BRAUER TREES FOR THE SCHUR COVER OF THE SYMMETRIC GROUP

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Abstract. We determine the Brauer trees of the faithful blocks of weight 1 of the Schur covers of the symmetric group and the alternating group in odd characteristic.

1. Introduction

1.1. Let $\tilde{\mathcal{S}}_n$ be a Schur cover of the symmetric group $\mathcal{S}_n$, and $\tilde{\mathcal{A}}_n$ be the Schur cover of the alternating group $\mathcal{A}_n$. Currently, parametrizations of the faithful modular irreducible characters of these groups and descriptions of their decomposition matrices are investigated. In this paper, we deal with the particular case of faithful blocks of cyclic defect.

More precisely, we determine the Brauer trees of the faithful blocks of weight 1 of $\tilde{\mathcal{S}}_n$ and $\tilde{\mathcal{A}}_n$ in odd characteristic. Our main result is stated in Theorem 4.4. As Corollary 6.3, we obtain a set of representatives of the Morita equivalence classes of faithful blocks of cyclic defect occurring for some $\tilde{\mathcal{S}}_n$. In particular, for fixed odd p, there are $\lfloor (p+3)/4 \rfloor$ Morita equivalence classes. This contrasts to the situation for the symmetric groups $\mathcal{S}_n$, where for fixed p there is exactly one such class.

Besides its own interest for the modular representation theory of $\tilde{\mathcal{S}}_n$ and $\tilde{\mathcal{A}}_n$, the result presented here is related to the following question: Which trees occur as Brauer trees of blocks of cyclic defect of finite groups? By [5, Thm.1.1], all Brauer trees are unfoldings of Brauer trees occurring for quasi-simple groups. Indeed, using the classification of finite simple groups, much is known for this class of groups; one of the few remaining gaps, the case of $\tilde{\mathcal{A}}_n$, is settled here.

1.2. We assume the reader familiar with the ordinary representation theory of the symmetric group and its Schur cover, see, e. g., [9] and [15], as well as with the decomposition theory of finite groups, in particular the theory of blocks of cyclic defect, see, e. g., [4, Sect.VII] or [7, Sect.11]. The latter is used to determine the shape of the Brauer trees, while we use a Scopes-Kessar reduction technique to determine the labelling of the vertices of the Brauer trees.

The paper is organized as follows: In Sections 2 and 3, we collect well-known definitions and facts from the combinatorics of partitions, see, e. g., [9, Sect.10], and the representation theory of $\tilde{\mathcal{S}}_n$ and $\tilde{\mathcal{A}}_n$, see, e. g., [9, Sect.10] and [15], respectively. In Propositions 3.8 and 3.9, we consider certain table automorphisms of $\tilde{\mathcal{S}}_n$ and $\tilde{\mathcal{A}}_n$. In Section 4, we introduce a labelling of the characters in a block of weight 1, see Proposition 4.3, state the main result Theorem 4.4, and give the first part of its proof. In Section 5 we prepare the Scopes-Kessar reduction mechanism needed

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for the second part of the proof. The exposition is largely based on [12, Sect.4.4–4.10], but we obtain somewhat sharper results, which in the weight 1 case lead to a stronger reduction, which is just suitable for our purposes, see Proposition 5.5. In Section 6 we give the second part of the proof of Theorem 4.4, and draw Corollaries 6.3, 6.4 and 6.6 on the Morita equivalence classes of blocks of weight 1 and on the ordering of the vertices of the Brauer trees.

1.3. **The setting.** Let p be an odd rational prime. Let $K' := \mathbb{Q}(Z) \subset \mathbb{C}$ be the field extension of $\mathbb{Q}$ generated by $Z := \{\zeta \in \mathbb{C}; \zeta^e = 1 \text{ for some } e \in \mathbb{N}, p \not| e\}$, and let $K := K'(\zeta_p) \subset \mathbb{C}$, where $1 \neq \zeta_p \in \mathbb{C}$ such that $\zeta_p^p = 1$. An element of K is called *p-rational*, if it is contained in K'. As all the algebraic field extensions $\mathbb{Q} \subset \mathbb{Q}(\zeta)$, for $\zeta \in Z$, are unramified over the prime p, see also [7, Thm.5.7], there is a discrete valuation ring R in K with maximal ideal $\wp \lhd R$ containing p. We once and for all fix such a p-modular system (K, R, k), where $k := R/\wp$ denotes the residue class field of R.

For a finite group G, we denote the usual scalar product of class functions χ, χ' on G by $\langle \chi, \chi' \rangle_G$. For a p-block B of G we denote the number of ordinary and Brauer characters in B by $k(B)$ and $l(B)$, respectively. If B has a cyclic defect group, then the multiplicity of the exceptional vertex is denoted by $m(B)$; in this case we have $m(B) = (p-1)/l(B)$ and $k(B) = l(B) + m(B)$.

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2. Operations with partitions

2.1. **Notation.** Let $\mathcal{P}_n$ denote the set of partitions of $n \in \mathbb{N}_0$. For $\lambda \in \mathcal{P}_n$ let $[\lambda_1, \lambda_2, \ldots, \lambda_l]$ be the sequence of its parts, where $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_l > 0$ and $\sum_{i=1}^{l} \lambda_i = n$. Let $l_\lambda = l$ be its *length*, let $\sigma_\lambda := n - l_\lambda$ be its *signature* and let $\epsilon_\lambda := (-1)^{\sigma_\lambda}$ be its *sign*. Let $\mathcal{P}_n^+ := \{\lambda \in \mathcal{P}_n; \epsilon_\lambda = 1\}$ and $\mathcal{P}_n^- := \mathcal{P}_n \setminus \mathcal{P}_n^+$. Let $\mathcal{O}_n := \{\lambda \in \mathcal{P}_n; \text{all } \lambda_i \text{ odd}\} \subseteq \mathcal{P}_n^+$ and $\mathcal{D}_n := \{\lambda \in \mathcal{P}_n; \lambda_i\text{'s pairwise distinct}\}$, as well as $\mathcal{D}_n^+ := \mathcal{D}_n \cap \mathcal{P}_n^+$ and $\mathcal{D}_n^- := \mathcal{D}_n \cap \mathcal{P}_n^-$, and $\mathcal{D} := \cup_{n \in \mathbb{N}_0} \mathcal{D}_n$.

2.2. **Bar operations on the abacus.** We record $\lambda \in \mathcal{D}_n$ by beads on an abacus with runners labelled by $0, 1, \ldots, p-1$ and rows labelled by $\mathbb{N}_0$ as follows: For $1 \leq i \leq l_\lambda$ we place the i-th bead in row x_i on runner y_i, where $\lambda_i = x_i p + y_i$. For $0 \leq j \leq p-1$ let $X_\lambda^j := \{x \in \mathbb{N}_0; xp + j \text{ is a part of } \lambda\}$.

Note that position $(0,0)$ of the abacus is never occupied, and that we have a bijection between $\mathcal{D}$ and the bead configurations on the abacus with position $(0,0)$ not occupied. Furthermore, λ is uniquely defined by the sets X_λ^j for $0 \leq j \leq p-1$.

Given $\lambda \in \mathcal{D}_n$ and its corresponding bead configuration on the abacus, performing either of the following steps on the abacus is called a *p-bar removal:* **Type '+':** Move a bead from position $(x, y) \neq (1, 0)$ to position $(x-1, y)$, if the latter is not yet occupied. **Type '0':** Remove a bead from position $(1, 0)$. **Type '−':** Remove two beads from positions $(0, y)$ and $(0, p-y)$, where $1 \leq y \leq (p-1)/2$.

Conversely, performing either of the following steps on the abacus is called a *p-bar addition:* **Type '+':** Move a bead from position (x, y) to position $(x+1, y)$, if the latter is not yet occupied. **Type '0':** Put a bead on position $(1, 0)$, if it is not yet occupied. **Type '$-$':** Put two beads on positions $(0, y)$ and $(0, p-y)$, where $1 \leq y \leq (p-1)/2$, if both are not yet occupied.

Let I_λ^+ and I_λ^0 be the sets of ordinals $1 \leq i \leq l_\lambda$ such that the i-th bead can be moved by a p-bar removal of type '+' and of type '0', respectively. Let I_λ^- be the sets of ordinals $1 \leq i \leq l_\lambda$ such that there is some $j > i$ such that the i-th and the j-th bead can be moved by a p-bar removal of type '$-$'. Note that I_λ^0 contains at most one element, and that the sets I_λ^+, I_λ^0 and I_λ^- are pairwise disjoint. Let $I_\lambda := I_\lambda^+ \cup I_\lambda^0 \cup I_\lambda^-$

2.3. **Related notions.** For $i \in I_\lambda$ let the *leg length* b_i be defined as follows: Let $\tilde{\lambda}$ be obtained from λ by p-bar removal involving the i-th bead. If $i \in I_\lambda^+$, let $b_i := j - i$, if the moved bead becomes the j-th bead of $\tilde{\lambda}$. If $i \in I_\lambda^0$, let $b_i := l_\lambda - i$. If $i \in I_\lambda^-$, let $b_i := j - i - \lambda_j - 1$, where the j-th bead, $j > i$, is the other bead involved in the p-bar removal. The reason for this terminology is an interpretation of b_i as the leg length of a hook in a certain Young diagram, see [9, Sect.10].

If $\tilde{\lambda}$ is obtained from λ by p-bar removal of type '+', then we have $l_\lambda - l_{\tilde{\lambda}} = 0$, while for the types '0' and '$-$' we have $l_\lambda - l_{\tilde{\lambda}} = 1$ and $l_\lambda - l_{\tilde{\lambda}} = 2$, respectively. Hence for the signatures and signs we have $\sigma_\lambda - \sigma_{\tilde{\lambda}} = p$ and $\epsilon_\lambda = -\epsilon_{\tilde{\lambda}}$ for type '+', while for the types '0' and '$-$' we have $\sigma_\lambda - \sigma_{\tilde{\lambda}} = p - 1$ and $\epsilon_\lambda = \epsilon_{\tilde{\lambda}}$, as well as $\sigma_\lambda - \sigma_{\tilde{\lambda}} = p - 2$ and $\epsilon_\lambda = -\epsilon_{\tilde{\lambda}}$, respectively.

The partition $\overline{\lambda} \in \mathcal{D}_{n-pw_\lambda}$ obtained by successive p-bar removal until no further p-bar removal is possible, is called the *p-bar core* of λ. The number $w_\lambda \in \mathbb{N}_0$ of steps needed to reach the p-bar core is called the *p-weight* of λ. If conversely $\overline{\lambda} \in \mathcal{D}_{n-pw}$ is a p-bar core, then we let $\mathcal{D}_{\overline{\lambda},w} := \{\pi \in \mathcal{D}_n; \overline{\pi} = \overline{\lambda}\}$, where $\overline{\pi}$ denotes the p-bar core of π, as well as $\mathcal{D}^+_{\overline{\lambda},w} := \mathcal{D}_{\overline{\lambda},w} \cap \mathcal{D}^+_n$ and $\mathcal{D}^-_{\overline{\lambda},w} := \mathcal{D}_{\overline{\lambda},w} \cap \mathcal{D}^-_n$.

3. Spin characters and blocks

3.1. **The groups $\tilde{\mathcal{S}}_n$ and $\tilde{\mathcal{A}}_n$.** For $n \in \mathbb{N}_0$ let $\mathcal{S}_n$ denote the symmetric group on $\{1, \ldots, n\}$. It is finitely presented as

$$\mathcal{S}_n \cong \langle s_1, \ldots, s_{n-1} | s_i^2 = (s_i s_{i+1})^3 = (s_i s_j)^2 = 1, 1 \leq i < j \leq n-1, |i-j| \geq 2 \rangle,$$

where the isomorphism maps $(i, i+1) \mapsto s_i$ for $1 \leq i \leq n-1$. Let the *Schur cover* $\tilde{\mathcal{S}}_n$ of $\mathcal{S}_n$ be defined as the finitely presented group, see [15, Sect.I.3.],

$$\begin{aligned} \tilde{\mathcal{S}}_n := \langle z, \tilde{s}_1, \ldots, \tilde{s}_{n-1} | \; & z^2 = 1, \tilde{s}_i^2 = (\tilde{s}_i \tilde{s}_{i+1})^3 = (\tilde{s}_i \tilde{s}_j)^2 = z, \\ & 1 \leq i < j \leq n-1, |i-j| \geq 2 \rangle. \end{aligned}$$

Thus $\tilde{\mathcal{S}}_n$ is a central extension $1 \to \langle z \rangle \to \tilde{\mathcal{S}}_n \stackrel{\alpha_n}{\to} \mathcal{S}_n \to 1$ of $\mathcal{S}_n$ by the cyclic group $\langle z \rangle$ order 2, where $\alpha_n : z \mapsto 1$ and $\alpha_n : \tilde{s}_i \mapsto s_i$ for $1 \leq i \leq n-1$. For the other central extension $\hat{\mathcal{S}}_n$ of $\mathcal{S}_n$ by a cyclic group order 2, which is isoclinic to $\tilde{\mathcal{S}}_n$, see Remark 6.7.

