ON A REMARKABLE PARTITION IDENTITY

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Abstract. The starting point of this note is a remarkable partition identity, concerning the parts of the partitions of a fixed natural number and the multiplicities with which these parts occur. This identity is related to the ordinary representation theory of the symmetric group. Our main result is a generalization of this identity, being related to the modular representation theory of the symmetric group.

Introduction. The starting point of this note is the remarkable partition identity stated below as Theorem 1. This identity seems to be well-known to combinatorialists, but it is also related to the ordinary representation theory of the symmetric group. We include a proof of Theorem 1 revealing this relationship. Our main result is a generalization of this identity, which is stated below as Theorem 2. While the original identity is related to the ordinary representation theory, its generalization is related to the modular representation theory of the symmetric group. The proof builds on the idea already used in our proof of Theorem 1. The assertion of Theorem 2 gives rise to some numbers called e_σ, whose significance, in particular in relation to the modular representation theory of the symmetric group, in most cases is not clear, except for $e_{[1^n]}$, which are related to the Cartan determinants of the symmetric group, and for which we give a closed combinatorial formula in Theorem 3. Finally, in Theorem 4, we give a similar formula for certain sums of e_σ's, which are related to alternating groups. Let us first fix the necessary

Notation. For $n \in \mathbb{N}_0$ let $\mathcal{P}_n$ denote the set of partitions of n and let $p_n := |\mathcal{P}_n|$ be its cardinality. For a partition $\lambda \in \mathcal{P}_n$ let $[\lambda_1, \lambda_2, \ldots, \lambda_l]$ be the list of its parts, where $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_l > 0$ and $\sum_{i=1}^{l} \lambda_i = n$. Let $l_\lambda = l$ be its length, and let $\lambda_i := 0$ for all $l < i \in \mathbb{N}$. Alternatively, we write $\lambda \in \mathcal{P}_n$ as $[1^{a_1}, 2^{a_2}, \ldots, n^{a_n}]$, where $a_j(\lambda) = a_j := |\{1 \leq i \leq l_\lambda; \lambda_i = j\}|$. The set $\mathcal{P}_n$ can be ordered lexicographically by letting $\lambda >_{lex} \mu$ if and only if for some $j \geq 1$ we have $\lambda_j > \mu_j$, while $\lambda_i = \mu_i$ for all $1 \leq i < j$. Furthermore, let $\mathcal{P}_n^+ := \{\lambda \in \mathcal{P}_n; n - l_\lambda \text{ even}\}$ and $\mathcal{P}_n^- := \mathcal{P}_n \setminus \mathcal{P}_n^+$ denote the sets of even and odd partitions, respectively, and let $p_n^\pm := |\mathcal{P}_n^\pm|$ be their cardinalities.

For $n \in \mathbb{N}$ let $\mathcal{S}_n$ be the symmetric group on n letters. For $\lambda \in \mathcal{P}_n$ let $\chi_\lambda \in \mathrm{Irr}(\mathcal{S}_n)$ denote the corresponding irreducible ordinary character of $\mathcal{S}_n$, see [7, Thm.2.1.11]. For $\mu \in \mathcal{P}_n$ let $\chi_\lambda(\mu) := \chi_\lambda(g_\mu)$, where $g_\mu \in \mathcal{S}_n$ has cycle type μ. Let $\mathcal{X}_n = [\chi_\lambda(\mu); \lambda, \mu \in \mathcal{P}_n] \in \mathbb{Z}^{p_n \times p_n}$ be the ordinary character table of $\mathcal{S}_n$, where its rows

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are ordered reversed lexicographically, i.e. the trivial character $1_{\mathcal{S}_n} = \chi_{[n]}$ is the first one, whereas its columns are ordered lexicographically, i.e. the first column corresponds to cycle type $[1^n]$.

For $\mu \in \mathcal{P}_n$ let $|C_{\mathcal{S}_n}(\mu)| := |C_{\mathcal{S}_n}(g_\mu)|$. For $\lambda \in \mathcal{P}_n$ let $\mathcal{S}_\lambda = \mathcal{S}_{\lambda_1} \times \ldots \times \mathcal{S}_{\lambda_{l(\lambda)}}$ denote a corresponding Young subgroup. Let $1_{\mathcal{S}_\lambda}$ be its trivial character, and let $\xi_\lambda = 1_{\mathcal{S}_\lambda}^{\mathcal{S}_n} \in \mathbb{Z}\mathrm{Irr}(\mathcal{S}_n)$ denote the corresponding permutation character. Let $\Xi_n = \{\xi_\lambda; \lambda \in \mathcal{P}_n\}$ and let $\mathcal{Y}_n = [\xi_\lambda(\mu); \lambda, \mu \in \mathcal{P}_n] \in \mathbb{Z}^{p_n \times p_n}$ be the corresponding character table, where again rows are ordered reversed lexicographically and columns lexicographically.

