THE 2-MODULAR DECOMPOSITION MATRICES OF THE SYMMETRIC GROUPS S_{15}, S_{16}, AND S_{17}

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Abstract. In this paper the 2-modular decomposition matrices of the symmetric groups S_{15}, S_{16}, and S_{17} are determined by application of methods from computational representation theory, in particular condensation techniques, and by using the computer algebra systems GAP, MOC, and the MeatAxe.

1. Introduction and statement of results

Currently, the modular representation theory of the symmetric groups is a very active field of research, and it is certainly worth while to have as many explicit results at hand as possible. The staring point of the present work was a paper by D. Benson [1], where all but one of the 2-modular Brauer characters of the symmetric group S_{15} have been determined. We are now able to fill this gap and to go beyond that, our results are stated at the end of this section. They are also available via GAP [3] or the Internet extension [16] of the ModularAtlas [7]. We remark that there is work in progress concerning decomposition matrices in odd characteristics and for even larger symmetric groups; in due course these results will also be made available in these places.

We interdependently apply a few of the methods which in recent years have been developed for the explicit computation of decomposition numbers and to solve other problems in computational representation theory. These encompass character theoretic computations, algorithms for which are implemented in the computer algebra systems MOC [5], see Section 2.1, and GAP, and which are complemented by a few particular techniques for the symmetric groups, see Section 2.2, as well as explicit construction and analysis of modules using the MeatAxe [13, 14], and applying a condensation technique, see Section 2.3. Due to space limitations it is not possible to give complete proofs for all the results presented here. Hence we are content with giving proofs only for Tables 5 and 1, see Sections 3 and 4, respectively, which show the methods at work.

As a general reference for the representation theory of the symmetric groups, see [6]. In particular, the irreducible ordinary characters of the symmetric group S_n are parametrized by the partitions of n, and the irreducible p-modular Brauer characters, p a rational prime, are parametrized by the p-regular partitions of n. Each of the groups S_{15}, S_{16}, and S_{17} has three 2-blocks, whose decomposition matrices are given in Tables 1–3, Tables 4–6, and Tables 7–9, respectively. We only print the rows of the decomposition matrices belonging to the irreducible ordinary characters being parametrized by 2-regular partitions, see Sections 2.1–2.2.

The decomposition matrices for S_{15}, which we restate here for convenience, have already been determined in [1] — except the underlined entries in Table 1. It has been shown in [1] that either $\varphi_{[7,4,3,1]}$ has degree 76830, which corresponds to the decomposition matrix as is printed in Table 1, or $\varphi_{[7,4,3,1]}$ has degree 76828, which means that the underlined entries in Table 1 are equal to '7' instead. In Section 4 we will show that the former case is the correct

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Table 1. S_{15} mod 2, principal block, defect 11

degree	partition	$\varphi_j, j = 1, \ldots, 15$														
1	[15]	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.
90	[13, 2]	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.
715	[12, 2, 1]	1	1	1	.	.	.	.	.	.	.	.	.	.	.	.
910	[11, 4]	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.
2835	[11, 3, 1]	1	.	1	1	1	.	.	.	.	.	.	.	.	.	.
7007	[10, 4, 1]	1	.	1	1	1	1	.	.	.	.	.	.	.	.	.
2002	[9, 6]	.	1	.	.	.	.	1	.	.	.	.	.	.	.	.
11375	[9, 5, 1]	1	1	1	.	.	1	1	1	.	.	.	.	.	.	.
22113	[9, 4, 2]	3	2	1	.	1	1	1	1	1	.	.	.	.	.	.
11583	[8, 6, 1]	1	1	.	.	.	1	1	1	.	1	.	.	.	.	.
35035	[8, 4, 3]	3	2	1	.	.	1	1	1	1	1	1	.	.	.	.
25025	[7, 6, 2]	1	1	.	.	.	1	1	1	1	1	.	1	.	.	.
45045	[7, 5, 3]	3	1	1	.	1	1	1	1	1	1	1	1	1	.	.
135135	[7, 4, 3, 1]	$\underline{5}$	3	1	2	1	1	2	1	2	1	1	1	1	1	.
175175	[6, 4, 3, 2]	$\underline{5}$	3	1	3	1	.	1	.	1	1	1	1	1	1	1