Let $\mathcal{A}_n$ denote the alternating group on n letters. Let $\tilde{\mathcal{A}}_n := \alpha_n^{-1}(\mathcal{A}_n) \trianglelefteq \tilde{\mathcal{S}}_n$, which for $n \geq 2$ is a normal subgroup of index 2, and equals $\tilde{\mathcal{S}}_n$ for $n \leq 1$. Hence $\tilde{\mathcal{A}}_n$ is a central extension $1 \to \langle z \rangle \to \tilde{\mathcal{A}}_n \stackrel{\alpha_n}{\to} \mathcal{A}_n \to 1$ of $\mathcal{A}_n$ by the cyclic group $\langle z \rangle$ order 2.

If $m \leq n$, let the monomorphism $\beta_n^m : \mathcal{S}_m \to \mathcal{S}_n$ be induced by the natural embedding $\{1, \ldots, m\} \to \{1, \ldots, n\}$. Then there is monomorphism $\tilde{\beta}_n^m : \tilde{\mathcal{S}}_m \to \tilde{\mathcal{S}}_n : \tilde{s}_i' \mapsto \tilde{s}_i$, for $1 \leq i \leq m-1$, where the $\tilde{s}_i'$ are the generators in the defining presentation for $\tilde{\mathcal{S}}_m$. Hence we have $\alpha_n \circ \tilde{\beta}_n^m = \beta_n^m \circ \alpha_m$. In the sequel, we will identify $\tilde{\mathcal{S}}_m$ and $\tilde{\beta}_n^m(\tilde{\mathcal{S}}_m) \leq \tilde{\mathcal{S}}_n$.

3.2. **Conjugacy classes.** Given a conjugacy class $C_\pi \subseteq \mathcal{S}_n$, which consists of elements of cycle type $\pi \in \mathcal{P}_n$, then $\alpha_n^{-1}(C_\pi)$ consists of two conjugacy classes C_π^1 and $C_\pi^2 = zC_\pi^1$ of $\tilde{\mathcal{S}}_n$, if and only if $\pi \in \mathcal{O}_n \dot\cup \mathcal{D}_n^-$. In this case, let $\pi = [\pi_1, \ldots, \pi_l]$ and $g_\pi := \prod_{i=1}^l (\pi_1 + \cdots + \pi_{i-1} + 1, \ldots, \pi_1 + \cdots + \pi_i) \in C_\pi \subseteq \mathcal{S}_n$. Hence we have $g_\pi = \prod_{i=1}^l (s_{\pi_1+\cdots+\pi_i-1} \cdot s_{\pi_1+\cdots+\pi_i-2} \cdot \cdots \cdot s_{\pi_1+\cdots+\pi_{i-1}+1})$. Following [15, Sect.III.11], we define $C_\pi^1 \subseteq \tilde{\mathcal{S}}_n$ to be the conjugacy class containing

$$\tilde{g}_\pi := \prod_{i=1}^{l} (\tilde{s}_{\pi_1+\cdots+\pi_i-1} \cdot \tilde{s}_{\pi_1+\cdots+\pi_i-2} \cdot \cdots \cdot \tilde{s}_{\pi_1+\cdots+\pi_{i-1}+1}) \in \tilde{\mathcal{S}}_n.$$

Given $\pi \in \mathcal{P}_n^+$, the conjugacy class C_π of $\mathcal{S}_n$ consists of two conjugacy classes of $\mathcal{A}_n$ if and only if $\pi \in \mathcal{O}_n \cap \mathcal{D}_n^+$. Given a conjugacy class $\tilde{C}_\pi$ of $\mathcal{A}_n$ consisting of elements of cycle type $\pi \in \mathcal{P}_n^+$, then $\alpha_n^{-1}(\tilde{C}_\pi)$ consists of two conjugacy classes of $\tilde{\mathcal{A}}_n$ if and only if $\pi \in \mathcal{O}_n \cup \mathcal{D}_n^+$.

3.3. **Spin characters.** Let $\mathrm{Irr}(\tilde{\mathcal{S}}_n)$ denote the set of irreducible ordinary characters of $\tilde{\mathcal{S}}_n$, and $\mathrm{Irr}^-(\tilde{\mathcal{S}}_n) := \{\chi \in \mathrm{Irr}(\tilde{\mathcal{S}}_n); \chi(z) = -\chi(1)\}$ denote the set of *spin characters.* Hence $\mathrm{Irr}^+(\tilde{\mathcal{S}}_n) := \mathrm{Irr}(\tilde{\mathcal{S}}_n) \setminus \mathrm{Irr}^-(\tilde{\mathcal{S}}_n)$ can be identified with $\mathrm{Irr}(\mathcal{S}_n)$. Let $\epsilon \in \mathrm{Irr}(\mathcal{S}_n)$ be the sign character, i. e., $\epsilon(\pi) = \epsilon_\pi$, and for $\chi \in \mathrm{Irr}(\tilde{\mathcal{S}}_n)$ let $\chi' := \chi \otimes \epsilon$. The spin characters $\mathrm{Irr}^-(\tilde{\mathcal{S}}_n)$ are parametrized by $\mathcal{D}_n$, see [15, Sect.X.41], where for $\lambda \in \mathcal{D}_n^+$ there is a *self-associate* spin character $\langle\lambda\rangle = \langle\lambda\rangle'$, while for $\lambda \in \mathcal{D}_n^-$ there is a pair of distinct *associate* spin characters $\langle\lambda\rangle$ and $\langle\lambda\rangle'$. If $\lambda \in \mathcal{D}_n^+$, let $\langle\lambda\rangle^* := \langle\lambda\rangle = \langle\lambda\rangle'$, while if $\lambda \in \mathcal{D}_n^-$, let $\langle\lambda\rangle^* := \langle\lambda\rangle + \langle\lambda\rangle'$.

We adopt the usual abuse of notation to write $\chi(\pi)$ for the value of $\chi \in \mathrm{Irr}(\tilde{\mathcal{S}}_n)$ on the elements contained in $\alpha_n^{-1}(C_\pi)$, for $\pi \in \mathcal{P}_n$, where for $\chi \in \mathrm{Irr}^-(\tilde{\mathcal{S}}_n)$ and $\pi \in \mathcal{O}_n \dot\cup \mathcal{D}_n^-$ we let $\chi(\pi) := \chi(\tilde{g}_\pi)$. We have $\langle\lambda\rangle(\pi) \neq 0$ possibly only for $\pi \in \mathcal{O}_n$ or for $\lambda = \pi \in \mathcal{D}_n^-$. If $\lambda = \pi \in \mathcal{D}_n^-$, then we define $\langle\lambda\rangle$ to be the character fulfilling $\langle\lambda\rangle(\lambda) = \sqrt{-1}^{(\sigma_\lambda+1)/2} \cdot \sqrt{(\prod_{i=1}^{l_\lambda} \lambda_i)/2}$, while of course $\langle\lambda\rangle'(\lambda) = -\langle\lambda\rangle(\lambda)$.

3.4. **Morris recursion formula.** If $\pi \in \mathcal{O}_n$, then $\langle\lambda\rangle(\pi) \in \mathbb{Z}$ can be computed by the following recursion formula, see [13, Thm.2]: Let $e \in \mathbb{N}$ be a part of π, hence e is odd. Let $I_\lambda^e \subseteq \{1, \ldots, l_\lambda\}$ denote the I-set for λ with respect to e-bar removal, see Section 2.2. For $i \in I_\lambda^e = I_\lambda^{e,+} \dot\cup I_\lambda^{e,0} \dot\cup I_\lambda^{e,-}$ let $\lambda^i \in \mathcal{D}_{n-e}$ denote the partition obtained by e-bar removal involving the i-th part of λ, and let b_i the corresponding leg length, see Section 2.3. Let $a_i \in \{0, 1\}$ be defined as follows: If $i \in I_\lambda^{e,+} \cup I_\lambda^{e,-}$ and $\epsilon_\lambda = 1$, then let $a_i := 1$, otherwise let $a_i = 0$. Then we have $\langle\lambda\rangle(\pi) = \sum_{i \in I_\lambda^e} (-1)^{b_i} 2^{a_i} \langle\lambda^i\rangle(\pi \setminus e)$, where $\pi \setminus e \in \mathcal{D}_{n-e}$ is the partition obtained from π by deleting its part e.

3.5. **Branching rule.** Let $\lambda \in \mathcal{D}_n$. Let $I_\lambda^1 \subseteq \{1, \ldots, l_\lambda\}$ denote the I-set for λ with respect to 1-bar removal, see Section 2.2. Note that only types '+' and '0' are relevant here, and that we have $l_\lambda \in I_\lambda^1$. For $i \in I_\lambda^1$ let $\lambda^i \in \mathcal{D}_{n-1}$ denote the partition obtained by 1-bar removal involving the i-th part of λ. Note that 1-bar removal of type '0' is recorded on an abacus with p-runners by removing a bead from position $(0, 1)$, while type '+' is recorded by moving the involved bead from position $(x, y) \neq (0, 1)$ to $(x, y - 1)$ if $y > 0$, and to $(x - 1, p - 1)$ if $y = 0$.

If $\lambda_{l_\lambda} \neq 1$, by [9, Thm.10.2] we have $\langle\lambda\rangle|_{\tilde{\mathcal{S}}_{n-1}} = \sum_{i\in I^1_\lambda}\langle\lambda^i\rangle^*$, while if $\lambda_{l_\lambda} = 1$, we have $\langle\lambda\rangle|_{\tilde{\mathcal{S}}_{n-1}} = \langle\lambda^{l_\lambda}\rangle + \sum_{i\in I^1_\lambda, i\neq l_\lambda}\langle\lambda^i\rangle^*$.

For $\lambda \in \mathcal{D}_n$ and $\tilde{\lambda} \in \mathcal{D}_m$, where $n \geq m$, let $\mathcal{R}^\lambda_{\tilde{\lambda}}$ denote the set of all sequences $[\lambda = \lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(n-m)} = \tilde{\lambda}]$, such that $\lambda^{(i)} \in \mathcal{D}_{n-i}$ is obtained from $\lambda^{(i-1)} \in \mathcal{D}_{n-i+1}$ by 1-bar removal, for $1 \leq i \leq n-m$. From the decomposition of $\langle\lambda\rangle|_{\tilde{\mathcal{S}}_{n-1}}$ we conclude by induction that for $m \leq n$ we have $\langle\langle\lambda\rangle|_{\tilde{\mathcal{S}}_m}, \langle\tilde{\lambda}\rangle\rangle_{\tilde{\mathcal{S}}_m} \neq 0$ if and only if $\mathcal{R}^\lambda_{\tilde{\lambda}} \neq \emptyset$. For $m \leq n-2$, from the character values given in Section 3.3, we conclude that $\langle\lambda\rangle'|_{\tilde{\mathcal{S}}_m} = (\langle\lambda\rangle \otimes \epsilon)|_{\tilde{\mathcal{S}}_m} = \langle\lambda\rangle|_{\tilde{\mathcal{S}}_m} \otimes \epsilon = \langle\lambda\rangle|_{\tilde{\mathcal{S}}_m}$.

3.6. **Spin characters for $\tilde{\mathcal{A}}_n$.** Analogously to the definitions in Section 3.3, we define the sets $\mathrm{Irr}(\tilde{\mathcal{A}}_n) = \mathrm{Irr}^+(\tilde{\mathcal{A}}_n) \,\dot\cup\, \mathrm{Irr}^-(\tilde{\mathcal{A}}_n)$, where $\mathrm{Irr}^+(\tilde{\mathcal{A}}_n)$ can be identified with $\mathrm{Irr}(\mathcal{A}_n)$. The characters in $\mathrm{Irr}^-(\tilde{\mathcal{A}}_n)$ are by Clifford theory described as follows: If $\lambda \in \mathcal{D}^-_n$, then the restriction $\langle\lambda\rangle|_{\tilde{\mathcal{A}}_n}$ of $\langle\lambda\rangle$ to $\tilde{\mathcal{A}}_n$ is irreducuible, and we have $\langle\lambda\rangle^{(\prime)} := \langle\lambda\rangle|_{\tilde{\mathcal{A}}_n} = \langle\lambda\rangle'|_{\tilde{\mathcal{A}}_n}$. If $\lambda \in \mathcal{D}^+_n$, then we have $\langle\lambda\rangle|_{\tilde{\mathcal{A}}_n} \cong \langle\lambda\rangle_1 + \langle\lambda\rangle_2$, where the latter are a pair of distinct irreducible characters which are conjugate under the outer automorphism of $\tilde{\mathcal{A}}_n$ induced by $\tilde{\mathcal{S}}_n$. In this case, for $\pi \in \mathcal{P}^+_n$, we have $\langle\lambda\rangle_1(\pi) - \langle\lambda\rangle_2(\pi) \neq 0$, if and only if $\lambda = \pi$, and we define $\langle\lambda\rangle_1$ and $\langle\lambda\rangle_2$ to fulfill $\langle\lambda\rangle_1(\lambda) - \langle\lambda\rangle_2(\lambda) = \sqrt{-1}^{\sigma_\lambda/2} \cdot \sqrt{\prod_{i=1}^{l_\lambda}\lambda_i}$.

