{A}_K) \setminus (\{\varphi_0\} \,\dot\cup\, \mathcal{O})$, where the latter is to be shown to be the empty set, and let $\psi := \sum_{\varphi'\in\mathcal{O}} \varphi'$ and $\psi' := \sum_{\varphi'\in\mathcal{O}'} \varphi'$. Hence we have $\psi(A_i), \psi'(A_i) \in \mathrm{Fix}_K(\mathrm{Aut}_\mathbb{Q}(K)) \cap R = \mathbb{Q} \cap R = \mathbb{Z}$, for all $i \in \{0,\dots,d\}$, and letting $d_\psi := \psi(A_0) = |\mathcal{O}|$ and $d_{\psi'} := \psi'(A_0) = |\mathcal{O}'|$ be the associated degrees we have $1 \leq d_\psi < |\mathrm{Irr}(\mathcal{A}_K)| \leq \sum_{\varphi\in\mathrm{Irr}(\mathcal{A}_K)} m_\varphi d_\varphi = n = p$ and similarly $0 \leq d_{\psi'} < p$.

Since the trivial character is the unique irreducible character of $\mathcal{A}_F$, decomposition theory implies that $\psi(A_i) - n_i d_\psi \in \wp$. Thus there are $a_i \in \mathbb{Z}$ such that $\psi(A_i) = n_i d_\psi - p a_i \in \mathbb{Z}$, and similarly there are $a_i' \in \mathbb{Z}$ such that and $\psi'(A_i) = n_i d_{\psi'} - p a_i' \in \mathbb{Z}$, for all $i \in \{0,\dots,d\}$. The orthogonality relations imply $0 = \sum_{i=0}^d \frac{\varphi_0(A_{i^*})\psi(A_i)}{n_i} = \sum_{i=0}^d \psi(A_i) = p(d_\psi - \sum_{i=0}^d a_i)$, thus $\sum_{i=0}^d a_i = d_\psi$ and similarly $\sum_{i=0}^d a_i' = d_{\psi'}$. Again by the orthogonality relations we get $0 = \sum_{i=0}^d \frac{\psi(A_{i^*})\psi'(A_i)}{n_i} = \sum_{i=0}^d n_i d_\psi d_{\psi'} - p\cdot\sum_{i=0}^d (d_\psi a_i' + d_{\psi'} a_{i^*}) + p^2 \cdot \sum_{i=0}^d \frac{a_{i^*} a_i'}{n_i} = -p d_\psi d_{\psi'} + p^2 \cdot \sum_{i=0}^d \frac{a_{i^*} a_i'}{n_i}$. Since $n_i < n = p$ for all $i \in \{0,\dots,d\}$, this implies $d_\psi d_{\psi'} = p \cdot \sum_{i=0}^d \frac{a_{i^*} a_i'}{n_i} \in \mathbb{Z}_{\langle p\rangle}$, hence $p \mid d_\psi d_{\psi'} \in \mathbb{Z}$, and thus $d_{\psi'} = 0$. ♯

(4.10) Example: Johnson scheme $\mathcal{J}(7,2)$. By (2.7) the adjacency algebra $\mathcal{A}_F$ is split for any field F. Hence we let $K := \mathbb{Q}$ and $R := \mathbb{Z}$ and consider the cases $F := \mathbb{F}_p$ for $p \in \{2,3,5,7\}$. We have $n = 21$ and $n_0 = 1$ and $n_1 = n_2 = 10$, hence $\Delta = \prod_{i=0}^2 n_i = 100 = 2^2 \cdot 5^2$. Moreover, we have $\mathcal{I} = \{0,\dots,2\}$, hence the Frame number is $\mathcal{N} = (-1)^0 \cdot \Delta \cdot \prod_{i=0}^2 c_i^{d_i^2} = 11025 = 3^2 \cdot 5^2 \cdot 7^2$, and the **decomposition matrices**, that is the matrices describing the various decomposition maps, are as follows, where $3, 7 \notin \Delta$ while $2, 5 \in \Delta$:

	m_i	c_i	d_i	$p = 7$		$p = 3$		$p = 5$		$p = 2$		
φ_0	1	21	1	1	.	1	.	1	.	1	.	.
φ_1	6	$\frac{7}{2}$	1	1	.	.	1	.	1	.	1	.
φ_2	14	$\frac{3}{2}$	1	.	1	1	.	.	1	.	.	1