Table 2. S_{15} mod 2, non-principal block of defect 10

degree	partition	$\varphi_j, j = 16, \ldots, 26$										
14	[14, 1]	1	.	.	.	.	.	.	.	.	.	.
350	[12, 3]	1	1	.	.	.	.	.	.	.	.	.
1638	[10, 5]	1	1	1	.	.	.	.	.	.	.	.
9100	[10, 3, 2]	2	1	1	1	.	.	.	.	.	.	.
42042	[9, 3, 2, 1]	5	2	2	1	1	.	.	.	.	.	.
1430	[8, 7]	1	.	1	.	.	1	.	.	.	.	.
32032	[8, 5, 2]	.	.	.	.	.	.	1	.	.	.	.
91000	[8, 4, 2, 1]	4	2	2	.	1	1	1	1	.	.	.
108108	[7, 5, 2, 1]	2	1	2	.	1	2	1	1	1	.	.
30030	[6, 5, 4]	1	.	1	1	.	1	.	.	.	1	.
128700	[6, 5, 3, 1]	6	3	2	1	1	2	.	1	1	1	1

Table 3. S_{15} mod 2, non-principal block of defect 0

degree	partition	φ_{27}
292864	[5, 4, 3, 2, 1]	1

one. We remark that we have independently cross-checked all the computational results in [1]. In particular $\varphi_{[7,6,2]}$ indeed has degree $4096 = 2^{12}$, which does not divide the group order $|S_{15}|$.

2. Methods

2.1. **Character theory.** Let G be a finite group and p a rational prime. Let $\mathrm{Cf}(G)$, $\mathrm{Cf}_p^0(G)$, and $\mathrm{Cf}_{p'}(G)$ be the the $\mathbb{C}$-vector spaces of all class functions on G, of all class functions vanishing on the p-singular classes, and of all class functions on the p-regular classes of G, respectively. Then the ordinary character theoretic scalar product induces a duality between the set $\mathbb{Z}\mathrm{IBr}_p(G) \subseteq \mathrm{Cf}_{p'}(G)$ of generalized Brauer characters of G and the set $\mathbb{Z}\mathrm{IPr}_p(G) \subseteq \mathrm{Cf}_p^0(G)$ of generalized projective characters of G, a pair of mutually dual bases being the sets $\mathrm{IBr}_p(G)$ and $\mathrm{IPr}_p(G)$ of irreducible Brauer characters and projective indecomposable characters, respectively. A $\mathbb{Z}$-basis $\mathcal{BS} \subseteq \mathbb{N}_0\mathrm{IBr}_p(G)$ of $\mathbb{Z}\mathrm{IBr}_p(G)$ is called a *basic set of*