3.7. **Table automorphisms.** Let G be a finite group and $\mathcal{C}l(G)$ the set of its conjugacy classes. For $d \in \mathbb{N}$, the *d-th powermap* of G is defined as $p_d : \mathcal{C}l(G) \to \mathcal{C}l(G) : g^G \mapsto (g^d)^G$. A *table automorphism* τ of G is a bijection from $\mathcal{C}l(G)$ to itself which commutes with all the powermaps of G and leaves the set $\mathrm{Irr}(G)$ invariant, where τ acts on $\mathrm{Irr}(G)$ via $\chi \mapsto \chi \circ \tau$.

3.8. **Proposition.** Let $\lambda \in \mathcal{D}^-_n$ and $\alpha_n^{-1}(C_\lambda)$ be the disjoint union of the conjugacy classes C_1 and $C_2 = zC_1$ of $\tilde{\mathcal{S}}_n$. Let $\tau : \mathcal{C}l(\tilde{\mathcal{S}}_n) \to \mathcal{C}l(\tilde{\mathcal{S}}_n)$ interchange C_1 and C_2 and leave all the other conjugacy classes of $\tilde{\mathcal{S}}_n$ fixed. Then τ is a table automorphism of $\tilde{\mathcal{S}}_n$. It acts on $\mathrm{Irr}(\tilde{\mathcal{S}}_n)$ by interchanging $\langle\lambda\rangle$ and $\langle\lambda\rangle'$ and leaving all the other characters in $\mathrm{Irr}(\tilde{\mathcal{S}}_n)$ fixed.

Proof. Let $d \in \mathbb{N}$ and $C, \tilde{C} \in \mathcal{C}l(\tilde{\mathcal{S}}_n)$ such that $p_d : C \mapsto \tilde{C}$. We may assume that C or $\tilde{C}$ are not fixed by τ. If $\tilde{C} = C_1$, say, then by the assumption on λ we have d odd and $C = C_1$ or $C = C_2$. As $C_2 = zC_1$, we conclude that p_d either fixes both of C_1 and C_2 or exchanges them, hence in any case commutes with τ. If $\tilde{C}$ is fixed by τ and $C = C_1$, say, then d divides one of the parts of λ. If d is even, then we have $p_d(C_2) = p_d(zC_1) = p_d(C_1)$. If d is odd, then $p_d(C_\lambda)$ is parametrized by a partition in $\mathcal{P}^-_n \setminus \mathcal{D}^-_n$, hence $\alpha_n^{-1}(p_d(C_\lambda))$ consists of a single conjugacy class of $\tilde{\mathcal{S}}_n$, thus again we have $p_d(C_1) = p_d(C_2)$. As τ induces the identity on $\mathrm{Irr}(\mathcal{S}_n)$, the second assertion follows using the spin character values in Section 3.3. ♯

3.9. **Proposition.** Let $\lambda \in \mathcal{D}^+_n$.
a) Let $\lambda \notin \mathcal{O}_n$ and $\alpha_n^{-1}(C_\lambda)$ be the disjoint union of the conjugacy classes C_1 and $C_2 = zC_1$. Let $\tau : \mathcal{C}l(\tilde{\mathcal{A}}_n) \to \mathcal{C}l(\tilde{\mathcal{A}}_n)$ interchange C_1 and C_2 and leave all the other conjugacy classes of $\tilde{\mathcal{A}}_n$ fixed. Then τ is a table automorphism of $\tilde{\mathcal{A}}_n$. It acts on $\mathrm{Irr}(\tilde{\mathcal{A}}_n)$ by interchanging $\langle\lambda\rangle_1$ and $\langle\lambda\rangle_2$ and leaving all the other characters in $\mathrm{Irr}(\tilde{\mathcal{A}}_n)$ fixed.
b) Let $\lambda \in \mathcal{O}_n$ and $\alpha_n^{-1}(C_\lambda)$ be the disjoint union of the conjugacy classes C_1,

$C_2 = zC_1$, C_3 and $C_4 = zC_3$ of $\tilde{\mathcal{A}}_n$, where C_1 and C_3, say, consist of elements of odd order. Let $\tau : \mathcal{C}l(\tilde{\mathcal{A}}_n) \to \mathcal{C}l(\tilde{\mathcal{A}}_n)$ interchange C_1 and C_3 as well as C_2 and C_4, and leave all the other conjugacy classes of $\tilde{\mathcal{A}}_n$ fixed. Then τ is a table automorphism of $\tilde{\mathcal{A}}_n$. It acts on $\mathrm{Irr}^-(\tilde{\mathcal{A}}_n)$ by interchanging $\langle\lambda\rangle_1$ and $\langle\lambda\rangle_2$ and leaving all the other characters in $\mathrm{Irr}^-(\tilde{\mathcal{A}}_n)$ fixed.

Proof. a) is proved similarly to Proposition 3.8.
b) Let $d \in \mathbb{N}$ and $C, \tilde{C} \in \mathcal{C}l(\tilde{\mathcal{S}}_n)$ such that $p_d : C \mapsto \tilde{C}$. We may assume that C or $\tilde{C}$ are not fixed by τ. If $\tilde{C} = C_2$, say, then necessarily d is odd and C_1 is in the image of p_d as well. Hence we may assume $\tilde{C} = C_1$. Then d is coprime to all the parts of λ. If d is odd, then p_d either interchanges C_1 and C_3 as well as C_2 and C_4, or leaves all of C_1, C_2, C_3, C_4 fixed. If d is even, then either $p_d(C_1) = p_d(C_2) = C_3$ and $p_d(C_3) = p_d(C_4) = C_1$, or $p_d(C_1) = p_d(C_2) = C_1$ and $p_d(C_3) = p_d(C_4) = C_3$. Hence in any case p_d commutes with τ.
If $\tilde{C}$ is fixed by τ and $C = C_1$, say, then d divides one of the parts of λ. Hence $p_d(C_\lambda)$ is parametrized by a partition in $\mathcal{O}_n \setminus \mathcal{D}_n^+$, thus $\alpha_n^{-1}(p_d(C_\lambda))$ consists of two conjugacy classes of $\tilde{\mathcal{A}}_n$, exactly one of which consists of elements of odd order. Hence again p_d commutes with τ.
As τ interchanges the conjugacy classes $\alpha_n(C_1)$ and $\alpha_n(C_3)$ of $\mathcal{A}_n$, it follows from [11, Thm.2.5.13] that τ induces a table automorphism of $\mathcal{A}_n$. The second assertion now follows from the description of the character values in Section 3.6. ♯

3.10. **Spin blocks.** The distribution of the spin characters $\mathrm{Irr}^-(\tilde{\mathcal{S}}_n)$ into p-blocks is described as follows: Let $\lambda \in \mathcal{D}_n$. If λ is a p-bar core, then by [9, Prop.10.6] $\langle\lambda\rangle$ and $\langle\lambda\rangle'$ are of p-defect 0, and hence form p-blocks of their own. If λ is not a p-bar core, then by [10], $\langle\lambda\rangle$ is of positive p-defect, $\langle\lambda\rangle$ and $\langle\lambda\rangle'$ belong to the same p-block, and for $\tilde{\lambda} \in \mathcal{D}_n$ the characters $\langle\lambda\rangle$ and $\langle\tilde{\lambda}\rangle$ are in the same p-block if and only if λ and $\tilde{\lambda}$ have the same p-bar core. In particular, it follows that all p-blocks of $\mathrm{Irr}^-(\tilde{\mathcal{S}}_n)$ of positive defect are invariant under complex conjugation $\chi \mapsto \overline{\chi}$ and under tensoring with the sign character $\chi \mapsto \chi \otimes \epsilon = \chi'$, for $\chi \in \mathrm{Irr}(\tilde{\mathcal{S}}_n)$.

For $w_\lambda > 0$, let $\overline{\lambda} \in \mathcal{D}_{n-pw_\lambda}$ be the p-bar core of λ. Hence all characters in the p-block $B_{\overline{\lambda},w_\lambda}$ of $\mathrm{Irr}^-(\tilde{\mathcal{S}}_n)$ containing $\langle\lambda\rangle$ have the same p-weight w_λ, which is also called the *weight* of $B_{\overline{\lambda},w_\lambda}$. Let $\sigma_{B_{\overline{\lambda},w_\lambda}} := \sigma_{\overline{\lambda}}$ and $\epsilon_{B_{\overline{\lambda},w_\lambda}} := \epsilon_{\overline{\lambda}}$ be the *signature* and the *sign* of $B_{\overline{\lambda},w_\lambda}$, respectively.

By [3, Thm.3.2.A] the p-Sylow subgroups of $\tilde{\mathcal{S}}_{pw_\lambda} \leq \tilde{\mathcal{S}}_n$, are defect groups of $B_{\overline{\lambda},w_\lambda}$. Hence the p-blocks of $\mathrm{Irr}^-(\tilde{\mathcal{S}}_n)$ of cyclic defect are precisely the p-blocks of weight 1, which in turn are of defect 1. These blocks have the groups $\langle g_{[p,1^{n-p}]}\rangle \leq \tilde{\mathcal{S}}_n$, where $g_{[p,1^{n-p}]}$ is as defined in Section 3.2, as defect groups.

3.11. **Spin blocks for $\tilde{\mathcal{A}}_n$.** By Section 2.3, p-bar addition of type '+' and '−' changes the sign of the partitions involved, while for type '0' the sign is unchanged. From that and the description of the characters in p-blocks $B_{\overline{\lambda},w}$ of $\mathrm{Irr}^-(\tilde{\mathcal{S}}_n)$ of positive weight w, it follows that $B_{\overline{\lambda},w}$ contains a character $\langle\lambda\rangle$ where $\lambda \in \mathcal{D}_n^-$.

As $\langle\lambda\rangle|_{\tilde{\mathcal{A}}_n}$ is irreducible, by [4, La.IV.4.10] there is exactly one p-block $\tilde{B}_{\overline{\lambda},w}$ of $\mathrm{Irr}^-(\tilde{\mathcal{A}}_n)$ which is covered by $B_{\overline{\lambda},w}$. Conversely, $B_{\overline{\lambda},w}$ is the only p-block of $\mathrm{Irr}^-(\tilde{\mathcal{S}}_n)$ covering $\tilde{B}_{\overline{\lambda},w}$. By [1, Thm.IV.15.1] this gives a bijection between the p-blocks of

$\mathrm{Irr}^-(\tilde{\mathcal{A}}_n)$ of positive weight and the p-blocks of $\mathrm{Irr}^-(\tilde{\mathcal{S}}_n)$ of positive weight, which preserves block defect groups.

4. Blocks of weight 1

4.1. **The s-invariant.** Let $\overline{\lambda} \in \mathcal{D}_{n-p}$ be a p-bar core. Thus to find the characters in the weight 1 block $B_{\overline{\lambda},1}$ of $\mathrm{Irr}^-(\tilde{\mathcal{S}}_n)$ we have to find $\mathcal{D}_{\overline{\lambda},1}$, hence we have to apply p-bar addition steps to the bead configuration for $\overline{\lambda}$.
Let the *s-invariant* $s_{\overline{\lambda}} \geq 0$ be the number of occupied positions in the 0-th row of the abacus, in the bead configuration for $\overline{\lambda}$. As for $1 \leq y \leq (p-1)/2$ at most one of the y-th and $(p-y)$-th runners might possibly be occupied, we have $s_{\overline{\lambda}} \leq (p-1)/2$.