Theorem 1. For a partition $\lambda \in \mathcal{P}_n$ let $\mathcal{A}_\lambda := \prod_{i=1}^{l_\lambda} \lambda_i$ denote the product of its parts, and $\mathcal{B}_\lambda := \prod_{j=1}^{n} a_j(\lambda)!$ denote the product of the factorials of its exponents. Then we have

$$\prod_{\lambda \in \mathcal{P}_n} \mathcal{A}_\lambda = \prod_{\lambda \in \mathcal{P}_n} \mathcal{B}_\lambda.$$

Proof. By the orthogonality relations, $\mathcal{X}_n^{tr} \cdot \mathcal{X}_n$ is a diagonal matrix with entries $|C_{\mathcal{S}_n}(\mu)|$, where $\mu \in \mathcal{P}_n$. We have $|C_{\mathcal{S}_n}(\mu)| = \mathcal{A}_\mu \cdot \mathcal{B}_\mu$, see [7, La.1.2.15], and hence $|\det(\mathcal{X}_n)|^2 = \prod_{\mu \in \mathcal{P}_n} \mathcal{A}_\mu \cdot \mathcal{B}_\mu$. By [7, Thm.2.2.10], we have $\mathbb{Z}\Xi = \mathbb{Z}\mathrm{Irr}(\mathcal{S}_n)$, and hence $|\det(\mathcal{X}_n)| = |\det(\mathcal{Y}_n)|$.

Let $g_\mu \in \mathcal{S}_\lambda \leq \mathcal{S}_n$. Then we have $\lambda \geq_{lex} \mu$, since clearly $\lambda_1 \geq \mu_1$ holds, and if we have $\lambda_1 = \mu_1$, we proceed by induction. Hence $\mathcal{Y}_n$ is a triangular matrix, the non-vanishing entries being concentrated in the upper left hand corner, and we have $|\det(\mathcal{Y}_n)| = \prod_{\lambda \in \mathcal{P}_n} \xi_\lambda(\lambda)$. We have $\xi_\lambda(\lambda) = \xi_\lambda(g_\lambda)$, which is the number of fixed points of g_λ of its action on the right cosets $\mathcal{S}_\lambda/\mathcal{S}_n$ of $\mathcal{S}_\lambda$ in $\mathcal{S}_n$. Hence $\xi_\lambda(\lambda) = |\{\mathcal{S}_\lambda h \in \mathcal{S}_\lambda/\mathcal{S}_n; g_\lambda \in \mathcal{S}_\lambda^h \cap \mathcal{S}_\lambda\}|$. For $h \in \mathcal{S}_n$ the subgroup $\mathcal{S}_\lambda^h \cap \mathcal{S}_\lambda$ again is a Young subgroup, $\mathcal{S}_\mu$ say, where $\mu \leq_{lex} \lambda$, and $\mu = \lambda$ if and only if $\mathcal{S}_\lambda^h = \mathcal{S}_\lambda$. Hence from $g_\lambda \in \mathcal{S}_\lambda^h \cap \mathcal{S}_\lambda$ having cycle type λ we conclude that $h \in N_{\mathcal{S}_n}(\mathcal{S}_\lambda)$. Thus $\xi_\lambda(\lambda) = [N_{\mathcal{S}_n}(\mathcal{S}_\lambda) : \mathcal{S}_\lambda]$. The normalizer $N_{\mathcal{S}_n}(\mathcal{S}_\lambda)$ permutes the $\mathcal{S}_\lambda$-orbits of equal length. Hence by [7, 4.1.25] we have $|N_{\mathcal{S}_n}(\mathcal{S}_\lambda)| = \prod_{j=1}^{n} (j!)^{a_j(\lambda)} \cdot a_j(\lambda)!$. Thus $[N_{\mathcal{S}_n}(\mathcal{S}_\lambda) : \mathcal{S}_\lambda] = \mathcal{B}_\lambda$ and $|\det(\mathcal{Y}_n)| = \prod_{\lambda \in \mathcal{P}_n} \mathcal{B}_\lambda$. ♯

Remark. A related proof of Theorem 1 is given in [9]. A related proof of the equation $|\det(\mathcal{X}_n)| = \prod_{\lambda \in \mathcal{P}_n} \mathcal{A}_\lambda$ is given in [5, Cor.6.5].

We now prepare the setting for our main result.

Notation and Definition. Let p be a fixed rational prime. For $g \in \mathcal{S}_n$ let $g_p, g_{p'} \in \mathcal{S}_n$ denote its p-part and its p'-part, respectively, i.e. g_p has p-power order, the order of $g_{p'}$ is coprime to p, and $g_p g_{p'} = g_{p'} g_p$ holds. If g has cycle type λ, we denote the cycle type of g_p and $g_{p'}$ by λ_p and $\lambda_{p'}$, respectively. The partitions λ_p and $\lambda_{p'}$ are called the *p-part of* λ and the *p'-part of* λ, respectively.

Let $\sigma \in \mathcal{P}_n$ such that $\sigma = \sigma_p$. Hence $\sigma = [(p^0)^{n_0}, (p^1)^{n_1}, \ldots, (p^k)^{n_k}]$, say, where $\sum_{i=0}^{k} n_i p^i = n$. The set $\mathcal{P}_\sigma = \{\lambda \in \mathcal{P}_n; \lambda_p = \sigma\}$ is called the *p-section of partitions* corresponding to σ. Let $p_\sigma = |\mathcal{P}_\sigma|$ denote its cardinality. The elements of $\mathcal{P}_\sigma$ are the cycle types of the elements in the p-section of $\mathcal{S}_n$ determined by some $g_\sigma \in \mathcal{S}_n$, see [4, Ch.IV.6,p.172]. In particular, $\mathcal{P}_{[1^n]}$ is the set of the cycle types of the p-regular elements of $\mathcal{S}_n$.