TABLE 4. S_{16} mod 2, principal block, defect 15

degree	partition	$\varphi_j, j = 1, \ldots, 22$																					
1	[16]	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
15	[15, 1]	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
104	[14, 2]	.	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
440	[13, 3]	.	1	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
1260	[12, 4]	.	1	.	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
3900	[12, 3, 1]	2	1	1	1	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
2548	[11, 5]	.	1	.	1	1	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
13860	[11, 3, 2]	2	2	.	1	1	1	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.
3640	[10, 6]	.	1	1	1	.	.	1	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.
20020	[10, 5, 1]	2	1	1	1	1	1	1	.	1	1	.	.	.	.	.	.	.	.	.	.	.	.
38220	[10, 4, 2]	4	2	2	1	1	1	1	1	1	1	1	.	.	.	.	.	.	.	.	.	.	.
3432	[9, 7]	.	1	1	.	.	.	1	.	1	.	.	1	.	.	.	.	.	.	.	.	.	.
65520	[9, 5, 2]	4	.	3	.	.	1	.	.	2	1	1	.	1	.	.	.	.	.	.	.	.	.
180180	[9, 4, 2, 1]	6	9	5	4	1	1	4	1	3	1	2	1	1	1	.	.	.	.	.	.	.	.
18018	[8, 7, 1]	2	1	1	.	.	.	1	.	1	1	.	1	.	.	1	.	.	.	.	.	.	.
68640	[8, 6, 2]	2	.	2	.	.	.	.	.	2	1	1	.	1	.	1	1	.	.	.	.	.	.
112112	[8, 5, 3]	4	.	3	.	.	1	.	.	2	1	2	.	1	.	1	1	1	.	.	.	.	.
318500	[8, 4, 3, 1]	6	7	5	3	2	1	3	.	3	1	3	2	1	1	1	1	1	1	.	.	.	.
210210	[7, 6, 2, 1]	2	3	3	1	.	.	3	.	3	1	2	2	1	1	1	2	.	.	1	.	.	.
100100	[7, 5, 4]	4	1	1	.	.	1	1	1	1	1	1	1	.	.	1	1	1	.	.	1	.	.
416988	[7, 5, 3, 1]	6	8	4	4	2	1	4	1	3	1	3	3	1	1	1	2	1	1	1	1	1	.
500500	[6, 5, 3, 2]	8	12	6	6	4	1	4	1	2	.	2	3	.	1	1	2	1	1	1	1	2	1

TABLE 5. S_{16} mod 2, non-principal block of defect 8

degree	partition	$\varphi_j, j = 23, \ldots, 29$						
896	[13, 2, 1]	1	.	.	.	.	.	.
10752	[11, 4, 1]	.	1	.	.	.	.	.
24960	[9, 6, 1]	1	.	1	.	.	.	.
69888	[9, 4, 3]	2	.	1	1	.	.	.
91520	[7, 6, 3]	1	.	1	1	1	.	.
512512	[7, 4, 3, 2]	.	1	.	.	.	1	.
1153152	[6, 4, 3, 2, 1]	3	2	1	1	1	1	1

TABLE 6. S_{16} mod 2, non-principal block of defect 4

degree	partition	φ_{30}	φ_{31}	φ_{32}
71680	[10, 3, 2, 1]	1	.	.
266240	[8, 5, 2, 1]	.	1	.
292864	[6, 5, 4, 1]	1	.	1

TABLE 7. S_{17} mod 2, principal block, defect 15

degree	partition	$\varphi_j, j = 1, \ldots, 22$																					
1	[17]	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
119	[15, 2]	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
1105	[14, 2, 1]	1	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
1700	[13, 4]	.	1	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
5236	[13, 3, 1]	2	1	1	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
15912	[12, 4, 1]	2	1	.	1	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
6188	[11, 6]	.	1	.	1	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
33320	[11, 5, 1]	2	1	.	1	1	1	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.
62832	[11, 4, 2]	4	2	.	1	2	1	1	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.
48620	[10, 6, 1]	2	1	1	1	1	.	1	1	.	1	.	.	.	.	.	.	.	.	.	.	.	.
129948	[10, 4, 3]	4	2	2	1	1	.	1	1	1	1	1	.	.	.	.	.	.	.	.	.	.	.
4862	[9, 8]	.	1	.	.	.	.	1	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.
46410	[9, 7, 1]	2	1	1	.	.	.	1	1	.	1	.	1	1	.	.	.	.	.	.	.	.	.
159120	[9, 6, 2]	4	.	1	.	1	.	.	2	.	1	.	.	1	1	.	.	.	.	.	.	.	.
247520	[9, 5, 3]	6	.	2	.	2	.	.	2	.	1	1	.	1	1	1	.	.	.	.	.	.	.
680680	[9, 4, 3, 1]	10	6	2	3	2	.	3	2	1	1	1	2	1	1	1	1	.	.	.	.	.	.
272272	[8, 6, 3]	4	.	1	.	1	.	.	2	.	1	1	.	2	1	1	.	1	.	.	.	.	.
1299480	[8, 4, 3, 2]	6	5	.	3	1	1	3	1	.	.	.	3	1	1	1	1	.	1	.	.	.	.
243100	[7, 6, 4]	4	1	1	.	1	.	1	1	1	1	1	1	1	.	1	.	1	.	1	.	.	.
918918	[7, 6, 3, 1]	6	3	1	2	1	.	3	2	1	1	1	3	2	1	1	1	1	.	1	1	.	.
1750320	[7, 5, 3, 2]	10	8	.	5	2	1	4	1	1	.	.	4	2	1	2	1	.	1	1	1	1	.
3573570	[6, 5, 3, 2, 1]	28	13	6	7	5	3	5	3	2	2	2	5	6	1	5	1	2	1	2	1	2	1

