On the ring of invariants of ordinary quartic curves in characteristic 2

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Abstract

The moduli space of the ordinary non-singular quartic curves over fields of characteristic 2 is isomorphic to a certain open subset of an affine variety, whose coordinate ring in turn is given as the invariant algebra of a certain module of the finite group $GL_3(\mathbb{F}_2)$. We derive a complete description of this invariant algebra by combining theoretical analysis with application of specially tailored computational techniques.
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1 Introduction

Traditionally, non-singular curves of a fixed genus g are classified in two categories, those which are hyperelliptic and those which are not. The first ones are usually considered to be the simplest, since their geometry relies on their Weierstraß points and is condensed on a line. Thus, the hyperelliptic locus $\mathcal{M}_g^{\mathrm{h}}$ of the corresponding moduli space $\mathcal{M}_g$ is easily described and well understood. Seen from an invariant theoretical viewpoint, this is reflected by the fact that one only needs to deal with binary forms. To the contrary, non-hyperelliptic curves are more involved. Even in the simplest case, non-singular quartic curves over $\mathbb{C}$, no complete description of the relevant invariant algebra is known. More precisely, for the invariant algebra of the natural action of $SL_3(\mathbb{C})$ on the vector space of homogeneous polynomials of degree 4 in 3 variables a set of primary invariants, see [6], but no complete algebra generating set is known; a conjecture of Shioda says that the invariant algebra is generated as an algebra by 13 elements.

Hence the question arises whether we can describe the situation more precisely over other fields? If F is a finite field of characteristic 2, in [10] a complete classification of the F-isomorphism types of non-singular quartic curves defined over F has been obtained. Moreover, the stratification of the non-hyperelliptic locus $\mathcal{M}_3^{\mathrm{nh}}$ of the moduli space $\mathcal{M}_3$ with respect to the 2-rank of the Jacobian, and the F-rational points on the various strata, have been described there. Here, the generic case is the one of ordinary non-singular quartic curves, where the 2-rank of the Jacobian is maximal, hence equal to 3. In [10], a precise description of the ordinary non-singular quartic curves is given, and it is shown that the invariant algebra associated to their moduli space $\mathcal{M}_3^{\mathrm{ord}}$ is given by a linear action of the finite group $G := GL_3(\mathbb{F}_2)$ on a certain 6-dimensional $\mathbb{F}_2$-vector

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space W'^*. In the present paper we give a complete description of the invariant algebra $S[W'^*]^G$, answering the corresponding question posed in [10].

Despite the precise description of G and W'^*, being suitable to be handled by computer algebra systems, brute force computer calculations to find primary and secondary invariants of $S[W'^*]^G$, using the standard press-button algorithms available in computer algebra systems as well as up to 2 Gigabytes of memory and several hours of computing time, had to be abandoned unsuccessfully. Hence the strategy employed here is to intertwine theoretical and computational analysis of $S[W'^*]^G$, which in effect leads both to some structural understanding of $S[W'^*]^G$ and finally to explicitly given invariants. Actually, the theoretical analysis indicates how to combine ideas from computational invariant theory and available tools to obtain specially tailored techniques applicable to the examples at hand. The computations have been carried out using the computer algebra systems MAGMA [4] and GAP [8]. After all, to check the correctness of the results only needs a few seconds of computing time and approximately 10 Megabytes of memory. More details of the computations, a MAGMA input file, as well as the primary and secondary invariants calculated, can be found under `http://www.math.jussieu.fr/~ritzenth`.

More precisely, W'^* turns out to be a trivial source G-module, and the algebra $S[W'^*]^G$ turns out to be Cohen-Macaulay. An optimal set of primary invariants has degrees $\{2, 3, 3, 4, 6, 7\}$, and a corresponding minimal set of secondary invariants has cardinality 18. Moreover, $S[W'^*]^G$ is generated as an algebra by at most 11 invariants, namely the 6 primary invariants and 5 of the secondary invariants, the latter having degrees $\{4, 5, 5, 6, 7\}$. Hence in particular $S[W'^*]^G$ is generated by invariants of degree at most 7. By the way, the number of generators rings a bell: The authors wonder whether there is a connection to Shioda's conjecture mentioned above.