(4.11) Example. Let $G := J_1$ be the smallest **sporadic simple Janko group**, of order $|G| = 175560 = 2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19$, let $H := L_2(11) < G$ be the largest maximal subgroup, of order $|H| = 660 = 2^2 \cdot 3 \cdot 5 \cdot 11$, and let $\mathcal{X}$ be the Schurian scheme on $X := H \backslash G$. Then $\mathcal{X}$ is commutative such that $d = 4$ and $n = 266 = 2 \cdot 7 \cdot 19$. We have $[n_0, \ldots, n_4] = [1, 11, 110, 132, 12]$, and by (5.4) the character table $\Phi(\mathcal{A}_{\mathbb{C}})$ is given as follows:

	m_i	c_i	d_i	A_0	A_1	A_2	A_3	A_4
φ_0	1	266	1	1	11	110	132	12
φ_1	56	$\frac{19}{4}$	1	1	$\frac{-7-\sqrt{5}}{2}$	$\frac{5+7\sqrt{5}}{2}$	$\frac{3-9\sqrt{5}}{2}$	$\frac{-3+3\sqrt{5}}{2}$
φ_2	56	$\frac{19}{4}$	1	1	$\frac{-7+\sqrt{5}}{2}$	$\frac{5-7\sqrt{5}}{2}$	$\frac{3+9\sqrt{5}}{2}$	$\frac{-3-3\sqrt{5}}{2}$
φ_3	76	$\frac{7}{2}$	1	1	4	5	-8	-2
φ_4	77	$\frac{38}{11}$	1	1	1	-10	4	4

Since $\mathcal{A}_{\mathbb{Q}}$ is commutative, the character field $K := \mathbb{Q}(\sqrt{5})$ is the unique minimal splitting field of $\mathcal{A}_{\mathbb{Q}}$ in $\mathbb{C}$; the ring of integers in K is $R := \mathbb{Z}[\frac{1+\sqrt{5}}{2}]$, which is a principal ideal domain. The valencies $n_i = \varphi_0(A_i)$ are pairwise different, implying that $\mathcal{I} := \{0, \ldots, 4\}$, and yielding $\Delta = \prod_{i=0}^{4} n_i = 1916640 = 2^5 \cdot 3^2 \cdot 5 \cdot 11^3$. The multiplicities m_i are not pairwise different, and we have $\mathcal{N} = (-1)^0 \cdot \Delta \cdot n^5 \cdot \prod_{i=0}^{4} (\frac{1}{m_i})^{d_i^2} = 139081177620 = 2^2 \cdot 3^2 \cdot 5 \cdot 7^2 \cdot 11^2 \cdot 19^4$, which is not a square in $\mathbb{Z}$, but only is a square in R.

As for the decomposition maps, let $\wp \lhd R$ be a prime ideal such that $\wp \cap \mathbb{Z} = p\mathbb{Z}$ for $p \in \{2, 3, 5, 7, 11, 19\}$, where $11, 19 \notin \Delta$ while $2, 3, 5, 11 \in \Delta$. Appropriate choices of $\wp$ for $p \in \{11, 19\}$ yield:

	c_i	$p = 19$			$p = 7$			
φ_0	266	1	.	.	1	.	.	.
φ_1	$\frac{19}{4}$	1	.	.	.	1	.	.
φ_2	$\frac{19}{4}$	.	1	.	.	.	1	.
φ_3	$\frac{7}{2}$	.	.	1	1	.	.	.
φ_4	$\frac{38}{11}$	.	1	.	.	.	.	1

	c_i	$p = 11$				$p = 5$				$p = 3$				$p = 2$			
φ_0	266	1	.	.	.	1	.	.	.	1	.	.	.	1	.	.	.
φ_1	$\frac{19}{4}$	.	1	.	.	.	1	.	.	.	1	.	.	.	1	.	.
φ_2	$\frac{19}{4}$	.	.	1	.	.	1	.	.	.	.	1	.	.	.	1	.
φ_3	$\frac{7}{2}$	.	.	1	.	.	.	1	.	.	.	.	1	.	.	.	1
φ_4	$\frac{38}{11}$	.	.	.	1	.	.	.	1	.	.	.	1	1	.	.	.

5 Commutative schemes

In this section let $\mathcal{X}$ be a commutative association scheme.