Table 8. S_{17} mod 2, non-principal block of defect 11

degree	partition	$\varphi_j, j = 23, \dots, 37$														
16	$[16, 1]$	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.
544	$[14, 3]$	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.
3808	$[12, 5]$	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.
20400	$[12, 3, 2]$	1	.	1	1	.	.	.	.	.	.	.	.	.	.	.
116688	$[11, 3, 2, 1]$	1	.	2	1	1	.	.	.	.	.	.	.	.	.	.
7072	$[10, 7]$	.	1	.	.	.	1	.	.	.	.	.	.	.	.	.
123760	$[10, 5, 2]$	1	.	1	1	.	.	1	.	.	.	.	.	.	.	.
333200	$[10, 4, 2, 1]$	3	2	2	1	1	1	1	1	.	.	.	.	.	.	.
583440	$[9, 5, 2, 1]$	3	3	1	1	.	2	1	1	1	.	.	.	.	.	.
116688	$[8, 7, 2]$	1	.	.	.	.	.	1	.	.	1	.	.	.	.	.
618800	$[8, 6, 2, 1]$	1	2	.	.	.	2	1	1	1	1	1	.	.	.	.
272272	$[8, 5, 4]$	1	.	.	1	.	.	1	.	.	1	.	1	.	.	.
1113840	$[8, 5, 3, 1]$	3	3	2	1	.	2	1	2	1	1	1	1	1	.	.
1050192	$[7, 5, 4, 1]$	3	1	2	1	1	1	1	1	.	1	1	1	1	1	.
1361360	$[6, 5, 4, 2]$	5	3	3	1	1	1	.	1	.	1	1	1	1	1	1

Table 9. S_{17} mod 2, non-principal block of defect 1

degree	partition	φ_{38}
3620864	$[7, 4, 3, 2, 1]$	1

Brauer characters. Analogously, a $\mathbb{Z}$-basis $\mathcal{PS} \subseteq \mathbb{N}_0\mathrm{IPr}_p(G)$ of $\mathbb{Z}\mathrm{IPr}_p(G)$ is called a *basic set of projective characters.*

The general strategy implemented in the MOC system, see [5, 8], now is to find basic sets of Brauer characters and of projective characters in the first place, e.g. by using the irreducible ordinary characters of G and characters induced from subgroups. Then decomposing Brauer characters and projective characters into the basic sets $\mathcal{BS}$ and $\mathcal{PS}$, gives relations which can be used to improve the current basic sets and to find new characters, e.g. by tensoring. Ideally, after a series of steps we should obtain the basics sets $\mathcal{BS}_\infty = \mathrm{IBr}_p(G)$ and $\mathcal{PS}_\infty = \mathrm{IPr}_p(G)$. In practice, however, one usually ends up with basic sets that do not entirely consist of irreducible Brauer characters and projective indecomposable characters, but also contain characters which are sums of a very few irreducible Brauer characters or projective indecomposable characters, their possible decompositions being under control. This is where module theoretic methods come into play, e.g. condensation techniques, see Section 2.3.