This paper is organised as follows: In Section 2 we prepare the setting on ordinary quartic curves, recall the necessary facts from [10], and exhibit the G-module W'^* whose invariant algebra $S[W'^*]^G$ we are interested in. In Section 3 we give a general description of the specially tailored techniques needed to deal with $S[W'^*]^G$, whose analysis subsequently is carried out in Section 4. We assume the reader familiar with the basic notions of commutative algebra, in particular Cohen-Macaulay rings, and of the invariant theory of finite groups; as general references see e. g. [2, 5]. We consider right group actions throughout, in accordance with the assumptions in the computer algebra systems used.

2 Ordinary quartic curves

(2.1) Let $\mathbb{F}_2$ be the field with 2 elements, and let $\overline{\mathbb{F}}_2$ be its algebraic closure. Let $M := \mathbb{F}_2^3$, considered as row vector space, and let $\overline{M} := M \otimes_{\mathbb{F}_2} \overline{\mathbb{F}}_2$. Let $W := \mathbb{F}_2^7$ and $\overline{W} := W \otimes_{\mathbb{F}_2} \overline{\mathbb{F}}_2$, and let $\mathcal{C} := \{C_{a,b,c,d,e,f,g}; 0 \neq [a, b, c, d, e, f, g] \in \overline{W}\}$ be the family of quartic curves given by

$$C_{a,b,c,d,e,f,g} \colon Q^2_{a,b,c,d,e,f} = g^2 \cdot xyz(x + y + z),$$

where $Q_{a,b,c,d,e,f} := ax^2+by^2+cz^2+dxy+eyz+fzx$. This family is important because of the following

(2.2) Proposition. See [10, Prop.1.1].
Let C be a non-singular quartic curve defined over $\overline{\mathbb{F}}_2$. Then the following conditions are equivalent:
i) The Jacobian variety J_C of C is ordinary, i. e. we have $|J_C[2](\overline{\mathbb{F}}_2)| = 2^3$.
ii) The curve C has 7 bitangents.
iii) The curve C is isomorphic to some curve $C_{a,b,c,d,e,f,g} \in \mathcal{C}$ such that

$$(*) \quad abcg(a+b+d)(b+c+e)(a+c+f)(a+b+c+d+e+f+g) \neq 0. \qquad \sharp$$

Moreover, non-singular quartic curves C and C' are isomorphic, if and only if there is an element $\gamma \in GL_3(\overline{\mathbb{F}}_2)$ such that $C^\gamma = C'$, where $\mathbb{P}(\overline{M})$ is considered as the natural right module for the projective general linear group $PGL_3(\overline{\mathbb{F}}_2)$ of rank 3 over $\overline{\mathbb{F}}_2$, and C^γ is the curve defined by

$$V(C^\gamma) := \{[x,y,z] \in \mathbb{P}(\overline{M}); [x,y,z]\cdot\gamma^{-1} \in V(C)\},$$

where $V(\cdot)$ denotes the locus of points of the curve. Using the action on the 7 bitangents, the isomorphism issue is reduced to the finite group $G := GL_3(\mathbb{F}_2) = PGL_3(\mathbb{F}_2) < PGL_3(\overline{\mathbb{F}}_2)$. Indeed, non-singular curves $C_{a,b,c,d,e,f,g} \in \mathcal{C}$ and $C_{a',b',c',d',e',f',g'} \in \mathcal{C}$ are isomorphic, if and only if there is an element $\gamma \in G$ such that $(C_{a,b,c,d,e,f,g})^\gamma = C_{a',b',c',d',e',f',g'}$, see [10].