Each $\lambda \in \mathcal{P}_\sigma$ is uniquely determined by the tuple $(\lambda^0, \lambda^1, \ldots, \lambda^k)$ of partitions, where $\lambda^j \in \mathcal{P}_{[1^{n_j}]}$, such that $\lambda = [\lambda_1^k p^k, \lambda_2^k p^k, \ldots, \lambda_{l_{\lambda^k}}^k p^k, \lambda_1^{k-1} p^{k-1}, \ldots, \lambda_{l_{\lambda^0}}^0 p^0]$, up to a reordering of the parts. Hence in particular we have $p_\sigma = \prod_{j=1}^k p_{[1^{n_j}]}$.

Each $\lambda \in \mathcal{P}_\sigma$ is also uniquely determined by the pair (λ^0, λ') of partitions, where $\lambda^0 \in \mathcal{P}_{[1^{n_0}]}$ is as above, and $\lambda' \in \mathcal{P}_{n'}$, for $n = n_0 + n'p$, such that up to a reordering of the parts $\lambda = [\lambda_1^0, \ldots, \lambda_{l_{\lambda^0}}^0, p\lambda_1', \ldots, p\lambda_{l_{\lambda'}}']$. This might become clearer by an

Example. For $p = 2$ let $k = 2$ and $\sigma = \sigma_2 = [1^4, 2^3, 4^2] \in \mathcal{P}_{18}$. We have $\mathcal{P}_{[1^2]} = \{[1^2]\}$, $\mathcal{P}_{[1^3]} = \{[1^3], [3]\}$ and $\mathcal{P}_{[1^4]} = \{[1^4], [3, 1]\}$. Hence $p_\sigma = 4$ and the elements $\lambda \in \mathcal{P}_\sigma$ are as follows, where $\lambda^0 \in \mathcal{P}_{[1^4]}$, $\lambda^1 \in \mathcal{P}_{[1^3]}$, $\lambda^2 \in \mathcal{P}_{[1^2]}$ and $\lambda' \in \mathcal{P}_7$.

λ	$(\lambda^0, \lambda^1, \lambda^2)$	(λ^0, λ')
$[4^2, 2^3, 1^4]$	$([1^4], [1^3], [1^2])$	$([1^4], [2^2, 1^3])$
$[6, 4^2, 1^4]$	$([1^4], [3], [1^2])$	$([1^4], [3, 2^2])$
$[4^2, 3, 2^3, 1]$	$([3, 1], [1^3], [1^2])$	$([3, 1], [2^2, 1^3])$
$[6, 4^2, 3, 1]$	$([3, 1], [3], [1^2])$	$([3, 1], [3, 2^2])$

Theorem 2. Let $\sigma \in \mathcal{P}_n$ such that $\sigma = \sigma_p$. Then there exists $e_\sigma \in \mathbb{Z}$ such that

$$p^{e_\sigma} \cdot \prod_{\lambda \in \mathcal{P}_\sigma} \mathcal{A}_\lambda = \prod_{\lambda \in \mathcal{P}_\sigma} \mathcal{B}_\lambda.$$

Proof. Let $\mathcal{X}_\sigma$ denote the submatrix of $\mathcal{X}_n$ consisting of the columns corresponding to $\mathcal{P}_\sigma$, i.e. belonging to the p-section determined by g_σ. Let $C_{\mathcal{S}_n}(\sigma) := C_{\mathcal{S}_n}(g_\sigma)$ and let $\{g_i; 1 \leq i \leq p'_\sigma\}$ be a set of representatives of the p-regular conjugacy classes of $C_{\mathcal{S}_n}(\sigma)$. By [4, Ch.IV.6,p.173] the set $\{g_\sigma g_i; 1 \leq i \leq p'_\sigma\}$ is a set of representatives of the conjugacy classes of $\mathcal{S}_n$ belonging to the p-section determined by g_σ, hence in particular we have $p'_\sigma = p_\sigma$. As $C_{\mathcal{S}_n}(g_\sigma g_i) \leq C_{\mathcal{S}_n}(\sigma)$ we have $|C_{C_{\mathcal{S}_n}(\sigma)}(g_i)| = |C_{\mathcal{S}_n}(g_\sigma g_i)| = |C_{\mathcal{S}_n}(\mu)|$, where $g_\sigma g_i$ has cycle type $\mu \in \mathcal{P}_\sigma$, say.

By [7, 4.1.19] we have $C_{\mathcal{S}_n}(\sigma) \cong \prod_{j=0}^k C_{p^j} \wr \mathcal{S}_{n_j}$, where C_{p^j} denotes the cyclic group of order p^j. Hence the irreducible p-modular Brauer characters $\mathrm{IBr}_p(C_{\mathcal{S}_n}(\sigma))$ are rational valued, and each field of characteristic p is a splitting field for $C_{\mathcal{S}_n}(\sigma)$. Let $\mathcal{Z}_\sigma \in \mathbb{Z}^{p_\sigma \times p_\sigma}$ and $\mathcal{W}_\sigma \in \mathbb{Z}^{p_\sigma \times p_\sigma}$ denote the p-modular Brauer character table and the corresponding table of indecomposable projective characters of $C_{\mathcal{S}_n}(\sigma)$, respectively. By the orthogonality relations we have $\mathcal{Z}_\sigma \cdot M_\sigma^{-1} \cdot \mathcal{W}_\sigma^{tr} = E_{p_\sigma}$, where M_σ is a diagonal matrix with entries $|C_{\mathcal{S}_n}(\mu)|$, where $\mu \in \mathcal{P}_\sigma$. Hence we conclude $|\det(\mathcal{Z}_\sigma)| \cdot |\det(\mathcal{W}_\sigma)| = \prod_{\mu \in \mathcal{P}_\sigma} \mathcal{A}_\mu \cdot \mathcal{B}_\mu$. Furthermore, by [4, Thm.IV.3.11] we have $|\det(\mathcal{W}_\sigma)| = |\det(\mathcal{Z}_\sigma)| \cdot \prod_{\mu \in \mathcal{P}_\sigma} |C_{\mathcal{S}_n}(\mu)|_p = |\det(\mathcal{Z}_\sigma)| \cdot \prod_{\mu \in \mathcal{P}_\sigma} (\mathcal{A}_\mu \cdot \mathcal{B}_\mu)_p$, where the subscript denotes the maximal power of p dividing these rational integers. Hence we obtain $\det(\mathcal{Z}_\sigma)^2 = \prod_{\mu \in \mathcal{P}_\sigma} (\mathcal{A}_\mu \cdot \mathcal{B}_\mu)_{p'}$, where the subscript denotes the maximal divisor coprime to p.