All the processes of finding suitable basic sets, producing rather large numbers of characters, finding the relevant relations between them, and improving the basic sets, is done automatically in MOC. It produces a protocol of the steps it has taken, from which a 'classical' proof can be easily deduced. To give the reader an impression how this works in one of the easier examples, in Section 3 we give the complete proof of the correctness of Table 5, which has been produced automatically by MOC — except the typesetting, of course.

2.2. **Special techniques for the symmetric groups.** For the symmetric group S_n, a basic set of Brauer characters is given by the restrictions $\hat{\chi}_\lambda \in \mathrm{Cf}_{p'}(S_n)$ of the irreducible ordinary characters χ_λ to the p-regular classes, where λ is a p-regular partition of n, see [6, Corollary 7.1.16]. We remark that for an arbitrary finite group G it is an open problem whether there is a basic set of Brauer characters consisting of restrictions of irreducible ordinary characters to the p-regular classes.

A basic set of projective characters for S_n can be found as a subset of the set of r-induced projective indecomposable characters of S_{n-1}, see [6, Section 6.3]. Again, we remark that, to find a basic set of projective characters for an arbitrary finite group G, in general we need the induced projective indecomposable characters of all the conjugacy classes of maximal subgroups of G.

Let $\hat{S}_n$ denote a Schur covering group of S_n, where for our purposes it does not matter which isoclinism type we use. $\hat{S}_n$ is a central extension of S_n by a cyclic group of order 2. Hence $\mathrm{Cf}_{2'}(\hat{S}_n)$ and $\mathrm{Cf}_{2'}(\hat{S}_n)$ coincide, we have $\mathrm{IBr}_2(\hat{S}_n) = \mathrm{IBr}_2(S_n)$, and the restrictions $\hat{\vartheta} \in \mathrm{Cf}_{2'}(S_n)$ of the spin characters ϑ, i.e. the faithful irreducible ordinary characters of $\hat{S}_n$, to the 2-regular classes are Brauer characters of S_n. By [11], character formulae for the spin characters are known; here we have used ordinary character tables for $\hat{S}_n$ which have been kindly provided by G. Malle [9].

Furthermore, we have made use of the Jantzen-Schaper formula [15], see also [10]. This formula is related to the Jantzen filtration of Specht modules and gives certain upper estimates of decomposition numbers; for details see the references cited. We have used our own implementation of the Jantzen-Schaper formula, written in GAP, to exclude some candidate cases left over in the character theoretic computations.

2.3. **Condensation.** As many 'interesting' modules are too large to be constructed directly, one tries to 'condense' these modules to smaller ones which still reflect enough of the original structure but can be analysed explicitly. For more details on the range of applications of condensation we refer to [2, 12]. The following functorial description of the theoretical background is inspired by [4].

Let k be a field, A be a finite-dimensional k-algebra, $0 \neq e \in A$ be an idempotent and mod–A be the category of finitely generated unital right A-modules. Then the functor

$$C_e := ? \otimes_A Ae : \text{mod–}A \longrightarrow \text{mod–}eAe$$

is called the *condensation functor* with respect to e. This means, $M \in \text{mod–}A$ is mapped to the *condensed module* $M \otimes_A Ae \in \text{mod–}eAe$, which can be identified with the subset $Me \subseteq M$. Using this identification, a homomorphism $\alpha \in \mathrm{Hom}_A(M, N)$ is mapped to $\alpha|_{Me} \in \mathrm{Hom}_A(Me, Ne)$. As $C_e \cong \mathrm{Hom}_A(eA, ?)$ as functors, C_e is an exact functor.