(5.1) The natural module. Endowing the natural $\mathcal{A}_{\mathbb{C}}$-module $\mathbb{C}^n$ with the standard scalar product $\langle v, w\rangle := v \cdot \overline{w}^{\mathrm{tr}} \in \mathbb{C}$, for all $v, w \in \mathbb{C}^n$, the adjoint of $A \in \mathbb{C}^{n\times n}$ is given as $A^{\mathrm{adj}} = \overline{A}^{\mathrm{tr}}$, where $\overline{}\colon \mathbb{C} \to \mathbb{C}$ is complex conjugation. Since $A_j^{\mathrm{adj}} = \overline{A}_j^{\mathrm{tr}} = A_{j^*} \in \mathcal{A}$, for all $j \in \{0, \ldots, d\}$, we conclude that A_j is a normal matrix, that is A_j commutes with its adjoint. Hence by the spectral theorem for complex normal matrices there is an orthogonal direct sum decomposition $\mathbb{C}^n = \bigoplus_{i=0}^t V_i$ as $\mathbb{C}$-vector spaces, for some $t \in \mathbb{N}_0$, such that the **strata** V_i are maximal simultaneous eigenspaces of all elements of $\mathcal{A}_{\mathbb{C}}$.

Letting $\varphi_i(A_j) \in \mathbb{C}$ be the associated eigenvalue of A_j, for all $i \in \{0, \ldots, t\}$ and $j \in \{0, \ldots, d\}$, we have $A_j|_{V_i} = \varphi_i(A_j) \cdot \mathrm{id}_{V_i}$. Since $(A^{\mathrm{adj}})|_{V_i} = (A|_{V_i})^{\mathrm{adj}}$ for all $A \in \mathcal{A}_{\mathbb{C}}$, we conclude that $\varphi_i(A_{j^*}) = \overline{\varphi}_i(A_j) \in \mathbb{C}$; in particular if $j^* = j$ then we have $\varphi_i(A_j) \in \mathbb{R}$. For the minimum and characteristic polynomials of A_j we have $\mu_{A_j} = \prod_{i=0}^t (T - \varphi_i(A_j)) \in \mathbb{C}[T]$ and $\chi_{A_j} = \prod_{i=0}^t (T - \varphi_i(A_j))^{m_i} \in \mathbb{C}[T]$, where $m_i := \dim_{\mathbb{C}}(V_i) \in \mathbb{N}$. Since the natural representation is faithful, the character table of $\mathcal{A}_{\mathbb{C}}$ is given as $\Phi := \Phi(\mathcal{A}_{\mathbb{C}}) = [\varphi_i(A_j)]_{ij} \in \mathbb{C}^{(t+1)\times(d+1)}$ with associated multiplicities $m_i \in \mathbb{N}$ and degrees $d_i = 1$, for all $i \in \{0, \ldots, t\}$.

If $\mathcal{X}$ is symmetric, then $A_i^{\mathrm{tr}} = A_i \in \mathcal{A} \subseteq \mathbb{R}^{n\times n}$ shows that A_i is a symmetric real matrix, for all $i \in \{0, \ldots, d\}$. Letting $\mathbb{R}^n$ be endowed with the standard scalar product $\langle v, w\rangle := v \cdot w^{\mathrm{tr}} \in \mathbb{R}$, for all $v, w \in \mathbb{R}^n$, by the spectral theorem for symmetric real matrices there is an orthogonal direct sum decomposition $\mathbb{R}^n = \bigoplus_{i=0}^t W_i$ as $\mathbb{R}$-vector spaces such that $V_i = W_i \otimes_{\mathbb{R}} \mathbb{C}$.

(5.2) Characters. Let K be an algebraic number field such that $\mathcal{A}_K$ is split, and let R be the ring of integers in K. The algebra $\mathcal{A}_K$ is split semisimple, this implies $t = d$, hence we have $\mathcal{A}_K \cong \bigoplus_{i\in\{0,\ldots,d\}} \mathrm{End}_K(S_i) \cong \bigoplus_{i\in\{0,\ldots,d\}} K$, where S_i is the irreducible $\mathcal{A}_K$-module affording $\varphi_i \in \mathrm{Irr}(\mathcal{A}_K)$. We have $\varphi_i(A_j) \in R$ for all $i, j \in \{0, \ldots, d\}$, and $\Phi \in R^{(d+1)\times(d+1)}$ is a square matrix. We may assume that $K := \mathbb{Q}(\varphi_i(A_j); i, j \in \{0, \ldots, d\})$ is the character field, being the unique minimal splitting field of $\mathcal{A}_{\mathbb{Q}}$ in $\mathbb{C}$.