Let the set $\Xi_\sigma = \{\xi_\lambda; \lambda \in \mathcal{P}_\sigma\}$ of permutation characters have the corresponding partial character table $\mathcal{Y}_\sigma = [\xi_\lambda(\mu); \lambda, \mu \in \mathcal{P}_\sigma] \in \mathbb{Z}^{p_\sigma \times p_\sigma}$. As in the proof of Theorem 1 we have $|\det(\mathcal{Y}_\sigma)| = \prod_{\mu \in \mathcal{P}_\sigma} \mathcal{B}_\mu \neq 0$. As the irreducible p-modular Brauer characters $\mathrm{IBr}_p(C_{\mathcal{S}_n}(\sigma))$ are rational valued, we have a generalized decomposition map $\mathbb{Z}\mathrm{Irr}(\mathcal{S}_n) \to \mathbb{Z}\mathrm{IBr}_p(C_{\mathcal{S}_n}(\sigma))$, see [4, Ch.IV.6,p.172]. Hence we conclude that

$\mathcal{Y}_\sigma = D_\sigma \cdot \mathcal{Z}_\sigma$, for some $D_\sigma \in \mathbb{Z}^{p_\sigma \times p_\sigma}$. Hence $\det(\mathcal{Z}_\sigma)$ divides $\det(\mathcal{Y}_\sigma)$. Because of [4, Thm.IV.3.11], we have $\det(\mathcal{Z}_\sigma)_p = 1$, hence $\det(\mathcal{Z}_\sigma)$ even divides $\det(\mathcal{Y}_\sigma)_{p'}$.

Thus $\det(\mathcal{Z}_\sigma)$ divides $\prod_{\mu \in \mathcal{P}_\sigma} (\mathcal{B}_\mu)_{p'}$, and because of $\det(\mathcal{Z}_\sigma)^2 = \prod_{\mu \in \mathcal{P}_\sigma} (\mathcal{A}_\mu \cdot \mathcal{B}_\mu)_{p'}$ we conclude that $\prod_{\mu \in \mathcal{P}_\sigma} (\mathcal{A}_\mu)_{p'}$ divides $\det(\mathcal{Z}_\sigma)$. Taking the product over all p-sections we find that $\prod_{\mu \in \mathcal{P}_n} (\mathcal{A}_\mu)_{p'}$ divides $\prod_{\sigma \in \mathcal{P}_n \sigma = \sigma_p} \det(\mathcal{Z}_\sigma)$, which in turn divides $\prod_{\mu \in \mathcal{P}_n} (\mathcal{B}_\mu)_{p'}$. By Theorem 1 we have $\prod_{\mu \in \mathcal{P}_n} (\mathcal{A}_\mu)_{p'} = \prod_{\mu \in \mathcal{P}_n} (\mathcal{B}_\mu)_{p'}$, from which we finally conclude that $\prod_{\mu \in \mathcal{P}_\sigma} (\mathcal{A}_\mu)_{p'} = |\det(\mathcal{Z}_\sigma)| = \prod_{\mu \in \mathcal{P}_\sigma} (\mathcal{B}_\mu)_{p'}$ holds. ♯

Remark. The above proof reveals some representation theoretic relevance of the numbers $\mathcal{A}_\lambda$ and $\mathcal{B}_\lambda$, if we consider $C_{\mathcal{S}_n}(\sigma) \cong \prod_{j=0}^k C_{p^j} \wr \mathcal{S}_{n_j}$ a bit more closely. Let $D_\sigma := \prod_{j=0}^k C_{p^j}^{n_j} \trianglelefteq C_{\mathcal{S}_n}(\sigma)$ be the product of the base groups, and let $\mathcal{C}_\sigma \in \mathbb{Z}^{p_\sigma \times p_\sigma}$ and $\overline{\mathcal{C}}_\sigma \in \mathbb{Z}^{p_\sigma \times p_\sigma}$ denote the Cartan matrices of $C_{\mathcal{S}_n}(\sigma)$ and $C_{\mathcal{S}_n}(\sigma)/D_\sigma \cong \prod_{j=0}^k \mathcal{S}_{n_j}$, respectively. Then we have

$$\det(\mathcal{C}_\sigma) = |\det(\mathcal{W}_\sigma)|/|\det(\mathcal{Z}_\sigma)| = \prod_{\lambda \in \mathcal{P}_\sigma} (\mathcal{A}_\lambda \cdot \mathcal{B}_\lambda)_p,$$