(2.3) In the sequel let $G := GL_3(\mathbb{F}_2)$ be the general linear group of degree 3 over $\mathbb{F}_2$, which up to isomorphism is the unique simple group of order 168, see [3, p.3]. Let $A, B, C \in G$ be the elements of order 2, 3 and 7, respectively, defined as

$$A := \begin{bmatrix} 1 & . & 1 \\ . & 1 & . \\ . & . & 1 \end{bmatrix}, \quad B := \begin{bmatrix} . & . & 1 \\ 1 & . & . \\ . & 1 & . \end{bmatrix}, \quad C := \begin{bmatrix} . & 1 & . \\ . & . & 1 \\ 1 & 1 & . \end{bmatrix}.$$

It is easily checked using GAP or MAGMA that $G = \langle A, B\rangle$. The $\mathbb{F}_2$-vector space M can be considered as the natural right module for G. As the above action of $PGL_3(\overline{\mathbb{F}}_2)$ restricts to an action of G on $\mathcal{C}$, it is easily checked that we get an $\mathbb{F}_2$-linear action of G on W as

$$D_W: \quad A \mapsto \left[\begin{array}{ccc|ccc|c} 1 & . & . & . & . & . & . \\ . & 1 & . & . & . & . & . \\ 1 & . & 1 & . & . & . & . \\ \hline . & . & . & 1 & . & . & . \\ . & . & . & 1 & 1 & . & . \\ 1 & . & . & . & . & 1 & . \\ \hline . & . & . & 1 & . & . & 1 \end{array}\right], \quad B \mapsto \left[\begin{array}{ccc|ccc|c} . & . & 1 & . & . & . & . \\ 1 & . & . & . & . & . & . \\ . & 1 & . & . & . & . & . \\ \hline . & . & . & . & . & 1 & . \\ . & . & . & 1 & . & . & . \\ . & . & . & . & 1 & . & . \\ \hline . & . & . & . & . & . & 1 \end{array}\right].$$

The $\mathbb{F}_2$-subspace $W' := \{[a,b,c,d,e,f,g] \in W; g = 0\} < W$ is a G-submodule of W, and the above matrices show that $W/W' \cong \mathbb{F}_2$ is the trivial G-module. Hence we have an extension of G-modules

$$(**) \qquad \{0\} \to W' \to W \to \mathbb{F}_2 \to \{0\}.$$

Moreover, there is a G-submodule $W'' < W'$, such that $\dim_{\mathbb{F}_2}(W'') = 3$, as is also indicated above. The characteristic polynomials of the action of $C \in G$ on W'' and W'/W'' are $t^3+t+1 \in \mathbb{F}_2[t]$ and $t^3+t^2+1 \in \mathbb{F}_2[t]$, respectively, which both are irreducible. Hence by [9, p.3] we conclude that W'' and W'/W'' are non-isomorphic absolutely irreducible G-modules, where $W'' \cong M$ is isomorphic to the natural representation of G, and W'/W'' is obtained from W'' by applying the automorphism of G given by inverting and transposing matrices.

(2.4) Let $\eta \in Z^1(G, W')$ be the cocycle describing the extension $(**)$, and let $\mathcal{S}_4 \cong H < G$ be the subgroup permuting the set $\{x^*, y^*, z^*, (x+y+z)^*\} \subseteq M^*$, where $M^* := \mathrm{Hom}_{\mathbb{F}_2}(M, \mathbb{F}_2)$ is the G-module contragredient to M and $\{x^*, y^*, z^*\} \subseteq M^*$ is the $\mathbb{F}_2$-basis dual to the standard basis of M.

By construction, for the restriction of η to the subgroup H we have $\eta|_H = 0 \in Z^1(H, W')$. As $[G\colon H] = 7$ is invertible in $\mathbb{F}_2$, we by [1, Cor.3.6.18] conclude that $\eta = 0 \in H^1(G, W') \cong \mathrm{Ext}^1_G(\mathbb{F}_2, W')$. Thus the extension $(**)$ splits, and we have $W = W' \oplus \mathbb{F}_2$ as G-modules. More concretely, going over from the standard basis of W to the $\mathbb{F}_2$-basis where the last standard basis vector is replaced by $[1,1,1,1,1,1,1] \in W$, we indeed obtain