This is applied to the following situation: Let $A := k[G]$ be the group algebra of a finite group G. Let $K \leq G$ be a subgroup such that $|K| \neq 0 \in k$, the *condensation subgroup*, giving rise to the *condensation idempotent*

$$e = e_K := \frac{1}{|K|} \cdot \sum_{g \in K} g \in k[G].$$

We have $e \cdot k[G] \cong (1_K)^G \in \text{mod–}k[G]$, where $(1_K)^G$ is the permutation representation of G on the cosets of K. Hence for $M \in \text{mod–}k[G]$, we have

$$Me \cong \mathrm{Hom}_{k[G]}((1_K)^G, M) \cong \mathrm{Hom}_{k[K]}(1_K, M_K) \cong \mathrm{Fix}_M(K)$$

as vector spaces, where $\mathrm{Fix}_M(K) \subseteq M$ denotes the subspace of elements of M being fixed by K. Thus this technique is called *fixed point condensation*. As all $M \in \text{mod–}k[K]$ are semisimple, the dimension of the condensed module of M can be computed as the ordinary character theoretic scalar product of the trivial K-character and the restriction to K of the Brauer character of M.

If $\mathcal{A}$ is a subset of A, the subalgebra $\mathcal{C} := \langle eae; a \in \mathcal{A} \rangle_{k-\text{algebra}}$ of eAe is called the corresponding *condensation algebra*. But even if $\mathcal{A}$ generates A as a k-algebra, $\mathcal{C}$ is not necessarily equal to eAe. In practice we can only compute the action of a finite set of *condensed elements* eae, $a \in A$, on a condensed module. Hence we have to draw conclusions about the structure of a condensed module from its analysis as a module for a suitable condensation algebra.

There are several implementations of fixed point condensation available, suitable for different types of modules. We will use the InducedCondense package [12] for fixed point condensation of modules induced from a subgroup H, provided the $k[H]$-module to be induced and the action of G on the cosets of H are given.

3. Proving the correctness of table 5

We first fix a basic set $\mathcal{BS}$ of Brauer characters, see Section 2.1, as follows:

$$\hat{\chi}_{[13,2,1]},\ \hat{\chi}_{[11,4,1]},\ \mathrm{Ind}(\varphi_{[9,6]}),\ \mathrm{Ind}(\varphi_{[9,4,2]}),$$
$$\mathrm{Ind}(\varphi_{[7,6,2]}),\ \mathrm{Ind}(\varphi_{[7,4,3,1]}),\ \mathrm{Ind}(\varphi_{[6,4,3,2]}),$$

where the $\hat{\chi}$'s denote the restrictions of the irreducible ordinary characters of S_{16} to the 2-regular classes, the φ's denote the irreducible Brauer characters of S_{15} according to Tables 1–3, and 'Ind' denotes induction from S_{15} to S_{16} and restriction to the present block.

We then find a basic set $\mathcal{PS} := \{\Psi_1, \ldots, \Psi_7\}$ of projective characters as follows:

$$\mathrm{Ind}(\Phi_{[13,2]})/3,\ \mathrm{Ind}(\Phi_{[11,4]})/3,\ \mathrm{Ind}(\Phi_{[8,6,1]})/2,\ \mathrm{Ind}(\Phi_{[8,4,3]})/3,$$
$$\mathrm{Ind}(\Phi_{[7,6,2]})/3,\ \mathrm{Ind}(\Phi_{[6,4,3,2]})/4,\ \mathrm{Ind}(\Phi_{[5,4,3,2,1]}),$$

where the $\Phi_.$'s denote the projective indecomposable characters of S_{15}, and '$\cdot/d$' means that we have divided all the values of the induced character by d, because all of its scalar products with the characters in $\mathcal{BS}$ had been divisible by d.