as $\det(\mathcal{C}_\sigma)$ is positive by [4, Cor.I.17.9]. In particular, for $\sigma = [1^n]$ we have $C_{\mathcal{S}_n}([1^n]) = \mathcal{S}_n$ and $\mathcal{C}_{[1^n]}$ is the Cartan matrix of $\mathcal{S}_n$. As $(\mathcal{A}_\lambda)_p = 1$ for $\lambda \in \mathcal{P}_{[1^n]}$, we obtain an expression for the Cartan determinant of $\mathcal{S}_n$ in terms of $e_{[1^n]}$ as

$$\det(\mathcal{C}_{[1^n]}) = \prod_{\lambda \in \mathcal{P}_{[1^n]}} (\mathcal{B}_\lambda)_p = p^{e_{[1^n]}}.$$

If $\lambda \in \mathcal{P}_\sigma$ is given by $(\lambda^0, \lambda^1, \ldots, \lambda^k)$, we have $\mathcal{A}_\lambda = \prod_{j=0}^k (p^{j \cdot l_{\lambda^j}} \cdot \mathcal{A}_{\lambda^j})$ and $\mathcal{B}_\lambda = \prod_{j=0}^k \mathcal{B}_{\lambda^j}$. Thus in general we get

$$\det(\overline{\mathcal{C}}_\sigma) = \prod_{\lambda \in \mathcal{P}_\sigma} (\mathcal{B}_\lambda)_p,$$

since the left hand side equals $\det(\mathcal{C}_{[1^{n_0}]} \otimes \cdots \otimes \mathcal{C}_{[1^{n_k}]}) = \prod_{j=0}^k \det(\mathcal{C}_{[1^{n_j}]})^{p_\sigma / p_{[1^{n_j}]}} = \prod_{j=0}^k \prod_{\lambda^j \in \mathcal{P}_{[1^{n_j}]}} (\mathcal{B}_{\lambda^j})_p^{p_\sigma / p_{[1^{n_j}]}}$, which equals the right hand side. Furthermore, if $g_i \in C_{\mathcal{S}_n}(\sigma)$ such that $g_\sigma g_i$ has cycle type $\lambda \in \mathcal{P}_\sigma$, then we have $|C_{D_\sigma}(g_i)| = \prod_{j=0}^k p^{j \cdot l_{\lambda^j}} = (\mathcal{A}_\lambda)_p$, hence

$$\prod_{i=1}^{p_\sigma} |C_{D_\sigma}(g_i)| = \prod_{\lambda \in \mathcal{P}_\sigma} (\mathcal{A}_\lambda)_p.$$

Thus altogether we obtain $\det(\mathcal{C}_\sigma) = \det(\overline{\mathcal{C}}_\sigma) \cdot \prod_{i=1}^{p_\sigma} |C_{D_\sigma}(g_i)|$, which of course is a special case of the Alperin-Collins-Sibley Theorem, see [2] or [8, Thm.III.8.1].

However, the significance of the integers e_σ for arbitrary $\sigma = \sigma_p$ is unclear. Anyway, the first few small cases for $p = 2$ are given in Table 1. In general, we have

$$\prod_{\lambda \in \mathcal{P}_\sigma} \frac{\mathcal{B}_\lambda}{\mathcal{A}_\lambda} = \prod_{\lambda^0 \in \mathcal{P}_{[1^{n_0}]}} \cdots \prod_{\lambda^k \in \mathcal{P}_{[1^{n_k}]}} \left(\prod_{j=0}^k p^{-j \cdot l_{\lambda^j}} \cdot \frac{\mathcal{B}_{\lambda^j}}{\mathcal{A}_{\lambda^j}} \right)$$

and thus

$$e_\sigma = \sum_{j=0}^k \left(\frac{p_\sigma}{p_{[1^{n_j}]}} \cdot \left(e_{[1^{n_j}]} - j \cdot \sum_{\lambda^j \in \mathcal{P}_{[1^{n_j}]}} l_{\lambda^j} \right) \right).$$

TABLE 1. Small cases for $p = 2$ and $1 \leq n \leq 8$.

$\sigma = \sigma_2$	e_σ
$[1]$	0
$[1^2]$	1
$[2]$	-1
$[1^3]$	1
$[2, 1]$	-1
$[1^4]$	3
$[2, 1^2]$	0
$[2^2]$	-1
$[4]$	-2
$[1^5]$	4
$[2, 1^3]$	-1
$[2^2, 1]$	-1
$[4, 1]$	-2

$\sigma = \sigma_2$	e_σ
$[1^6]$	6
$[2, 1^4]$	1
$[2^2, 1^2]$	0
$[2^3]$	-3
$[4, 1^2]$	-1
$[4, 2]$	-3
$[1^7]$	9
$[2, 1^5]$	1
$[2^2, 1^3]$	-1
$[2^3, 1]$	-3
$[4, 1^3]$	-3
$[4, 2, 1]$	-3

$\sigma = \sigma_2$	e_σ
$[1^8]$	13
$[2, 1^6]$	2
$[2^2, 1^4]$	1
$[2^3, 1^2]$	-1
$[2^4]$	-3
$[4, 1^4]$	-1
$[4, 2, 1^2]$	-2
$[4, 2^2]$	-3
$[4^2]$	-3
$[8]$	-3