4.2. **Partitions.** For $s := s_{\overline{\lambda}}$ let $\{y_1, \ldots, y_s\} \subseteq \{1, \ldots, p-1\}$ be the set of runners occupied by at least one bead, and for $1 \leq i \leq s$ let $x_i \in \mathbb{N}_0$ be the largest element of $X^{y_i}_{\overline{\lambda}}$. We assume the ordering chosen such that $x_1 p + y_1 > x_2 p + y_2 > \ldots > x_s p + y_s$.
Let $\overline{\lambda}^i_+ \in \mathcal{D}_{\overline{\lambda},1}$ denote the partition obtained by p-bar addition of type '+' involving the bead on position (x_i, y_i). We have $I^0_{\overline{\lambda}^i_+} = \emptyset = I^-_{\overline{\lambda}^i_+}$, but $I^+_{\overline{\lambda}^i_+}$ contains one element. The corresponding leg length, see Section 2.3, is easily seen to be $b^i_+ := i-1$.
Let $\overline{\lambda}^0 \in \mathcal{D}_{\overline{\lambda},1}$ denote the partition obtained by p-bar addition of type '0', i. e., $\overline{\lambda}^0$ has the parts of $\overline{\lambda}$ as its parts, and an additional part equal to p. Note that hence $\overline{\lambda}^0$ is the only partition in $\mathcal{D}_{\overline{\lambda},1}$ having a part divisible by p. We have $I^+_{\overline{\lambda}^0} = \emptyset = I^-_{\overline{\lambda}^0}$, but $I^0_{\overline{\lambda}^0}$ contains one element. The corresponding leg length is easily seen to be $b^0 := s$.
Let $\{\tilde{y}_1, \ldots, \tilde{y}_t\} := \{1, \ldots, (p-1)/2\} \setminus \{y_1, (p-y_1), \ldots, y_s, (p-y_s)\}$, where $t := (p-1)/2 - s_{\overline{\lambda}}$ and where we assume the ordering chosen such that $\tilde{y}_1 > \ldots > \tilde{y}_t$.
For $1 \leq i \leq t$ let $\overline{\lambda}^i_- \in \mathcal{D}_{\overline{\lambda},1}$ denote the partition obtained by p-bar addition of type '$-$' involving runners $\tilde{y}_i$ and $(p - \tilde{y}_i)$. We have $I^+_{\overline{\lambda}^i_-} = \emptyset = I^0_{\overline{\lambda}^i_-}$, but $I^-_{\overline{\lambda}^i_-}$ contains one element. The corresponding leg length is easily seen to be $b^i_- := (p-1)/2 - i + 1$.
Hence we have proved:

4.3. **Proposition.** Let $\overline{\lambda} \in \mathcal{D}_{n-p}$ be a p-bar core, $s := s_{\overline{\lambda}}$ and $t := (p-1)/2 - s_{\overline{\lambda}}$. Then we have $\mathcal{D}_{\overline{\lambda},1} = \{\overline{\lambda}^1_+, \ldots, \overline{\lambda}^s_+, \overline{\lambda}^0, \overline{\lambda}^t_-, \ldots, \overline{\lambda}^1_-\}$. For all $\lambda \in \mathcal{D}_{\overline{\lambda},1}$ the set I_λ contains exactly one element, and the corresponding leg lengths are pairwise different and equal to $0, \ldots, s-1, s, s+1, \ldots, (p-1)/2$, respectively. ♯

We can now state our main result.

4.4. **Theorem.** Let $\overline{\lambda} \in \mathcal{D}_{n-p}$ be a p-bar core with s-invariant $s_{\overline{\lambda}}$.
a) Let $B_{\overline{\lambda},1}$ denote the p-block of $\mathrm{Irr}^-(\tilde{\mathcal{S}}_n)$ having weight 1 and p-bar core $\overline{\lambda}$, see Section 3.10. Depending on the cases $\epsilon_{B_{\overline{\lambda},1}} = 1$, where in turn $\sigma_{B_{\overline{\lambda},1}} - p \equiv 1 \pmod 4$ or $\sigma_{B_{\overline{\lambda},1}} - p \equiv -1 \pmod 4$, or $\epsilon_{B_{\overline{\lambda},1}} = -1$ the Brauer tree of $B_{\overline{\lambda},1}$ is as depicted in Table 1, 2 and 3, respectively, up to the table automorphisms in Proposition 3.8.
b) Let $\tilde{B}_{\overline{\lambda},1}$ denote the p-block of $\mathrm{Irr}^-(\tilde{\mathcal{A}}_n)$ being covered by $B_{\overline{\lambda},1}$, see Section 3.11. Depending on the cases $\epsilon_{\tilde{B}_{\overline{\lambda},1}} = 1$ or $\epsilon_{\tilde{B}_{\overline{\lambda},1}} = -1$, where in turn $\sigma_{\tilde{B}_{\overline{\lambda},1}} + p \equiv 0 \pmod 4$ or $\sigma_{\tilde{B}_{\overline{\lambda},1}} + p \equiv 2 \pmod 4$ the Brauer tree of $\tilde{B}_{\overline{\lambda},1}$ is as depicted in Table 4, 5 and 6, respectively, up to the table automorphisms in Proposition 3.9.

TABLE 1. Brauer tree of $B_{\overline{\lambda},1}$, $\epsilon_{B_{\overline{\lambda},1}} = 1$, $\sigma_{B_{\overline{\lambda},1}} - p \equiv 1 \pmod 4$, $s := s_{\overline{\lambda}}$ and $t = (p-1)/2 - s_{\overline{\lambda}}$.

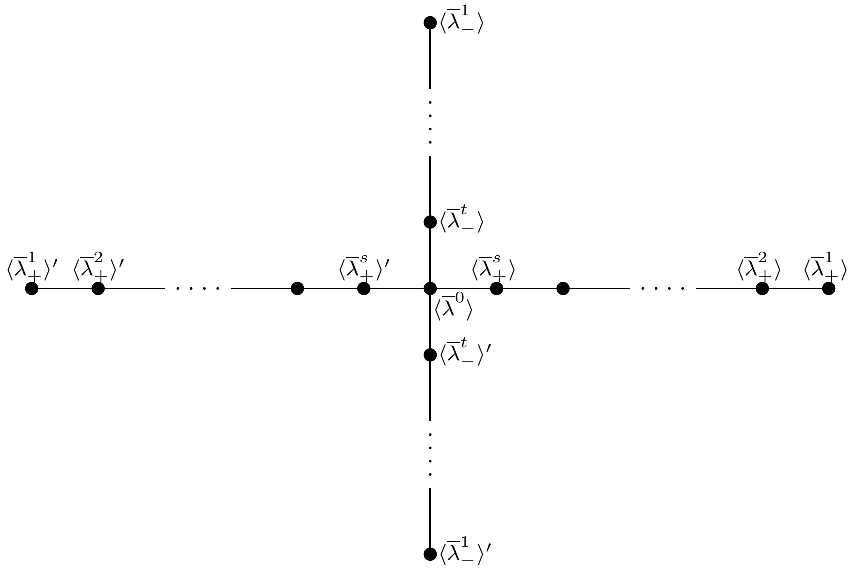


TABLE 2. Brauer tree of $B_{\overline{\lambda},1}$, $\epsilon_{B_{\overline{\lambda},1}} = 1$, $\sigma_{B_{\overline{\lambda},1}} - p \equiv -1 \pmod 4$, $s := s_{\overline{\lambda}}$ and $t = (p-1)/2 - s_{\overline{\lambda}}$.

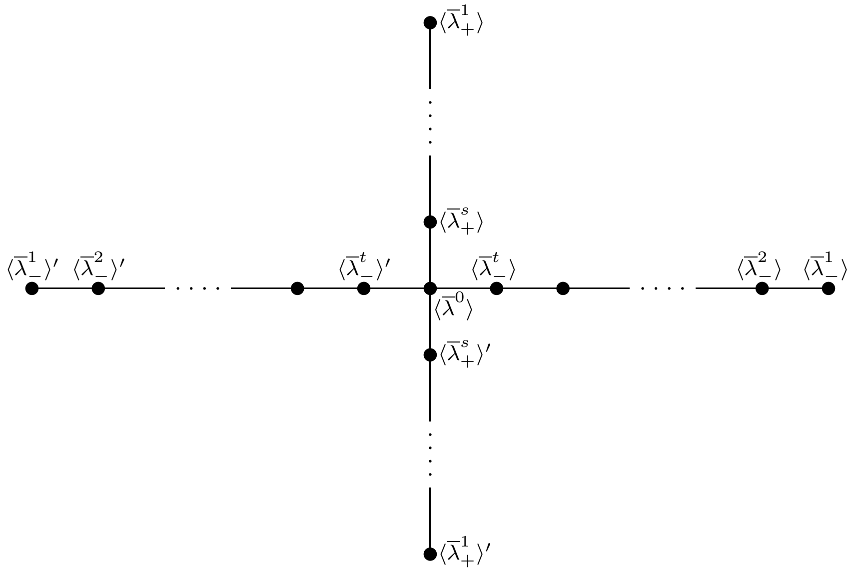


TABLE 3. Brauer tree of $B_{\overline{\lambda},1}$, $\epsilon_{B_{\overline{\lambda},1}} = -1$, $s := s_{\overline{\lambda}}$ and $t = (p-1)/2 - s_{\overline{\lambda}}$.

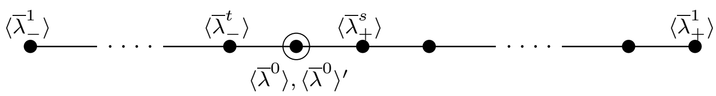


TABLE 4. Brauer tree of $\tilde{B}_{\overline{\lambda},1}$, $\epsilon_{\tilde{B}_{\overline{\lambda},1}} = 1$, $s := s_{\overline{\lambda}}$ and $t = (p-1)/2 - s_{\overline{\lambda}}$.

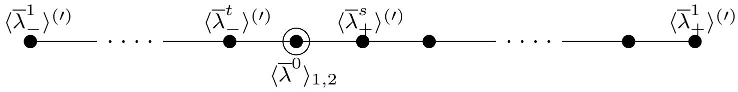


TABLE 5. Brauer tree of $\tilde{B}_{\overline{\lambda},1}$, $\epsilon_{\tilde{B}_{\overline{\lambda},1}} = -1$, $\sigma_{\tilde{B}_{\overline{\lambda},1}} + p \equiv 0 \pmod 4$, $s := s_{\overline{\lambda}}$ and $t = (p-1)/2 - s_{\overline{\lambda}}$.

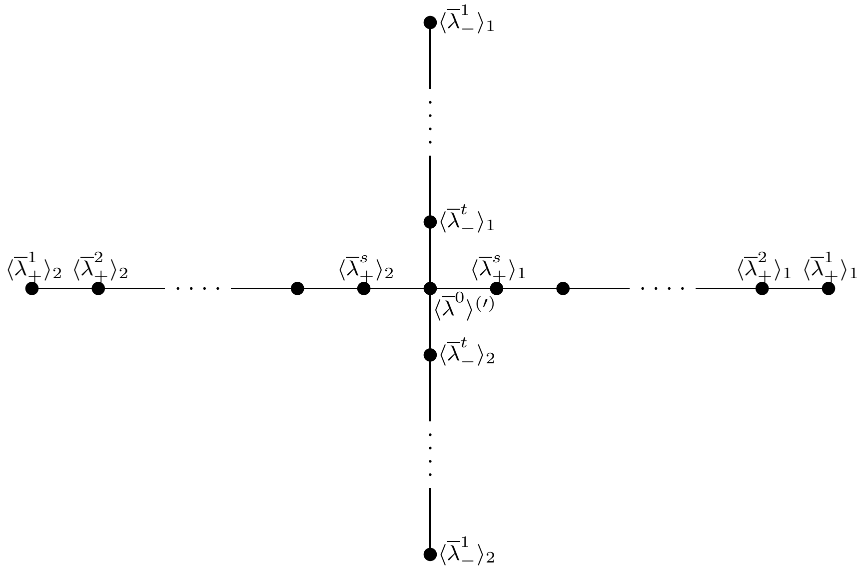
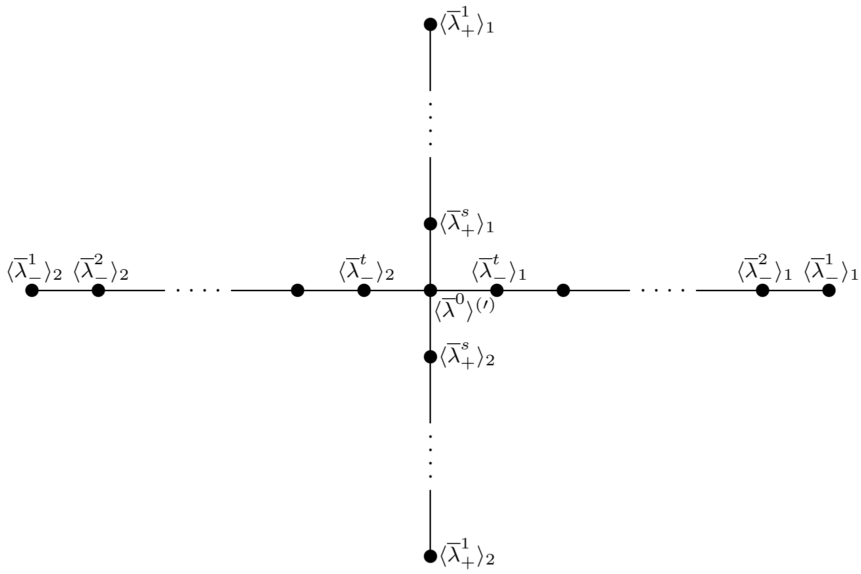


TABLE 6. Brauer tree of $\tilde{B}_{\overline{\lambda},1}$, $\epsilon_{\tilde{B}_{\overline{\lambda},1}} = -1$, $\sigma_{\tilde{B}_{\overline{\lambda},1}} + p \equiv 2 \pmod 4$, $s := s_{\overline{\lambda}}$ and $t = (p-1)/2 - s_{\overline{\lambda}}$.