The matrix of scalar products between the characters in $\mathcal{BS}$ and $\mathcal{PS}$, the columns being indexed by $\Psi_1, \ldots, \Psi_7$, is given as follows:

$$\begin{bmatrix} 1 & . & . & . & . & . & . \\ . & 1 & . & . & . & . & . \\ . & . & 1 & . & . & . & . \\ . & . & . & 1 & . & . & . \\ . & . & . & . & 1 & . & . \\ 4 & 2 & 1 & 2 & 1 & 1 & . \\ . & . & . & . & . & . & 1 \end{bmatrix}$$

Hence Ψ_6 is a projective indecomposable character. Next, we find that $\mathrm{Ind}(\Phi_{[12,2,1]})$, $\mathrm{Ind}(\Phi_{[11,3,1]})$, and $\mathrm{Ind}(\Phi_{[10,4,1]})$ decompose into $\mathcal{PS}$ as follows:

$$\begin{bmatrix} 1 & 2 & . & . & . & -8 & . \\ . & 2 & . & 2 & . & -8 & . \\ . & 1 & 2 & . & . & -4 & . \end{bmatrix} \cdot \mathcal{PS}$$

From this we conclude that

$$\begin{array}{lllllll} \Psi'_1 & := & \Psi_1 - 4\cdot\Psi_6, & \qquad & \Psi'_2 & := & \Psi_2 - 2\cdot\Psi_6, \\ \Psi'_3 & := & \Psi_3 - \Psi_6, & & \Psi'_4 & := & \Psi_4 - 2\cdot\Psi_6 \end{array}$$

are projective characters, and hence are projective indecomposable ones. Leaving Ψ_5, Ψ_6, and Ψ_7 unchanged, we obtain a new basic set $\mathcal{PS}' := \{\Psi'_1, \ldots, \Psi'_7\}$, whose decomposition into the irreducible ordinary characters in the present block is as follows:

degree	partition	$\Psi'_j, j=1,\ldots,7$						
896	$[13,2,1]$	1	.	.	.	.	.	.
10752	$[11,4,1]$	.	1	.	.	.	.	.
24960	$[9,6,1]$	1	.	1	.	.	.	.
69888	$[9,4,3]$	2	.	1	1	.	.	.
91520	$[7,6,3]$	1	.	1	1	1	.	.
512512	$[7,4,3,2]$	.	1	.	.	1	1	.
1153152	$[6,4,3,2,1]$	3	2	1	1	2	1	1

The last remaining question now is whether Ψ'_5 is a projective indecomposable character or whether $\Psi'_5 - \Psi'_6$ is. But the Jantzen-Schaper formula, see Section 2.2, shows that $\varphi_{[7,6,3]}$

Table 10. Dimensions of the condensed modules of $\mathcal{BS}$

φ_j	degree	dim.
1	1	1
2	90	2
3	910	6
4	1912	4
5	624	4
6	1300	4
7	4172	8
8	4576	8
9	832	.
10	9346	6
11	4096	.
12	13390	2
13	4704	.
14	76830	38
15	59136	12
16	14	.
17	336	4
18	1288	4
19	128	.
20	7448	16
21	32032	44
22	21152	.
23	31276	40
24	24260	12
25	17344	.
26	23296	.
27	292864	152

is not a 2-modular constituent of $\hat{\chi}_{[7,4,3,2]}$. Hence $\Psi'_5 - \Psi'_6$ is a projective indecomposable character, and we obtain the decomposition matrix shown in Table 5.

4. Proving the correctness of Table 1

We only have to show the correctness of the entries underlined in Table 1. Let $\mathcal{BS} := \{\varphi_1, \ldots, \varphi_{15}, \varphi_{16}, \ldots, \varphi_{26}, \varphi_{27}\}$ be the basic set of Brauer characters underlying the matrix shown in Table 1. Hence φ_{14} has degree 76830, and either it is an irreducible Brauer character or $\varphi_{14} - 2 \cdot \varphi_1$ is.