Letting $N := \mathrm{diag}[n_0, \ldots, n_d] \in \mathbb{Z}^{(d+1)\times(d+1)}$ and $M := \mathrm{diag}[m_0, \ldots, m_d] \in \mathbb{Z}^{(d+1)\times(d+1)}$, the orthogonality relations yield **row orthogonality** $\overline{\Phi} N^{-1} \Phi^{\mathrm{tr}} = nM^{-1} \in K^{(d+1)\times(d+1)}$; in particular, Φ is invertible, and we have **Biggs' formula (1974)** $\sum_{j=0}^d \frac{1}{n_j} \overline{\varphi}_i(A_j)\varphi_i(A_j) = \frac{n}{m_i}$ relating the character φ_i and its multiplicity m_i. Letting $Q \in \mathbb{Z}^{(d+1)\times(d+1)}$ be the permutation matrix describing the involution $*\colon i \mapsto i^*$ on $\{0, \ldots, d\}$, we have $\overline{\Phi} = \Phi Q$. By inverting, row orthogonality implies $\Phi^{-\mathrm{tr}} N \overline{\Phi}^{-1} = \frac{1}{n} M$, and thus we have **column orthogonality** $\Phi^{\mathrm{tr}} M \overline{\Phi} = nN \in R^{(d+1)\times(d+1)}$.

Let $\varphi_0 \in \mathrm{Irr}(\mathcal{A}_K)$ be the trivial character, that is we have $\varphi_0(A_i) = n_i \geq 0$

for all $i \in \{0, \dots, d\}$. From row orthogonality we infer that φ_0 is the unique irreducible character $\varphi \in \mathrm{Irr}(\mathcal{A}_K)$ such that $\varphi(A_i) \geq 0$ for all $i \in \{0, \dots, d\}$. Hence the valencies $n_i = \varphi_0(A_i)$ and thus the multiplicities m_i are determined from the character table Φ alone.

Let $\mathcal{E} = \{\epsilon_0, \dots, \epsilon_d\} \subseteq \mathcal{A}_K$ be the Wedderburn basis, where $\epsilon_i \in \mathcal{A}_K$ is the centrally primitive idempotent associated with φ_i. Hence we have ${}_{\mathcal{E}}\mathrm{id}_{\{A_k\}} = \Phi^{-\mathrm{tr}} \in K^{(d+1)\times(d+1)}$. The intersection matrices $P_j = {}_{\{A_k\}}(A_j)_{\{A_k\}} \in \mathbb{Z}^{(d+1)\times(d+1)}$, for $j \in \{0, \dots, d\}$, being the representing matrices of the right regular representation of $\mathcal{A}_K$ with respect to the Schur basis, we from $\epsilon_i A_j = \varphi_i(A_j)\epsilon_i$, for $i \in \{0, \dots, d\}$, get ${}_{\mathcal{E}}\mathrm{id}_{\{A_k\}} \cdot P_j = C_j \cdot {}_{\mathcal{E}}\mathrm{id}_{\{A_k\}}$, where $C_j := \mathrm{diag}[\varphi_i(A_j); i \in \{0, \dots, d\}] \in R^{(d+1)\times(d+1)}$. Hence we have $P_j = ({}_{\mathcal{E}}\mathrm{id}_{\{A_k\}})^{-1} \cdot C_j \cdot {}_{\mathcal{E}}\mathrm{id}_{\{A_k\}} = \Phi^{\mathrm{tr}} C_j \Phi^{-\mathrm{tr}}$, thus the intersection matrices P_j are determined from Φ alone.

Conversely, we show how Φ can be determined from the P_j: Since ${}_{\mathcal{E}}\mathrm{id}_{\{A_k\}}$ describes the Wedderburn isomorphism $\mathcal{A}_K \cong \bigoplus_{i=0}^d \mathcal{A}_K \epsilon_i \cong \bigoplus_{i=0}^d K$ in terms of the Schur basis, the rows of ${}_{\mathcal{E}}\mathrm{id}_{\{A_k\}} = \Phi^{-\mathrm{tr}}$ are simultaneous eigenvectors of all the P_j, with associated eigenvalues $\varphi_i(A_j)$; this is also seen from the equation $\Phi^{-\mathrm{tr}} P_j = C_j \Phi^{-\mathrm{tr}}$. Since $\mathcal{P}_K \cong \mathcal{A}_K$ as K-algebras, any non-zero simultaneous eigenspace of all the P_j corresponds to a submodule of $\mathcal{A}_K$ having only one constituent, thus is one-dimensional. Hence the non-zero simultaneous eigenspaces of all the P_j are the K-spans of the various rows of $\Phi^{-\mathrm{tr}}$. Thus $\Phi^{-\mathrm{tr}}$ can be determined up to scalars from the P_j, by computing eigenspaces and intersecting them, where it is sufficient to determine only a subset of the P_j and their eigenspaces such that one-dimensional simultaneous eigenspaces are obtained; finally Φ is found from transposing, inverting and using $\varphi_i(A_0) = 1$.