4.5. **Proof of Theorem 4.4, part I.** In part I we determine the shape of the Brauer trees and obtain preliminary information on the labelling of its vertices. In part II, see Section 6.2, we completely determine the labelling of the vertices.

a) We first consider $B_{\overline{\lambda},1}$ and distinguish the cases $\epsilon_{B_{\overline{\lambda},1}} = 1$ and $\epsilon_{B_{\overline{\lambda},1}} = -1$.

4.6. **Case** $\epsilon_{B_{\overline{\lambda},1}} = 1$**.** By Section 2.3 we have $\epsilon_{\overline{\lambda}^i_+} = -1 = \epsilon_{\overline{\lambda}^j_-}$ for all $1 \leq i \leq s := s_{\overline{\lambda}}$ and $1 \leq j \leq t := (p-1)/2 - s_{\overline{\lambda}}$, and $\epsilon_{\overline{\lambda}^0} = 1$. Hence we have $k(B_{\overline{\lambda},1}) = p$, where $\mathcal{D}^+_{\overline{\lambda},1} = \{\overline{\lambda}^0\}$. By Section 3.3, for $\lambda \in \mathcal{D}^-_{\overline{\lambda},1}$ and $\pi \in \mathcal{P}^-_n$, we have $\langle\lambda\rangle(\pi) = 0$, except for $\langle\lambda\rangle(\lambda) = -\langle\lambda\rangle'(\lambda) \neq 0$. As $\lambda \in \mathcal{D}^-_{\overline{\lambda},1}$ has no part divisible by p, we conclude that $l(B_{\overline{\lambda},1}) = p-1$, hence $m(B_{\overline{\lambda},1}) = 1$. We now have a further case distinction.

4.7. **Subcase** $\sigma_{B_{\overline{\lambda},1}} - p \equiv 1 \pmod 4$**.** By Sections 2.3 and 3.3 we find that $\langle\overline{\lambda}^0\rangle$ as well as $\langle\overline{\lambda}^i_+\rangle$ and $\langle\overline{\lambda}^i_+\rangle'$ are real-valued for all $1 \leq i \leq s$, while $\langle\overline{\lambda}^i_-\rangle$ and $\langle\overline{\lambda}^i_-\rangle'$ are pairs of complex conjugate characters for all $1 \leq i \leq t$. This determines the vertices on the real stem of the Brauer tree of $B_{\overline{\lambda},1}$.

Let $\kappa : \chi \mapsto \overline{\chi}' = \overline{\chi'}$ denote the concatenation of complex conjugation and tensoring with the sign character, for $\chi \in \mathrm{Irr}(\tilde{\mathcal{S}}_n)$. Again a consideration of character values shows that $\langle\overline{\lambda}^0\rangle$ as well as $\langle\overline{\lambda}^i_-\rangle$ and $\langle\overline{\lambda}^i_-\rangle'$ for all $1 \leq i \leq t$ are invariant under κ, while $\langle\overline{\lambda}^i_+\rangle$ and $\langle\overline{\lambda}^i_+\rangle'$ for all $1 \leq i \leq s$ are κ-orbits of length 2. By the arguments in the proof of [7, Thm.11.15,11.16] we conclude that the set of κ-fixed points also induces a subgraph of the Brauer tree which is a connected open polygon.

Hence the Brauer tree is a 4-fold star having $\langle\overline{\lambda}^0\rangle$ in its center; see Table 1. The real stem is depicted as horizontal line, whose vertices are labelled by $\langle\overline{\lambda}^0\rangle$ and the $\langle\overline{\lambda}^i_+\rangle$, $\langle\overline{\lambda}^i_+\rangle'$ for $1 \leq i \leq s$, while the vertices on the other branches are labelled by the $\langle\overline{\lambda}^i_-\rangle$, $\langle\overline{\lambda}^i_-\rangle'$ for $1 \leq i \leq t$. As tensoring with the sign character induces a graph automorphism of the Brauer tree of order 2, which leaves both the set of real characters as well as the set of κ-fixed points invariant, we conclude that $\langle\lambda\rangle$ and $\langle\lambda\rangle'$ have equal distance from the center $\langle\overline{\lambda}^0\rangle$ for all $\lambda \in \mathcal{D}_{\overline{\lambda},1}$. By Proposition 3.8 there are enough table automorphisms of $\tilde{\mathcal{S}}_n$ which allow to interchange the pairs of associate characters in $B_{\overline{\lambda},1}$ independently. In Table 1 we depict the Brauer tree where the $\langle\lambda\rangle$'s are concentrated on the right and upper branches of the Brauer tree, while the $\langle\lambda\rangle'$'s are concentrated on the left and lower branches. It remains to show that the ordering of the $\langle\overline{\lambda}^i_+\rangle$, for $1 \leq i \leq s$, on the right horizontal branch and the $\langle\overline{\lambda}^i_-\rangle$, for $1 \leq i \leq t$, on the upper vertical branch of the Brauer tree is as indicated, see Section 6.2.

4.8. **Subcase** $\sigma_{B_{\overline{\lambda},1}} - p \equiv -1 \pmod 4$**.** The same line of argument as in Section 4.7 works with the roles of the $\langle\overline{\lambda}^i_+\rangle$'s and $\langle\overline{\lambda}^i_-\rangle$'s interchanged, see Table 2.

4.9. **Case** $\epsilon_{B_{\overline{\lambda},1}} = -1$**.** By Section 2.3 we have $\epsilon_{\overline{\lambda}^i_+} = 1 = \epsilon_{\overline{\lambda}^j_-}$ for all $1 \leq i \leq s$ and $1 \leq j \leq t$, and $\epsilon_{\overline{\lambda}^0} = -1$. Hence we have $k(B_{\overline{\lambda},1}) = (p-1)/2 + 2$, where $\mathcal{D}^-_{\overline{\lambda},1} = \{\overline{\lambda}^0\}$. By Section 3.3, we have $\langle\overline{\lambda}^0\rangle(\overline{\lambda}^0) = -\langle\overline{\lambda}^0\rangle'(\overline{\lambda}^0) \neq 0$. As $\overline{\lambda}^0$ has p as a part, $\langle\overline{\lambda}^0\rangle$ is not p-rational. Hence by [7, Thm.11.5] we conclude that $\langle\overline{\lambda}^0\rangle$ and $\langle\overline{\lambda}^0\rangle'$

are exceptional characters of $B_{\overline{\lambda},1}$. Furthermore $\langle\lambda\rangle$ is integer-valued for $\lambda \in \mathcal{D}^+_{\overline{\lambda},1}$. Thus we have $l(B_{\overline{\lambda},1}) = (p-1)/2$, hence $m(B_{\overline{\lambda},1}) = 2$, and the Brauer tree is a real stem, see Table 3. From the results in Section 4.11, by induction from $\tilde{\mathcal{A}}_n$ to $\tilde{\mathcal{S}}_n$, we deduce that the vertices on one of the branches of the Brauer tree are labelled by the $\langle\overline{\lambda}^i_+\rangle$, for $1 \leq i \leq s$, while the vertices on the other branch are labelled by the $\langle\overline{\lambda}^i_-\rangle$, for $1 \leq i \leq t$. It remains to show that the ordering of the $\langle\overline{\lambda}^i_+\rangle$ and $\langle\overline{\lambda}^i_-\rangle$ on the branches of the Brauer tree is as indicated, see Section 6.2.

b) We now consider $\tilde{B}_{\overline{\lambda},1}$ and distinguish the cases $\epsilon_{\tilde{B}_{\overline{\lambda},1}} = 1$ and $\epsilon_{\tilde{B}_{\overline{\lambda},1}} = -1$.

4.10. **Case $\epsilon_{\tilde{B}_{\overline{\lambda},1}} = 1$.** As $\langle\overline{\lambda}^0\rangle$ is a self-associate character and $\overline{\lambda}^0$ has p as a part, by Section 3.6 the characters $\langle\overline{\lambda}^0\rangle_1$ and $\langle\overline{\lambda}^0\rangle_2$ in $\mathrm{Irr}^-(\tilde{\mathcal{A}}_n)$ are not p-rational, and hence exceptional. For $\lambda \in \mathcal{D}^-_{\overline{\lambda},1}$ the characters $\langle\lambda\rangle$ and $\langle\lambda\rangle'$ restrict irreducibly to $\tilde{\mathcal{A}}_n$, and hence $\langle\lambda\rangle^{(\prime)}$ is integer-valued. Thus we have $k(\tilde{B}_{\overline{\lambda},1}) = (p-1)/2 + 2$, as well as $l(\tilde{B}_{\overline{\lambda},1}) = (p-1)/2$ and $m(\tilde{B}_{\overline{\lambda},1}) = 2$. Furthermore, all the characters in $\tilde{B}_{\overline{\lambda},1}$ are real-valued. From the results in Sections 4.7 and 4.8, by restriction from $\tilde{\mathcal{S}}_n$ to $\tilde{\mathcal{A}}_n$, we deduce the Brauer tree as depicted in Table 4.

4.11. **Case $\epsilon_{\tilde{B}_{\overline{\lambda},1}} = -1$.** The characters $\langle\overline{\lambda}^0\rangle$ and $\langle\overline{\lambda}^0\rangle'$ restrict to the p-rational irreducible character $\langle\overline{\lambda}^0\rangle^{(\prime)}$ of $\tilde{\mathcal{A}}_n$. For the character $\langle\lambda\rangle$, where $\lambda \in \mathcal{D}^+_{\overline{\lambda},1}$, we have $\langle\lambda\rangle|_{\tilde{\mathcal{A}}_n} = \langle\lambda\rangle_1 + \langle\lambda\rangle_2$, which by Section 3.6 are p-rational as well. We conclude $k(\tilde{B}_{\overline{\lambda},1}) = p$, as well as $l(\tilde{B}_{\overline{\lambda},1}) = p-1$ and $m(\tilde{B}_{\overline{\lambda},1}) = 1$, by [7, Thm.11.5].
Again we have to distinguish two subcases $\sigma_{\tilde{B}_{\overline{\lambda},1}} + p \equiv 0 \pmod 4$ and $\sigma_{\tilde{B}_{\overline{\lambda},1}} + p \equiv 2 \pmod 4$. For both subcases, using the description of the character values in Section 3.6, we determine the real valued characters in $\tilde{B}_{\overline{\lambda},1}$, and the characters fixed under $\tilde{\kappa} : \chi \mapsto \overline{\chi}^\sigma = \overline{\chi^\sigma}$, for $\chi \in \mathrm{Irr}(\tilde{\mathcal{A}}_n)$, which is the concatenation of complex conjugation and the action σ induced by the outer automorphism of $\tilde{\mathcal{A}}_n$ induced by $\tilde{\mathcal{S}}_n$. Again using [7, Thm.11.15,11.16], we conclude that the Brauer tree is a 4-fold star having $\langle\overline{\lambda}^0\rangle^{(\prime)}$ in its center, see Tables 5 and 6, the vertices on the branches being labelled by the $\langle\overline{\lambda}^i_+\rangle_1$, $\langle\overline{\lambda}^i_+\rangle_2$ for $1 \leq i \leq s$ and the $\langle\overline{\lambda}^i_-\rangle_1$, $\langle\overline{\lambda}^i_-\rangle_2$ for $1 \leq i \leq t$, respectively.
As σ induces a graph automorphism of the Brauer tree of order 2, which leaves both the set of real characters as well as the set of $\tilde{\kappa}$-fixed points invariant, we conclude that $\langle\lambda\rangle_1$ and $\langle\lambda\rangle_2$ have equal distance from the center $\langle\overline{\lambda}^0\rangle$ for all $\lambda \in \mathcal{D}^+_{\overline{\lambda},1}$. Using Proposition 3.9, in Tables 5 and 6 we depict the Brauer trees where the $\langle\lambda\rangle_1$'s are concentrated on the right and upper branches of the Brauer tree, while the $\langle\lambda\rangle_2$'s are concentrated on the left and lower branches. The ordering of the $\langle\overline{\lambda}^i_+\rangle_1$, for $1 \leq i \leq s$, and the $\langle\overline{\lambda}^i_-\rangle_1$, for $1 \leq i \leq t$, on the right and upper branches of the Brauer tree follows from the corresponding result for $B_{\overline{\lambda},1}$, see Section 4.9. $\sharp$

5. Scopes-Kessar reduction

5.1. **Proposition.** Let $\lambda \in \mathcal{D}_n$ have p-weight w_λ, let $\overline{\lambda} \in \mathcal{D}_{n-pw_\lambda}$ be its p-bar core, and let $1 \leq j \leq p-1$.

a) We have $\{0,\ldots,|X^j_{\overline{\lambda}}|-w_\lambda-1\}\subseteq X^j_\lambda\subseteq\{0,\ldots,|X^j_{\overline{\lambda}}|+w_\lambda-1\}$, where for $|X^j_{\overline{\lambda}}|\le w_\lambda$ the left hand side is the empty set; and $X^0_\lambda\subseteq\{1,\ldots,w_\lambda\}$.
b) We have $|X^j_\lambda|-|X^j_{\overline{\lambda}}|\le w_\lambda$; and if $|X^j_{\overline{\lambda}}|\ge w_\lambda+1$, then $|X^j_\lambda|=|X^j_{\overline{\lambda}}|$ and $X^{p-j}_\lambda=\emptyset$.