Corollary. There exists $e_n^+ \in \mathbb{Z}$ such that

$$2^{e_n^+} \cdot \prod_{\lambda \in \mathcal{P}_n^+} \mathcal{A}_\lambda = \prod_{\lambda \in \mathcal{P}_n^+} \mathcal{B}_\lambda.$$

Proof. As $\lambda \in \mathcal{P}_n$ is an even partition if and only if its 2-part is, $\mathcal{P}_n^+$ is a union of 2-sections. Hence the assertion follows from Theorem 2 for $p = 2$, and we have

$$e_n^+ = \sum_{\sigma = \sigma_2 \in \mathcal{P}_n^+} e_\sigma.$$

♯

In the rest of this note we determine closed combinatorial formulas for $e_{[1^n]}$ in Theorem 3, and e_n^+ in Theorem 4. The main tool are so-called generating functions, see e.g. [1, Ch.V]. The necessary prerequisites are stated next. To make this note sufficiently self-contained, we include proofs for the non-obvious facts on generating functions needed here, which probably are well-known to combinatorialists.

Remarks on generating functions. If $\mathbb{N}_0 \to \mathbb{Q}\colon n \mapsto \alpha_n$ is an arbitrary function, then its ordinary generating function is $\alpha(X) := \sum_{k \geq 0} \in \alpha_k X^k \in \mathbb{Q}[[X]]$, where $\mathbb{Q}[[X]]$ is the ring of formal power series in the indeterminate X over $\mathbb{Q}$. In particular, $\mathbb{Q}[[X]]$ is a complete discrete valuation ring in its quotient field $\mathbb{Q}((X))$, with maximal ideal $X \cdot \mathbb{Q}[[X]]$. In particular, $\alpha(X) \in \mathbb{Q}[[X]]$ is invertible if and only if $\alpha_0 \neq 0$. Note that by this formalism the elements of $\mathbb{Q}[X]]$ are not functions in the analytical sense, which saves us from convergence considerations. But usually we will consider infinite sums or products, hence in these cases we have to ensure that the coefficients of the resulting function are determined by a finite number of these summands or factors.

For $k \in \mathbb{N}$ we have $\frac{1}{1-X^k} = \sum_{l \geq 0} X^{kl} \in \mathbb{Q}[[X]]$. From that we immediately deduce the following: The generating function for the number $d_n := |\{m \in \mathbb{N}; m \mid n\}|$ of divisors of $n \in \mathbb{N}$ is given as $d(X) := \sum_{k \geq 1} \frac{X^k}{1-X^k} \in \mathbb{Q}[[X]]$. The generating function for the number p_n of partitions of $n \in \mathbb{N}_0$ is given as $p(X) := \prod_{k \geq 1} \frac{1}{1-X^k} \in \mathbb{Q}[[X]]$, see also [1, Ex.V.2.6]. The generating function for the number $p_{[1^n]}$ of partitions of $n \in \mathbb{N}_0$ into parts which are not divisible by the rational prime p is given as $p_1(X) := \prod_{k \geq 1, p \nmid k} \frac{1}{1-X^k} \in \mathbb{Q}[[X]]$. The generating function for the number $p_{n,\mathrm{sd}}$ of self-conjugate partitions of $n \in \mathbb{N}_0$, which by [1, Thm.III.1.20] equals the number of partitions of n into pairwise distinct odd parts, is by [1, Ex.V.2.8] given as $p_{\mathrm{sd}}(X) := \prod_{k \geq 1, 2 \nmid k} (1 + X^k) \in \mathbb{Q}[[X]]$.

Lemma. a) For $n \in \mathbb{N}_0$ let $\sigma_{n,l} := \sum_{\lambda \in \mathcal{P}_n} l_\lambda$ be the sum of the lengths of the partitions of n. Then the generating function for $\sigma_{n,l}$ is given as

$$\sigma_l(X) := \left(\sum_{k \geq 1} \frac{X^k}{1-X^k} \right) \cdot \left(\prod_{k \geq 1} \frac{1}{1-X^k} \right) \in \mathbb{Q}[[X]].$$

In particular, we have $\sigma_l(X) = d(X) \cdot p(X)$.

b) For $n \in \mathbb{N}_0$ let $p_{n,\mathrm{even}} := |\{\lambda \in \mathcal{P}_n; l_\lambda \text{ even}\}|$ be the number of partitions of n of even length. Then the generating function for $p_{n,\mathrm{even}}$ is given as

$$p_{\mathrm{even}}(X) := \frac{1}{2} \left(\prod_{k \geq 1} \frac{1}{1-X^k} + \prod_{k \geq 1} \frac{1}{1+X^k} \right) \in \mathbb{Q}[[X]].$$

c) For $n \in \mathbb{N}_0$ let $\sigma_{n,\mathrm{even}} := \sum_{\lambda \in \mathcal{P}_n, l_\lambda \mathrm{even}} l_\lambda$ be the sum of the lengths of the partitions of n having even length. Then the generating function for $\sigma_{n,\mathrm{even}}$ is given as

$$\sigma_{\mathrm{even}}(X) = \frac{1}{2} \left(\sum_{k \geq 1} \frac{X^k}{1-X^k} \cdot \prod_{k \geq 1} \frac{1}{1-X^k} - \sum_{k \geq 1} \frac{X^k}{1+X^k} \cdot \prod_{k \geq 1} \frac{1}{1+X^k} \right) \in \mathbb{Q}[[X]].$$