The P_j^{tr} are the representing matrices of the left regular representation of $\mathcal{A}_K$, that is the right regular representation of the opposite algebra $\mathcal{A}_K^{\mathrm{opp}} \cong \mathcal{A}_K$, with respect to the dual Schur basis. The Frobenius-Schur relations imply ${}_{\mathcal{E}}\mathrm{id}_{\{A_k^*\}} = \mathrm{diag}[c_i; i \in \{0, \dots, d\}]^{-1} \cdot \Phi \in K^{(d+1)\times(d+1)}$, where $c_i \in K$ is the Schur element associated with φ_i. Hence the rows of Φ are simultaneous eigenvectors of all the P_j^{tr}, with associated eigenvalues $\varphi_i(A_j)$; this is also seen in terms of matrices, since from $\Phi^{-\mathrm{tr}} P_j = C_j \Phi^{-\mathrm{tr}}$ we get $C_j^{-1} \Phi^{-\mathrm{tr}} = \Phi^{-\mathrm{tr}} P_j^{-1}$, and thus inverting and transposing yields $\Phi P_j^{\mathrm{tr}} = C_j \Phi$. As above we conclude that the non-zero simultaneous eigenspaces of all the P_j^{tr} are the K-spans of the various rows of Φ. Thus Φ can be determined directly from the P_j^{tr}, avoiding a matrix inversion.

(5.3) Central Schurian schemes. Let $X := G$ be a finite group, let $G = \coprod_{i=0}^d C_i$ be its conjugacy classes, and let $x_i \in C_i$ for all $i \in \{0, \dots, d\}$, where $x_0 := 1$. For $i \in \{0, \dots, d\}$ let $R_i := \{[x, y] \in G^2; x^{-1}y \in C_i\}$. Then we have $G^2 = \coprod_{i=0}^d R_i$ where $R_0 = \{[x, x]; x \in G\}$, and letting $i^* \in \{0, \dots, d\}$ such that $C_i^{-1} = C_{i^*}$ we have $R_i^{\mathrm{tr}} = R_{i^*}$; hence $R_i^{\mathrm{tr}} = R_i$ if and only if C_i is self-inverse. Indeed $\mathcal{X} := [G, \{R_0, \dots, R_d\}]$ is a Schurian association scheme:

We consider the transitive action of $G \times G$ on G via $[g, h]\colon x \mapsto g^{-1}xh$. We have $\mathrm{Stab}_{G\times G}(1) = \Delta(G) := \{[g, g]; g \in G\} \cong G$, and hence the $\Delta(G)$-suborbits are given by $x^{\Delta(G)} = \{g^{-1}xg; g \in G\}$ for $x \in G$, that is the conjugacy classes of G. The diagonal action of $G \times G$ on G^2 is given by $[g,h]\colon [x,y] \mapsto [g^{-1}xh, g^{-1}yh]$. For $i \in \{0,\ldots,d\}$ let $[1,x_i]^{G\times G} = \{[g^{-1}h, g^{-1}x_ih]; g,h \in G\}$ be an orbital. Then $(g^{-1}h)^{-1}(g^{-1}x_ih) = h^{-1}x_ih \in C_i$ implies $[1,x_i]^{G\times G} \subseteq R_i$. Hence from $\coprod_{i=0}^d [1,x_i]^{G\times G} = G^2 = \coprod_{i=0}^d R_i$ we infer $[1,x_i]^{G\times G} = R_i$ for all $i \in \{0,\ldots,d\}$.