Proof. The inclusions $X^j_\lambda\subseteq\{0,\ldots,|X^j_{\overline{\lambda}}|+w_\lambda-1\}$ and $X^0_\lambda\subseteq\{1,\ldots,w_\lambda\}$ follow from a consideration of p-bar removal steps of type '+' and '0'. A consideration of p-bar removal steps of type '$-$' yields $|X^j_\lambda|\le|X^j_{\overline{\lambda}}|+w_\lambda$.

Let $|X^j_{\overline{\lambda}}|\ge w_\lambda+1$. Then, if one of the rows $0,\ldots,|X^j_{\overline{\lambda}}|-w_\lambda-1$ were not occupied on the j-th runner, at least $w_\lambda+1$ steps of type '+' would be possible, a contradiction. Furthermore, if a step of type '$-$' involving the j-th runner were possible, then at least w_λ additional steps of type '+' would be possible as well, a contradiction. $\sharp$

5.2. **Definition.** For $1\le j\le p-1$ the j-th *Scopes-Kessar map* $S^j:\mathcal{D}\to\mathcal{D}$ is defined as follows: For $j\ne1,(p+1)/2$ the bead configuration of $S^j(\lambda)$ is obtained from the one of λ by exchanging runners j and $j-1$ as well as exchanging runners $p-j$ and $p-j+1$. For $j=(p+1)/2$ runners $(p+1)/2$ and $(p-1)/2$ are exchanged. For $j=1$ runners 1 and $p-1$ are exchanged, and subsequently the beads on runner $p-1$ are moved one position towards decreasing row numbers.

5.3. **Remark.**
a) Let $\lambda\in\mathcal{D}$ have p-bar core $\overline{\lambda}$, and for $1\le j\le p-1$ let $S^j(\lambda)$ have p-bar core $\overline{S^j(\lambda)}$. If $j=1$, then let additionally $0\in X^1_\lambda$ and $X^{p-1}_\lambda=\emptyset$. Then, as is easily seen or by [12, La.4.7], we have $w_{S^j(\lambda)}=w_\lambda$ and $\overline{S^j(\lambda)}=S^j(\overline{\lambda})$.
b) For $1\le j,j'\le p-1$ we have $S^j=S^{j'}$ if and only if $j+j'=p+1$. For $j\ne1$ we have a weight-preserving involutory bijection $S^j:\mathcal{D}\to\mathcal{D}$, which commutes with taking p-bar cores. For $j=1$ the map S^1 induces a bijection from $\{\lambda\in\mathcal{D};0\in X^1_\lambda\}$ to $\mathcal{D}$, since for $\lambda\in\mathcal{D}$ the preimage $(S^1)^{-1}(\lambda)$ consists of two elements, $\tilde{\lambda}$ and $\tilde{\tilde{\lambda}}$ say, where $0\in X^1_{\tilde{\lambda}}$ and $\tilde{\tilde{\lambda}}=\tilde{\lambda}\setminus 1$.

5.4. **Proposition.** Let $\overline{\lambda}\in\mathcal{D}_n$ be a p-bar core. Let $1\le j\le p-1$ such that $X^j_{\overline{\lambda}}\ne\emptyset$ and $S^j(\overline{\lambda})\in\mathcal{D}_m$.
a) Then we have $n-m\ge|X^j_{\overline{\lambda}}|-|X^{j-1}_{\overline{\lambda}}|$. In particular, for $j=(p+1)/2$ we have $n-m=|X^{(p+1)/2}_{\overline{\lambda}}|$, and for $j=1$ we have $n-m=2|X^1_{\overline{\lambda}}|-1$.
b) If $j=1$ and $|X^1_{\overline{\lambda}}|\ge2$, then for $w\le|X^1_{\overline{\lambda}}|-1$ the map S^1 induces a bijection between the sets $\mathcal{D}_{\overline{\lambda},w}$ and $\mathcal{D}_{S^1(\overline{\lambda}),w}$.

Proof. a) For $j\ne1,(p+1)/2$ we have $n-m=|X^j_{\overline{\lambda}}|-|X^{j-1}_{\overline{\lambda}}|+|X^{p-j+1}_{\overline{\lambda}}|-|X^{p-j}_{\overline{\lambda}}|$. As $|X^j_{\overline{\lambda}}|\ne0$, we have $X^{p-j}_{\overline{\lambda}}=\emptyset$. For $j=(p+1)/2$ we have $n-m=|X^{(p+1)/2}_{\overline{\lambda}}|-|X^{(p-1)/2}_{\overline{\lambda}}|=|X^{(p+1)/2}_{\overline{\lambda}}|$. For $j=1$ we have $n-m=1+2\cdot(|X^1_{\overline{\lambda}}|-|X^{p-1}_{\overline{\lambda}}|)$. As $|X^1_{\overline{\lambda}}|\ne0$, we have $X^{p-1}_{\overline{\lambda}}=\emptyset$, hence $n-m=2|X^1_{\overline{\lambda}}|-1\ge|X^1_{\overline{\lambda}}|$.
b) Let $\pi\in\mathcal{D}_{\overline{\lambda},w}$. Then $|X^1_\pi|\ge|X^1_{\overline{\lambda}}|\ge w+1=w_\pi+1$. Hence by Proposition 5.1 we have $0\in X^1_\pi$ and $X^{p-1}_\pi=\emptyset$. By Remark 5.3 we conclude that S^1 maps $\mathcal{D}_{\overline{\lambda},w}$ injectively to $\mathcal{D}_{S^1(\overline{\lambda}),w}$.

Conversely, let $\pi\in\mathcal{D}_{S^1(\overline{\lambda}),w}$. Then $|X^{p-1}_\pi|\ge|X^{p-1}_{S^1(\overline{\lambda})}|\ge w=w_\pi$. If $|X^{p-1}_\pi|=|X^{p-1}_{S^1(\overline{\lambda})}|=w_\pi$, then in the p-bar removal process from π to $S^1(\overline{\lambda})$ the $(p-1)$-st runner is involved only for steps of type '+' as $X^1_{S^1(\overline{\lambda})} = \emptyset$, we have $X^1_\pi = \emptyset$. If $|X^{p-1}_\pi| \geq w_\pi + 1$, by Proposition 5.1 we have $X^1_\pi = \emptyset$. Let $\tilde{\pi} \in \mathcal{D}$ such that $S^1(\tilde{\pi}) = \pi$ and $0 \in X^1_{\tilde{\pi}}$. Hence $X^{p-1}_{\tilde{\pi}} = \emptyset$, and by Remark 5.3 we have $w_{\tilde{\pi}} = w_\pi$. Let $\tilde{\pi}$ have p-bar core $\overline{\tilde{\pi}}$. Again by Remark 5.3 we have $S^1(\overline{\tilde{\pi}}) = S^1(\overline{\lambda})$. Since $0 \in X^1_{\overline{\lambda}} \cap X^1_{\overline{\tilde{\pi}}}$, we have $\overline{\tilde{\pi}} = \overline{\lambda}$. ♯

5.5. **Proposition.** Let $\overline{\lambda} \in \mathcal{D}_{n-p}$ be a p-bar core, let $1 \leq j \leq p-1$ such that $|X^j_{\overline{\lambda}}| > |X^{j-1}_{\overline{\lambda}}|$, and if $j = 1$ let additionally $|X^1_{\overline{\lambda}}| \geq 2$. For $\lambda \in \mathcal{D}_{\overline{\lambda},1}$ and $\tilde{\lambda} \in \mathcal{D}_{S^j(\overline{\lambda}),1}$ we then have $\mathcal{R}^\lambda_{\tilde{\lambda}} \neq \emptyset$ if and only if $\tilde{\lambda} = S^j(\lambda)$, where $\mathcal{R}^\lambda_{\tilde{\lambda}}$ is as in Section 3.5.

Proof. Note that by Remark 5.3 and Proposition 5.4 indeed $S^j(\lambda) \in \mathcal{D}_{S^j(\overline{\lambda}),1}$. We consider different cases:

5.6. **Case** $j \neq 1$**.** We have $|X^j_{\overline{\lambda}}| \geq 1$. Assume $|X^j_\lambda| > |X^j_{\overline{\lambda}}|$, hence $|X^j_\lambda| \geq 2$, then by Proposition 5.1 we have $|X^j_\lambda| = |X^j_{\overline{\lambda}}|$, a contradiction. Thus indeed we have $|X^j_\lambda| = |X^j_{\overline{\lambda}}|$, and by the same argument as in the proof of Proposition 5.1 we conclude that $X^{p-j}_\lambda = \emptyset$. Furthermore we have $|X^{j-1}_{S^j(\overline{\lambda})}| = |X^j_{\overline{\lambda}}|$, and hence the same argument shows $|X^{j-1}_{\tilde{\lambda}}| = |X^{j-1}_{S^j(\overline{\lambda})}|$ and $X^{p-j+1}_{\tilde{\lambda}} = \emptyset$.

Let $k := |X^{j-1}_\lambda| - |X^{j-1}_{\overline{\lambda}}|$. By Proposition 5.1 we have $0 \leq k \leq 1$. If $k = 1$, the p-bar removal step from λ to $\overline{\lambda}$ is of type '$-$' and involves the $(j-1)$-st runner of the abacus. Hence we have $X^{j-1}_\lambda = \{0\}$ and $X^{j-1}_{\overline{\lambda}} = \emptyset$, as well as $X^j_\lambda = X^j_{\overline{\lambda}} \neq \emptyset$. If $k = 0$, the p-bar removal step from λ to $\overline{\lambda}$ either is of type '+' and involves the $(j-1)$-st runner, in which case we have $X^{j-1}_\lambda = \{0, \ldots, |X^{j-1}_{\overline{\lambda}}| - 2, |X^{j-1}_{\overline{\lambda}}|\} \subseteq \{0, \ldots, |X^j_{\overline{\lambda}}| - 1\} = X^j_{\overline{\lambda}} = X^j_\lambda$; or it is of type '+' and involves the j-th runner, in which case we have $X^{j-1}_\lambda = X^{j-1}_{\overline{\lambda}} = \{0, \ldots, |X^{j-1}_{\overline{\lambda}}| - 1\} \subseteq \{0, \ldots, |X^j_{\overline{\lambda}}| - 2, |X^j_{\overline{\lambda}}|\} = X^j_\lambda$; or the $(j-1)$-st and j-th runners are not involved, in which case we have $X^{j-1}_\lambda = X^{j-1}_{\overline{\lambda}} = \{0, \ldots, |X^{j-1}_{\overline{\lambda}}| - 1\} \subseteq \{0, \ldots, |X^j_{\overline{\lambda}}| - 1\} = X^j_{\overline{\lambda}} = X^j_\lambda$.

Hence in either case we have $X^{j-1}_\lambda \subseteq X^j_\lambda$. As $X^{p-j}_\lambda = \emptyset$, we also have $X^{p-j}_\lambda \subseteq X^{p-j+1}_\lambda$, hence $\mathcal{R}^\lambda_{S^j(\lambda)} \neq \emptyset$. Let conversely $[\lambda = \lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(n-m)} = \tilde{\lambda}] \in \mathcal{R}^\lambda_{\tilde{\lambda}}$.