Let $k := \mathbb{F}_2$. As a condensation subgroup we choose

$$K := \langle \{(5, 10, 15), (1, 4, 7, 10, 13)(2, 5, 8, 11, 14)(3, 6, 9, 12, 15)\} \rangle < G := S_{15},$$

and let $e = e_K$ denote the corresponding condensation idempotent. K is isomorphic to the natural wreath product $3 \wr 5 \cong 3^5 : 5$ and has order 1215. The distribution of the elements of K into the conjugacy classes of G is found using their permutation cycle types. From this we compute the permutation character $(1_K)^G$ on the 2-regular classes, and its scalar products with the characters in $\mathcal{BS}$. This gives the dimensions of the condensed modules of $k[G]$-modules whose Brauer characters are in $\mathcal{BS}$, see Table 10. In particular, φ_{14} condenses to a module of dimension 38; and if $\varphi_{14} - 2 \cdot \varphi_1$ were a Brauer character, it would condense to a module of dimension 36.

We are now going to induced-condense a suitable $k[H]$-module, where H denotes the Young subgroup $H := S_7 \times S_8$. The action of G on the cosets of H is equivalent to the action of G on the subsets of cardinality 7 of $\{1, \ldots, 15\}$. The Brauer character tables of S_7 and S_8 can be accessed from [7] or from GAP. Let $1a$ denote the trivial $k[S_7]$-module, and $40a$ denote the irreducible $k[S_8]$-module $\varphi_{[4,3,1]}$, which has degree 40. We obtain explicit matrices for $40a$ starting from the 2-modular reduction of the natural permutation representation of S_8 by a series of standard applications of the MeatAxe, such as tensoring and chopping. The Brauer character $(\varphi_{[1^7]} \otimes \varphi_{[4,3,1]})^G$ of the induced module $V := (1a \otimes 40a)^G$ decomposes into $\mathcal{BS}$ as follows:

$$(\varphi_{[1^7]} \otimes \varphi_{[4,3,1]})^G = [2, 0, 0, 0, 0, 1, 0, 0, 0, 2, 0, 0, 0, 1, 0, 6, 2, 2, 0, 2, 0, 0, 3, 2, 0, 0, 0] \cdot \mathcal{BS}$$

In particular, V has φ_{14} and hence $\varphi_{[7,4,3,1]}$ as constituents, and by Table 10 it condenses to a module of dimension 248.

The condensed module Ve of V is then computed using the InducedCondense package, i.e. we fix a vector space basis of Ve and, for a few group elements we compute the action of

$ege \in ek[G]e$ with respect to this basis. We choose a 15-cycle $g_1 \in G \setminus K$, and a 7-cycle $g_2 \in S_7 < G$, and let $\mathcal{C}'$ denote the subalgebra of $ek[G]e$ generated by $\{eg_1e, eg_2e\}$. Using the MeatAxe, we find the dimensions and multiplicities of the constituents of Ve as a $\mathcal{C}'$-module. But while $\{g_1, g_2\}$ already is a generating set for G, these results show that $\mathcal{C}'$ is too small a subalgebra to serve our purposes. By additionally condensing a few random products of g_1 and g_2, we finally end up with a condensation algebra $\mathcal{C}$, such that the MeatAxe finds the following constituents of Ve as a $\mathcal{C}$-module:

$$1a^2, 4a^2, 4b^2, 4c, 6a^2, 12a^2, 16a^2, 38a, 40a^3,$$

where as usual the constituents are denoted by their dimension and multiplicities are given as exponents.

Using the decomposition of $(\varphi_{[1^7]} \otimes \varphi_{[4,3,1]})^G$ into $\mathcal{BS}$ and the data from Table 10, we conclude that φ_{14} condenses to an irreducible $\mathcal{C}$-module of dimension 38, which hence is the restriction of an irreducible $ek[G]e$-module of the same dimension. Hence φ_{14} is an irreducible Brauer character, and we are done.

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