For $i,j,k \in \{0,\ldots,d\}$, since $[1,x_k] \in R_k$ we have $p_{ij}^k = |\{x \in G; [1,x] \in R_i, [x,x_k] \in R_j\}| = |\{x \in G; x \in C_i, x^{-1}x_k \in C_j\}| = |C_i \cap x_kC_j^{-1}| = |\{[x,y] \in C_i \times C_j; xy = x_k\}| =: a_{ij}^k$, the **central structure constants** of G:

Let R be a ring, and let $Z(R[G])$ be the centre of the group algebra $R[G]$. Then $\{C_i^+; \in \{0,\ldots,d\}\} \subseteq Z(R[G])$ is an R-basis, where $C_i^+ := \sum_{x\in C_i} x \in R[G]$ is the associated **conjugacy class sum**. Then we have $C_i^+C_j^+ = \sum_{k=0}^d a_{ij}^k C_k^+$, for all $i,j,k \in \{0,\ldots,d\}$. Thus we infer that $Z(R[G]) \to \mathcal{A}_R\colon C_i^+ \mapsto A_i$ is an isomorphism of R-algebras; in particular $\mathcal{X}$ is commutative.

Let K be a field such that $\mathrm{char}(K) = 0$ and $K[G]$ is split. For $\chi \in \mathrm{Irr}(K[G])$ let $\omega_\chi \in \mathrm{Irr}(Z(K[G]))$ be the associated **central character** given by $\omega_\chi(C_i^+) = \frac{|C_i|\chi(x_i)}{\chi(1)} \in K$, for all $i \in \{0,\ldots,d\}$, describing the scalar action of $Z(K[G])$ on the irreducible module affording χ. Hence the character table $[\omega_\chi(C_i^+); \chi \in \mathrm{Irr}(K[G]), i \in \{0,\ldots,d\}]_{\chi,i}$ of $Z(K[G])$ with respect to the class sum basis coincides with the character table of $\mathcal{A}_K$ with respect to the Schur basis. Thus the character table of $Z(K[G])$, and hence the irreducible characters of G, can be determined from the central structure constants.

(5.4) Example. Let $\mathcal{X}$ be the Schurian scheme on $X := H\backslash G$ where $G := J_1$ and $H := L_2(11) < G$ from (4.11). Still letting $[n_0,\ldots,n_4] = [1, 11, 110, 132, 12]$, the intersection matrix $P_1 \in \mathbb{Z}^{5\times 5}$ associated to $A_1 \in \mathcal{A}$ is given as follows:

$$P_1 = \begin{bmatrix} . & 1 & . & . & . \\ 11 & . & 1 & . & . \\ . & 10 & 4 & 5 & . \\ . & . & 6 & 5 & 11 \\ . & . & . & 1 & . \end{bmatrix}$$

Hence for the minimum polynomial of $b := [1,0,0,0,0] \in K^5$ with respect to P_1 we have $\mu_{P_1,b} = T^5 - 9T^4 - 42T^3 + 193T^2 + 341T - 484 = (T-11)(T - \frac{-7-\sqrt{5}}{2})(T - \frac{-7+\sqrt{5}}{2})(T-4)(T-1) \in K[T]$, thus for the minimum polynomial of P_1 we have $\mu_{P_1} = \mu_{P_1,b}$. Hence we have $\langle P_1^i; i \in \mathbb{N}_0\rangle_K = \langle P_1^0,\ldots,P_1^4\rangle_K$, where $\{P_1^0,\ldots,P_1^4\}$ is K-linearly independent. Hence we have $\langle P_1^0,\ldots,P_1^4\rangle_K = \mathcal{P}_K$, and it suffices to compute the eigenspaces of P_1, which are as follows:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 110+110\sqrt{5} & -60-40\sqrt{5} & 20+6\sqrt{5} & \frac{-35-5\sqrt{5}}{2} & 55 \\ 110-110\sqrt{5} & -60+40\sqrt{5} & 20-6\sqrt{5} & \frac{-35+5\sqrt{5}}{2} & 55 \\ 66 & 24 & 3 & -4 & -11 \\ 33 & 3 & -3 & 1 & 11 \end{bmatrix}$$

The character table Φ is then found from inverting, transposing and normalising the above matrix. Equivalently, it suffices to compute the eigenspaces of P_1^{tr}, where normalising eigenvectors directly yields the character table Φ:

$$\begin{bmatrix} 1 & 11 & 110 & 132 & 12 \\ 1 & \frac{-7-\sqrt{5}}{2} & \frac{5+7\sqrt{5}}{2} & \frac{3-9\sqrt{5}}{2} & \frac{-3+3\sqrt{5}}{2} \\ 1 & \frac{-7+\sqrt{5}}{2} & \frac{5-7\sqrt{5}}{2} & \frac{3+9\sqrt{5}}{2} & \frac{-3-3\sqrt{5}}{2} \\ 1 & 4 & 5 & -8 & -2 \\ 1 & 1 & -10 & 4 & 4 \end{bmatrix}$$

6 References

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