Proof. a) For $n, l \in \mathbb{N}_0$ let $p_{n,l} := |\{\lambda \in \mathcal{P}_n; l_\lambda = l\}|$. Its two-variable generating function is $\sum_{n \geq 0, l \geq 0} p_{n,l} X^n Y^l = \prod_{k \geq 1} \frac{1}{1-X^k Y} \in \mathbb{Q}[[X,Y]]$. Hence, using the formal partial derivative $\frac{\partial}{\partial Y} : \mathbb{Q}[[X,Y]] \to \mathbb{Q}[[X,Y]]$, we obtain

$$\sum_{n \geq 0, l \geq 0} l \cdot p_{n,l} X^n Y^l = Y \cdot \frac{\partial}{\partial Y} \left(\sum_{n \geq 0, l \geq 0} p_{n,l} X^n Y^l \right) = Y \cdot \frac{\partial}{\partial Y} \left(\prod_{k \geq 1} \frac{1}{1-X^k Y} \right).$$

Using the Leibniz rule, we get

$$\frac{\partial}{\partial Y} \left(\prod_{k \geq 1} \frac{1}{1-X^k Y} \right) = \left(\sum_{k \geq 1} \frac{X^k}{1-X^k Y} \right) \cdot \left(\prod_{k \geq 1} \frac{1}{1-X^k Y} \right).$$

As $\sigma_l(X) = \sum_{n \geq 0} (\sum_{l \geq 0} l \cdot p_{n,l}) X^n$, the assertion is obtained by substituting $\mathbb{Q}[[X,Y]] \to \mathbb{Q}[[X]] \colon X \mapsto X, Y \mapsto 1$.

b) We have $\sum_{n \geq 0, l \geq 0} p_{n,2l} X^n Y^{2l} = \frac{1}{2} \sum_{n \geq 0, l \geq 0} p_{n,l} X^n (Y^l + (-Y)^l)$, which equals $\frac{1}{2} (\prod_{k \geq 1} \frac{1}{1-X^k Y} + \prod_{k \geq 1} \frac{1}{1+X^k Y})$. As $p_{n,\mathrm{even}} = \sum_{l \geq 0} p_{n,2l}$, the assertion follows again from substituting $Y \mapsto 1$.

c) Applying the same technique as in the proof of a) to $\sum_{n\geq 0, l\geq 0} 2l \cdot p_{n,2l} X^n Y^{2l}$ and using the proof of b) yields the assertion. ♯

Theorem 3. The generating function $e_1(X) \in \mathbb{Q}[[X]]$ for $e_{[1^n]}$ is given as

$$e_1(X) = p_1(X) \cdot d(X^p),$$

i.e. for $n \in \mathbb{N}_0$ we have $e_{[1^n]} = \sum_{n'=1}^{\lfloor \frac{n}{p} \rfloor} p_{[1^{n-pn'}]} \cdot d_{n'}$.

Proof. We use the above description of $\lambda \in \mathcal{P}_n$ by a pair (λ^0, λ') of partitions $\lambda^0 \in \mathcal{P}_{[1^{n_0}]}$ and $\lambda' \in \mathcal{P}_{n'}$, where $n = n_0 + pn'$. By Theorem 1 we have

$$1 = \prod_{\lambda \in \mathcal{P}_n} \frac{\mathcal{B}_\lambda}{\mathcal{A}_\lambda} = \prod_{n'=0}^{\lfloor \frac{n}{p} \rfloor} \left(\prod_{\lambda' \in \mathcal{P}_{n'}, \lambda^0 \in \mathcal{P}_{[1^{n_0}]}, n = n_0 + pn'} p^{-l_{\lambda'}} \cdot \frac{\mathcal{B}_{\lambda'}}{\mathcal{A}_{\lambda'}} \cdot \frac{\mathcal{B}_{\lambda^0}}{\mathcal{A}_{\lambda^0}} \right).$$

The bracketed term equals

$$\left(\prod_{\lambda' \in \mathcal{P}_{n'}} p^{-l_{\lambda'}} \right)^{p_{[1^{n_0}]}} \cdot \left(\prod_{\lambda' \in \mathcal{P}_{n'}} \frac{\mathcal{B}_{\lambda'}}{\mathcal{A}_{\lambda'}} \right)^{p_{[1^{n_0}]}} \cdot \left(\prod_{\lambda^0 \in \mathcal{P}_{[1^{n_0}]}} \frac{\mathcal{B}_{\lambda^0}}{\mathcal{A}_{\lambda^0}} \right)^{p_{n'}},$$

where, by Theorems 1 and 2, the second factor equals 1 and the third factor equals $p^{e_{[1^{n_0}]} \cdot p_{n'}}$. This yields

$$1 = \prod_{n'=0}^{\lfloor \frac{n}{p} \rfloor} \left(p^{e_{[1^{n_0}]} \cdot p_{n'}} \cdot \left(\prod_{\lambda' \in \mathcal{P}_{n'}} p^{-l_{\lambda'} \cdot p_{[1^{n_0}]}} \right) \right).$$

Taking logarithms we hence have

$$\sum_{n'=0}^{\lfloor \frac{n}{p} \rfloor} p_{n'} \cdot e_{[1^{n-pn'}]} = \sum_{n'=0}^{\lfloor \frac{n}{p} \rfloor} \left(p_{[1^{n-pn'}]} \cdot \sum_{\lambda' \in \mathcal{P}_{n'}} l_{\lambda'} \right).$$