5.7. **Subcase** $j \neq (p+1)/2$**.** We have $|X^{j-1}_{\tilde{\lambda}}| - |X^{j-1}_\lambda| = |X^j_{\overline{\lambda}}| - |X^{j-1}_{\overline{\lambda}}| - k \geq 0$. Hence there is a set $I_j \subseteq \{0, \ldots, n-m-1\}$ of cardinality $|I_j| = |X^j_{\overline{\lambda}}| - |X^{j-1}_{\overline{\lambda}}| - k$, such that for $i \in I_j$ there is $x_i \in X^{j-1}_{\tilde{\lambda}} \setminus X^{j-1}_\lambda$ such that in the 1-bar removal step from $\lambda^{(i)}$ to $\lambda^{(i+1)}$ a bead is moved from position (x_i, j) to position $(x_i, j-1)$. Note that we have $|X^{p-j+1}_\lambda| - |X^{p-j+1}_{\overline{\lambda}}| = k$, and $X^{p-j+1}_{\tilde{\lambda}} = \emptyset$. Hence there is a set $I_{p-j+1} \subseteq \{0, \ldots, n-m-1\}$ of cardinality $|I_{p-j+1}| = |X^{p-j+1}_{\overline{\lambda}}| + k$, such that for $i \in I_{p-j+1}$ there is $x_i \in X^{p-j+1}_\lambda$ such that in the 1-bar removal step from $\lambda^{(i)}$ to $\lambda^{(i+1)}$ a bead is moved from position $(x_i, p-j+1)$ to position $(x_i, p-j)$. We have $I_j \cap I_{p-j+1} = \emptyset$ and $|I_j| + |I_{p-j+1}| = n - m$, as $X^{p-j}_{\tilde{\lambda}} = \emptyset$. Hence $I_j \cup I_{p-j+1} = \{0, \ldots, n-m-1\}$, which means that in all the 1-bar removal steps only beads on the j-th and $(p-j+1)$-st runners are moved to the left.

Let $i \in I_j$. Assume $x_i \notin X^j_\lambda$, then a bead on the $(j+1)$-st runner must be moved to the left, a contradiction. Thus we have $\{x_i; i \in I_j\} \subseteq X^j_\lambda$, and hence $\{x_i; i \in I_j\} =$

$X_{\lambda}^{j} \setminus X_{\lambda}^{j-1}$, since the latter set has cardinality $|X_{\tilde{\lambda}}^{j}| - |X_{\tilde{\lambda}}^{j-1}| - k = |I_j|$. Furthermore we have $\{x_i; i \in I_{p-j+1}\} = X_{\lambda}^{p-j+1}$. Hence we conclude that $\tilde{\lambda} = S^j(\lambda)$.

5.8. **Subcase** $j = (p+1)/2$**.** Since $(p+1)/2 = p-(p+1)/2+1$, we have $X_{\tilde{\lambda}}^{(p+1)/2} = \emptyset$. Hence there is a set $I_{(p+1)/2} \subseteq \{0, \ldots, n-m-1\}$ of cardinality $|I_{(p+1)/2}| = |X_{\tilde{\lambda}}^{(p+1)/2}|$, such that for $i \in I_{(p+1)/2}$ there is $x_i \in X_{\lambda}^{(p+1)/2}$ such that in the 1-bar removal step from $\lambda^{(i)}$ to $\lambda^{(i+1)}$ a bead is moved from position $(x_i, (p+1)/2)$ to position $(x_i, (p-1)/2)$. By Proposition 5.4 we have $|X_{\tilde{\lambda}}^{(p+1)/2}| = n-m$, hence $I_{(p+1)/2} = \{0, \ldots, n-m-1\}$, which means that in all the 1-bar removal steps only beads on the $(p+1)/2$-nd runner are moved to the left. Hence we conclude that $\tilde{\lambda} = S^{(p+1)/2}(\lambda)$.

5.9. **Case** $j = 1$**.** We have $|X_{\tilde{\lambda}}^{p-1}| \geq |X_{S^1(\overline{\lambda})}^{p-1}| \geq 1 = w_{\tilde{\lambda}}$. If $|X_{\tilde{\lambda}}^{p-1}| = |X_{S^1(\overline{\lambda})}^{p-1}| = 1$, then in the p-bar removal step from $\tilde{\lambda}$ to $S^1(\overline{\lambda})$ the $(p-1)$-st runner is involved possibly only in a step of type '+'; as $X_{S^1(\overline{\lambda})}^{1} = \emptyset$, we have $X_{\tilde{\lambda}}^{1} = \emptyset$. If $|X_{\tilde{\lambda}}^{p-1}| \geq 2$, by Proposition 5.1 we have $|X_{\tilde{\lambda}}^{p-1}| = |X_{S^1(\overline{\lambda})}^{p-1}|$ and $X_{\tilde{\lambda}}^{1} = \emptyset$. By Proposition 5.1 we have $|X_{\lambda}^{1}| = |X_{\tilde{\lambda}}^{1}|$ and $X_{\lambda}^{p-1} = \emptyset$. By Propositions 5.1 and 5.4 we have $X_{\lambda}^{0} \subseteq \{1\}$ and $\{0, \ldots, |X_{\lambda}^{1}| - 2\} \subseteq X_{\lambda}^{1}$. Hence if $|X_{\lambda}^{1}| \geq 3$ or $X_{\lambda}^{1} = \{0, 1\}$, then we have $X_{\lambda}^{0} \subseteq X_{\lambda}^{1}$. If $|X_{\lambda}^{1}| = 2$, but $X_{\lambda}^{1} \neq \{0, 1\}$, we conclude from $w_{\lambda} = 1$ that $X_{\lambda}^{1} = \{0, 2\}$ and $X_{\lambda}^{1} = \emptyset$, hence in any case we have $X_{\lambda}^{0} \subseteq X_{\lambda}^{1}$ and $0 \in X_{\lambda}^{1}$.
Thus we have $\tilde{\lambda} = S^1(\lambda)$ if and only if $X_{\tilde{\lambda}}^{p-1} = \{x-1; 0 \neq x \in X_{\lambda}^{1}\}$. Since $X_{\lambda}^{0} \subseteq X_{\lambda}^{1}$, we have $\mathcal{R}_{S^1(\lambda)}^{\lambda} \neq \emptyset$. Let conversely $[\lambda = \lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(n-m)} = \tilde{\lambda}] \in \mathcal{R}_{\tilde{\lambda}}^{\lambda}$.
There is a set $I_1 \subseteq \{0, \ldots, n-m-1\}$ of cardinality $|I_1| = |X_{\tilde{\lambda}}^{1}|$ such that for $i \in I_1$ there is $x_i \in X_{\lambda}^{1}$ such that in the 1-bar removal step from $\lambda^{(i)}$ to $\lambda^{(i+1)}$ a bead is moved from position $(x_i, 1)$ to position $(x_i, 0)$. And there is a set $I_0 \subseteq \{0, \ldots, n-m-1\}$ of cardinality $|I_0| = |X_{\tilde{\lambda}}^{1}| - 1$, such that for $i \in I_0$ there is $x_i \in X_{\lambda}^{0}$ such that in the 1-bar removal step from $\lambda^{(i)}$ to $\lambda^{(i+1)}$ a bead is moved from position $(x_i, 0)$ to position $(x_{i-1}, p-1)$. We have $I_0 \cap I_1 = \emptyset$ and, by Proposition 5.4, $|I_0| + |I_1| = n-m$, hence $I_0 \cup I_1 = \{0, \ldots, n-m-1\}$.
Let $x \in X_{\tilde{\lambda}}^{p-1}$; we are going to show that $x+1 \in X_{\lambda}^{1}$. We may assume $x+1 \notin X_{\lambda}^{0}$, and hence there is some $i \in I_1$ such that $x+1 = x_i \in X_{\lambda}^{1}$. Thus $\{x+1; x \in X_{\tilde{\lambda}}^{p-1}\} \subseteq X_{\lambda}^{1} \setminus \{0\}$. As these sets both have cardinality $|X_{\tilde{\lambda}}^{1}| - 1$, we have equality, and hence $\tilde{\lambda} = S^1(\lambda)$. ♯

5.10. **Remark.** By the branching rule, see Section 3.5, we have $\langle \langle \lambda \rangle|_{\tilde{\mathcal{S}}_m}, \langle \tilde{\lambda} \rangle \rangle_{\tilde{\mathcal{S}}_m} \neq 0$, if and only if $\mathcal{R}_{\tilde{\lambda}}^{\lambda} \neq \emptyset$. Indeed, the cardinality $|\mathcal{R}_{\tilde{\lambda}}^{\lambda}|$ is closely related to the value of the above scalar product, see, e. g., [12, Prop.3.7]. It is possible to determine $|\mathcal{R}_{\tilde{\lambda}}^{\lambda}|$ precisely, but we will not need this information.

6. The trees

6.1. **Proposition.** Let $\overline{\lambda} \in \mathcal{D}_{n-p}$ be a p-bar core, with s-invariant $s_{\overline{\lambda}}$. Let $1 \leq j \leq p-1$ such that $|X_{\overline{\lambda}}^{j}| > |X_{\overline{\lambda}}^{j-1}|$, and if $j = 1$ let additionally $|X_{\overline{\lambda}}^{1}| \geq 2$. Then $S^j(\overline{\lambda})$ has s-invariant $s_{\overline{\lambda}}$ and the map S^j induces a bijection between the sets $\mathcal{D}_{\overline{\lambda},1}$

and $\mathcal{D}_{S^j(\overline{\lambda}),1}$. Precisely, we have $S^j(\overline{\lambda}^i_+) = (S^j(\overline{\lambda}))^i_+$ for $1 \leq i \leq s_{\overline{\lambda}}$, as well as $S^j(\overline{\lambda}^0) = (S^j(\overline{\lambda}))^0$, and $S^j(\overline{\lambda}^i_-) = (S^j(\overline{\lambda}))^i_-$ for $1 \leq i \leq (p-1)/2 - s_{\overline{\lambda}}$.

Proof. The invariance of the s-invariant follows directly from Definition 5.2. The bijectivity of S^j follows from Remark 5.3 and Proposition 5.4.

Using the definitions in Section 4.2, the assertion on $S^j(\overline{\lambda}^0)$ is clear. Let the p-bar addition step from $\overline{\lambda}$ to $\overline{\lambda}^i_+$ involve a bead on runner y, where $1 \leq y \leq p-1$ and $y \neq p-j$. If $j = 1$, then in both cases $y \neq 1$ and $y = 1$ the assertion on $S^j(\overline{\lambda}^i_+)$ is clear. Let $j \neq 1$. If $y \notin \{j-1, j, p-j+1\}$, then the assertion on $S^j(\overline{\lambda}^i_+)$ is clear. If $y = j$ or $y = j-1$, then the assertion follows from $X^{j-1}_{\overline{\lambda}} \subset X^j_{\overline{\lambda}}$. If $y = p-j+1$, then we use analogously $\emptyset = X^{p-j}_{\overline{\lambda}} \subset X^{p-j+1}_{\overline{\lambda}}$. Finally, let the p-bar addition step from $\overline{\lambda}$ to $\overline{\lambda}^i_-$ involve beads on runners y and $p-y$, where $1 \leq y \leq (p-1)/2$ and $y \notin \{j, p-j\}$. For both cases $j \neq 1$ and $j = 1$, the assertion on $S^j(\overline{\lambda}^i_-)$ is clear. ♯

6.2. **Proof of Theorem 4.4, part II.** We now determine the ordering of the $\langle\overline{\lambda}^i_+\rangle$, for $1 \leq i \leq s := s_{\overline{\lambda}}$, and the $\langle\overline{\lambda}^i_-\rangle$, for $1 \leq i \leq t := (p-1)/2 - s_{\overline{\lambda}}$, on the branches of the Brauer tree of $B_{\overline{\lambda},1}$ by induction on n.

If there is $2 \leq j \leq p-1$ such that $|X^j_{\overline{\lambda}}| > |X^{j-1}_{\overline{\lambda}}|$, or if $|X^1_{\overline{\lambda}}| \geq 2$, in which case we let $j = 1$, then, by Proposition 5.4, $S^j(\overline{\lambda}) \in \mathcal{D}_{m-p}$, where $n - m \geq |X^j_{\overline{\lambda}}| - |X^{j-1}_{\overline{\lambda}}| > 0$. By induction the Brauer tree of $B_{S^j(\overline{\lambda}),1}$ is as asserted, and hence $\langle S^j(\overline{\lambda})^i_+\rangle^* + \langle S^j(\overline{\lambda})^{i+1}_+\rangle^*$, for $1 \leq i \leq s-1$, and $\langle S^j(\overline{\lambda})^i_-\rangle^* + \langle S^j(\overline{\lambda})^{i+1}_-\rangle^*$, for $1 \leq i \leq t-1$, as well as $\langle S^j(\overline{\lambda})^s_+\rangle + \langle S^j(\overline{\lambda})^0\rangle^*$ and $\langle S^j(\overline{\lambda})^t_-\rangle + \langle S^j(\overline{\lambda})^0\rangle^*$ are projective characters in $B_{S^j(\overline{\lambda}),1}$. Inducing these projective characters to $\tilde{\mathcal{S}}_n$ and taking $B_{\overline{\lambda},1}$-components yields a set of projective characters, whose decomposition into ordinary irreducible characters is found using the branching rule, see Section 3.5, Frobenius reciprocity, and Propositions 5.5 and 6.1. From that the Brauer tree of $B_{\overline{\lambda},1}$ is seen to be as asserted.