Thus by the convolution multiplication on $\mathbb{Q}[[X]]$, the last formula translates into $e_1(X) \cdot p(X^p) = p_1(X) \cdot \sigma_l(X^p)$. By the Lemma this gives $e_1(X) = p_1(X) \cdot d(X^p)$, which by the convolution multiplication yields the second assertion. ♯

Theorem 4. The generating function $e^+(X) \in \mathbb{Q}[[X]]$ for e_n^+ is given as

$$e^+(X) = p_{\text{sd}}(X) \cdot d(X^2),$$

i.e. for $n \in \mathbb{N}_0$ we have $e_n^+ = \sum_{n'=1}^{\lfloor \frac{n}{2} \rfloor} p_{n-2n',\text{sd}} \cdot d_{n'}$.

Proof. As in the proof of Theorem 3, we use the description of $\lambda \in \mathcal{P}_n$ by a pair (λ^0, λ') of partitions $\lambda^0 \in \mathcal{P}_{[1^{n_0}]}$ and $\lambda' \in \mathcal{P}_{n'}$, where this time $p = 2$ and hence $n = n_0 + 2n'$. We have $\lambda \in \mathcal{P}_n^+$ if and only if $l_{\lambda'}$ is even. Hence we get

$$2^{e_n^+} = \prod_{n'=0}^{\lfloor \frac{n}{2} \rfloor} \left(\prod_{\lambda' \in \mathcal{P}_{n'}, l_{\lambda'} \text{ even}, \lambda^0 \in \mathcal{P}_{[1^{n_0}]}, n = n_0 + 2n'} 2^{-l_{\lambda'}} \cdot \frac{\mathcal{B}_{\lambda'}}{\mathcal{A}_{\lambda'}} \cdot \frac{\mathcal{B}_{\lambda^0}}{\mathcal{A}_{\lambda^0}} \right),$$

where the bracketed term equals

$$\left(\prod_{\lambda' \in \mathcal{P}_{n'}, l_{\lambda'} \text{ even}} 2^{-l_{\lambda'}} \right)^{p_{[1^{n_0}]}} \cdot \left(\prod_{\lambda' \in \mathcal{P}_{n'}, l_{\lambda'} \text{ even}} \frac{\mathcal{B}_{\lambda'}}{\mathcal{A}_{\lambda'}} \right)^{p_{[1^{n_0}]}} \cdot \left(\prod_{\lambda^0 \in \mathcal{P}_{[1^{n_0}]}} \frac{\mathcal{B}_{\lambda^0}}{\mathcal{A}_{\lambda^0}} \right)^{p_{n',\text{even}}}.$$

As $l_{\lambda'}$ is even, we have $\lambda' \in \mathcal{P}^+_{n'}$ if and only if n' is even. Hence by Theorem 1 we have

$$\prod_{\lambda' \in \mathcal{P}_{n'}, l_{\lambda'} \text{ even}} \frac{\mathcal{B}_{\lambda'}}{\mathcal{A}_{\lambda'}} = 2^{(-1)^{n'} \cdot e^+_n}.$$

Taking logarithms we thus obtain

$$e^+_n = \sum_{n'=0}^{\lfloor \frac{n}{2} \rfloor} \left(e_{[1^{n-2n'}]} \cdot p_{n',\text{even}} + \left((-1)^{n'} \cdot e^+_{n'} - \sum_{\lambda' \in \mathcal{P}_{n'}, l_{\lambda'} \text{ even}} l_{\lambda'} \right) \cdot p_{[1^{n-2n'}]} \right).$$

We have $e^+(-X^2) = \sum_{k \geq 0} (-1)^k e^+_k X^{2k}$. From that, we translate the last formula into a recursion formula for the generating function $e^+(X)$ as

$$e^+(X) = e_1(X) \cdot p_{\text{even}}(X^2) + \left(e^+(-X^2) - \sigma_{\text{even}}(X^2) \right) \cdot p_1(X).$$

We let $g(X) := e^+(X)/p_{\text{sd}}(X)$. Note that indeed $p_{\text{sd}}(X) \in \mathbb{Q}[[X]]$ is invertible. Using the explicit expressions for the generating functions involved, the above recursion formula becomes $g(X) = g(-X^2) + \sum_{k \geq 1} \frac{X^{2k}}{1-X^{4k}}$.

Thus $g(X)$ has a non-vanishing coefficient at X^k only if $k \equiv 0 \bmod 2$, and hence $g(-X^2)$ has a non-vanishing coefficient at X^k only if $k \equiv 0 \bmod 4$. Furthermore, $\sum_{k \geq 1} \frac{X^{2k}}{1-X^{4k}}$ has a non-vanishing coefficient at X^k only if $k \equiv 2 \bmod 4$. Thus the coefficients of $g(X)$ at X^k for $k \equiv 2 \bmod 4$ are equal to those of $\sum_{k \geq 1} \frac{X^{2k}}{1-X^{4k}}$. By induction this determines the coefficients of $g(X)$ at X^k for $k \equiv 0 \bmod 4$. Hence the above recursion formula for $g(X)$ admits a unique solution. Finally, it is easily verified that the recursion formula is fulfilled by $d(X^2)$. ♯

Remark. The formula in Theorem 3 has also been proved independently in [3, Thm.3.3]. G. James [6] has let me know that he also has a proof of the formula in Theorem 4.

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