If such j does not exist, then we have $\overline{\lambda} = \overline{\lambda}^{(s)} := [s, s-1, \ldots, 1] \in \mathcal{D}_{n-p}$, hence $n - p = s(s+1)/2$. For $s \geq 1$ let $\tilde{\lambda}^{(s)} := [s, s-1, \ldots, 2] = \overline{\lambda}^{(s)} \setminus 1 \in \mathcal{D}_{n-p-1}$, which is a p-bar core with s-invariant $s_{\tilde{\lambda}^{(s)}} = s-1$. By induction the Brauer tree of $B_{\tilde{\lambda}^{(s)},1}$ is as asserted, and hence $\langle(\tilde{\lambda}^{(s)})^i_+\rangle^* + \langle(\tilde{\lambda}^{(s)})^{i+1}_+\rangle^*$ for $1 \leq i \leq s-2$, and $\langle(\tilde{\lambda}^{(s)})^i_-\rangle^* + \langle(\tilde{\lambda}^{(s)})^{i+1}_-\rangle^*$ for $1 \leq i \leq t$, as well as $\langle(\tilde{\lambda}^{(s)})^{s-1}_+\rangle + \langle(\tilde{\lambda}^{(s)})^0\rangle^*$ and $\langle(\tilde{\lambda}^{(s)})^{t+1}_+\rangle + \langle(\tilde{\lambda}^{(s)})^0\rangle^*$ are projective characters in $B_{\tilde{\lambda}^{(s)},1}$. By the branching rule the $B_{\tilde{\lambda}^{(s)},1}$-components of the restrictions of the characters in $B_{\overline{\lambda}^{(s)},1}$ to $\tilde{\mathcal{S}}_{n-1}$ are as follows: $\langle(\overline{\lambda}^{(s)})^i_+\rangle|_{\tilde{\mathcal{S}}_{n-1}} \equiv \langle(\tilde{\lambda}^{(s)})^i_+\rangle$ for $1 \leq i \leq s-1$ and $\langle(\overline{\lambda}^{(s)})^s_+\rangle|_{\tilde{\mathcal{S}}_{n-1}} \equiv \langle(\tilde{\lambda}^{(s)})^0\rangle^*$, as well as $\langle(\overline{\lambda}^{(s)})^i_-\rangle|_{\tilde{\mathcal{S}}_{n-1}} \equiv \langle(\tilde{\lambda}^{(s)})^i_-\rangle$ for $1 \leq i \leq t$ and $\langle(\overline{\lambda}^{(s)})^0\rangle|_{\tilde{\mathcal{S}}_{n-1}} \equiv \langle(\tilde{\lambda}^{(s)})^0\rangle + \langle(\tilde{\lambda}^{(s)})^{t+1}_-\rangle^*$. Again by Frobenius reciprocity the Brauer tree of $B_{\overline{\lambda}^{(s)},1}$ is as asserted.

If $s = 0$ then $\overline{\lambda} = \overline{\lambda}^{(0)} = []$ and $n = p$. Hence p does not divide $|\tilde{\mathcal{S}}_{n-1}|$, and thus all characters in $\mathrm{Irr}^-(\tilde{\mathcal{S}}_{n-1})$ are projective. By the branching rule and Frobenius reciprocity we obtain the following induced projective characters, where again we only record the $B_{[],1}$-components: $\langle[(p-1)/2+i, (p-1)/2-i]\rangle \uparrow^{\tilde{\mathcal{S}}_p} \equiv \langle(\overline{\lambda}^{(0)})^i_-\rangle^* +$

$\langle(\overline{\lambda}^{(0)})_-^{i+1}\rangle^*$ for $1 \leq i \leq (p-1)/2-1 = t-1$, and $\langle[p-1]\rangle \uparrow^{\tilde{\mathcal{S}}_p} \equiv \langle(\overline{\lambda}^{(0)})^0\rangle^* + \langle(\overline{\lambda}^{(0)})_-^t\rangle$. This shows that the Brauer tree of $B_{[],1}$ is as asserted. ♯

6.3. **Corollary.** We consider the set of all Morita equivalence classes of p-blocks of $\mathrm{Irr}^-(\tilde{\mathcal{S}}_n)$ of weight 1 occurring for some $n \geq p$. Then, by Theorem 4.4, the set of representatives $B_{\overline{\lambda},1}$ for these classes with $n_{\overline{\lambda}}$ minimal is given as follows, where $\overline{\lambda}^{(s)} := [s, s-1, \ldots, 1] \in \mathcal{D}_{s(s+1)/2}$ for $1 \leq s \leq (p-1)/2$ is as defined in Section 6.2, and $\overline{\lambda}^{(s+)} := [s+1, s-1, s-2, \ldots, 1] \in \mathcal{D}_{s(s+1)/2+1}$ is the partition obtained from $\overline{\lambda}^{(s)}$ by exchanging its part s against $s+1$.

a) Let $p \equiv 1 \pmod 4$. For the cases $m(B_{\overline{\lambda},1}) = 1$ and $m(B_{\overline{\lambda},1}) = 2$, respectively, the p-bar core $\overline{\lambda}$ ranges over

$$\begin{array}{rl} & \{\overline{\lambda}^{(s)}; 0 \leq s \leq (p-1)/4, s \equiv 0,1 \pmod 4\} \\ \dot\cup & \{\overline{\lambda}^{(s+)}; 0 \leq s \leq (p-1)/4, s \not\equiv 0,1 \pmod 4\}, \end{array}$$

and

$$\begin{array}{rl} & \{\overline{\lambda}^{(s)}; 0 \leq s \leq (p-1)/4, s \not\equiv 0,1 \pmod 4\} \\ \dot\cup & \{\overline{\lambda}^{(s+)}; 0 \leq s \leq (p-1)/4, s \equiv 0,1 \pmod 4\}. \end{array}$$

b) Let $p \equiv -1 \pmod 4$. For the cases $m(B_{\overline{\lambda},1}) = 1$ and $m(B_{\overline{\lambda},1}) = 2$, respectively, the p-bar core $\overline{\lambda}$ ranges over

$$\begin{array}{rl} & \{\overline{\lambda}^{(s)}; 0 \leq s \leq (p-3)/4, s \equiv 0,1 \pmod 4\} \\ \dot\cup & \{\overline{\lambda}^{(s+)}; 0 \leq s \leq (p-3)/4, s \not\equiv 0,1 \pmod 4\}, \end{array}$$

and

$$\begin{array}{rl} & \{\overline{\lambda}^{(s)}; 0 \leq s \leq (p-3)/4, s \not\equiv 0,1 \pmod 4\} \\ \dot\cup & \{\overline{\lambda}^{(s+)}; 0 \leq s \leq (p-3)/4, s \equiv 0,1 \pmod 4\}. \end{array}$$

Note that this set of Morita equivalence classes coincides with the set of all Morita equivalence classes of p-blocks of $\mathrm{Irr}^-(\tilde{\mathcal{A}}_n)$ of weight 1 occurring for some $n \geq p$.

6.4. **Corollary.** If we walk along the Brauer tree of $B_{\overline{\lambda},1}$, from the leaf labelled by $\langle\overline{\lambda}_+^1\rangle$ inwards to the center, and then outwards again to the leaf labelled by $\langle\overline{\lambda}_-^1\rangle$, and record the leg lengths of the labelling partitions accordingly, then by Proposition 4.3 we get the increasing sequence $0, 1, \ldots, s, \ldots, (p-1)/2$.

6.5. **The δ-invariant.** Let $\overline{\lambda} \in \mathcal{D}_{n-p}$ be a p-bar core. If $\epsilon_{B_{\overline{\lambda},1}} = -1$, then, by Theorem 4.4, $\langle\overline{\lambda}^0\rangle^*$ is the sum of the exceptional characters in $B_{\overline{\lambda},1}$, while if $\epsilon_{B_{\overline{\lambda},1}} = 1$, then it is a self-associate irreducible character. Let $\delta_\chi, \delta_{\langle\overline{\lambda}^0\rangle^*} \in \{\pm 1\}$, where χ is a non-exceptional character in $B_{\overline{\lambda},1}$, denote the invariants defined in [4, Thm.VII.2.14], reflecting the lengths of the Green correspondents of indecomposable lattices for these characters.

6.6. **Corollary.** Let $\overline{\lambda} \in \mathcal{D}_{n-p}$ be a p-bar core with s-invariant $s_{\overline{\lambda}}$.

a) If $\lambda \in \mathcal{D}_{\overline{\lambda},1}$, where $\lambda \neq \overline{\lambda}^0$, then $\delta_{\langle\lambda\rangle} = \delta_{\langle\lambda\rangle'}$.

b) We have $\delta_{\langle\overline{\lambda}^0\rangle^*} = (-1)^{s_{\overline{\lambda}}}$, and $\delta_{\langle\overline{\lambda}_+^i\rangle} = (-1)^{i-1}$ for $1 \leq i \leq s_{\overline{\lambda}}$, as well as $\delta_{\langle\overline{\lambda}_-^i\rangle} = (-1)^{(p-1)/2-i+1}$ for $1 \leq i \leq (p-1)/2 - s_{\overline{\lambda}}$.

Proof. a) As $\langle\lambda\rangle' = \langle\lambda\rangle \otimes \epsilon$, the assertion follows from [4, Thm.VII.2.14].
b) By Section 3.10 and [8, La.4.4.6] we have $\langle\lambda\rangle([p,1^{n-p}])\cdot\delta_{\langle\lambda\rangle} > 0$, where $\lambda \in \mathcal{D}_{\overline{\lambda},1}$, $\lambda \neq \overline{\lambda}^0$. By Section 3.4, we have $\langle\lambda\rangle([p,1^{n-p}]) = \sum_{i\in I_\lambda}(-1)^{b_i}2^{a_i}\langle\lambda^i\rangle([1^{n-p}])$. Since $|I_\lambda| = 1$, we only have to consider the corresponding leg length b_i. Hence the assertion follows from Proposition 4.3. The same argument holds for $\langle\overline{\lambda}^0\rangle^*$. ♯

6.7. **Remark.** For $n \in \mathbb{N}_0$, let Let $\hat{\mathcal{S}}_n$ be defined as the finitely presented group, see [15, Sect.I.3.]:

$$\begin{aligned}\hat{\mathcal{S}}_n := \langle z, \hat{s}_1, \ldots, \hat{s}_{n-1} |\ & z^2 = \hat{s}_i^2 = (\hat{s}_i z)^2 = (\hat{s}_i\hat{s}_{i+1})^3 = 1, (\hat{s}_i\hat{s}_j)^2 = z,\\ & 1 \leq i < j \leq n-1, |i-j| \geq 2\rangle\end{aligned}$$

Thus $\hat{\mathcal{S}}_n$ also is a central extension $1 \to \langle z\rangle \to \hat{\mathcal{S}}_n \xrightarrow{\hat{\alpha}_n} \mathcal{S}_n \to 1$ of $\mathcal{S}_n$ by the cyclic group $\langle z\rangle$ order 2, where $\hat{\alpha}_n : z \mapsto 1$ and $\hat{\alpha}_n : \hat{s}_i \mapsto s_i$ for $1 \leq i \leq n-1$. For $n \neq 0, 1, 6$, the groups $\hat{\mathcal{S}}_n$ and $\tilde{\mathcal{S}}_n$ are not isomorphic, but for all n they are *isoclinic*, see [2, Def.III.1.1]. In particular, for $\hat{\mathcal{A}}_n := \hat{\alpha}_n^{-1}(\mathcal{A}_n) \trianglelefteq \hat{\mathcal{S}}_n$ we have $\hat{\mathcal{A}}_n \cong \tilde{\mathcal{A}}_n$. By [2, Thm.III.5.6] a short calculation shows that we have a bijection $\mathrm{Irr}^-(\tilde{\mathcal{S}}_n) \to \mathrm{Irr}^-(\hat{\mathcal{S}}_n) : \chi \mapsto \hat{\chi}$, where $\hat{\chi}(z^i \cdot \prod_{j=1}^l \hat{s}_{i_j}) := \sqrt{-1}^l \cdot \chi(z^i \cdot \prod_{j=1}^l \tilde{s}_{i_j})$, for $i \in \mathbb{Z}$ and $1 \leq i_j \leq n-1$ for all j, which induces a bijection between the p-blocks of $\mathrm{Irr}^-(\tilde{\mathcal{S}}_n)$ and the p-blocks of $\mathrm{Irr}^-(\hat{\mathcal{S}}_n)$. There is an analogous bijection on the set of p-modular Brauer characters in these blocks, which by [2, Thm.III.5.12] commutes with the p-modular decomposition map. Hence, by Theorem 4.4, the Brauer trees of the p-blocks of weight 1 of $\hat{\mathcal{S}}_n$ are also as depicted in Tables 1, 2 and 3, except that in Tables 1 and 2 the trees have to be reflected along the main diagonal.

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