$$D'_W\colon \quad A \mapsto \left[\begin{array}{ccc|ccc|c} 1 & . & . & . & . & . & . \\ . & 1 & . & . & . & . & . \\ 1 & . & 1 & . & . & . & . \\ \hline . & . & . & 1 & . & . & . \\ . & . & . & 1 & 1 & . & . \\ 1 & . & . & . & . & 1 & . \\ \hline . & . & . & . & . & . & 1 \end{array}\right], \quad B \mapsto \left[\begin{array}{ccc|ccc|c} . & . & 1 & . & . & . & . \\ 1 & . & . & . & . & . & . \\ . & 1 & . & . & . & . & . \\ \hline . & . & . & . & . & 1 & . \\ . & . & . & 1 & . & . & . \\ . & . & . & . & 1 & . & . \\ \hline . & . & . & . & . & . & 1 \end{array}\right].$$

Actually, this basis change amounts to substituting the curves $C_{a,b,c,d,e,f,g} \in \mathcal{C}$ by curves defined by

$$(ax^2 + by^2 + cz^2 + dxy + eyz + fzx)^2 = g^2 \cdot C_K(x,y,z),$$

where $C_K(x,y,z) = x^4+y^4+z^4+(xy)^2+(yz)^2+(zx)^2+xyz(x+y+z)$. We have $x^{*4}+y^{*4}+z^{*4}+(x^*y^*)^2+(y^*z^*)^2+(z^*x^*)^2+x^*y^*z^*(x^*+y^*+z^*) \in S[M^*]^G$ by construction, where $S[M^*]^G \subseteq S[M^*]$ denotes the algebra of G-invariants in the symmetric algebra $S[M^*]$ over M^*. As for the homogeneous component $S[M^*]^G_4$ of $S[M^*]^G$ of degree 4 we have $\dim_{\mathbb{F}_2}(S[M^*]^G_4) = 1$, see [7], the curve $C_K \subseteq \mathbb{P}(\overline{M})$ is a twist of the Klein quartic curve.

As the extension $(**)$ splits, for the corresponding contragredient G-modules we have $W^* \cong W'^* \oplus \mathbb{F}_2^*$. With respect to the $\mathbb{F}_2$-basis $\{a^*, b^*, c^*, d^*, e^*, f^*\} \subseteq W'^*$

dual to the standard basis of W', the G-action on W'^* is given as

$$D_{W'^*}\colon \quad A \mapsto \left[\begin{array}{ccc|ccc} 1 & . & 1 & . & . & 1 \\ . & 1 & . & . & . & . \\ . & . & 1 & . & . & . \\ \hline . & . & . & 1 & 1 & . \\ . & . & . & . & 1 & . \\ . & . & . & . & . & 1 \end{array}\right], \quad B \mapsto \left[\begin{array}{ccc|ccc} . & . & 1 & . & . & . \\ 1 & . & . & . & . & . \\ . & 1 & . & . & . & . \\ \hline . & . & . & . & . & 1 \\ . & . & . & 1 & . & . \\ . & . & . & . & 1 & . \end{array}\right].$$

Let $\overline{W'^*} := W'^* \otimes_{\mathbb{F}_2} \overline{\mathbb{F}}_2$ and $\overline{W'} := W' \otimes_{\mathbb{F}_2} \overline{\mathbb{F}}_2$. Moreover, let $S[\overline{W'^*}]^G$ be the algebra of G-invariants in the symmetric algebra $S[\overline{W'^*}]$ over W'^*. Hence we have $S[\overline{W'^*}] \cong S[W'^*] \otimes_{\mathbb{F}_2} \overline{\mathbb{F}}_2$, and by (3.1) we have $S[\overline{W'^*}]^G \cong S[W'^*]^G \otimes_{\mathbb{F}_2} \overline{\mathbb{F}}_2$. The embedding of affine algebras $S[\overline{W'^*}]^G \subseteq S[\overline{W'^*}]$ defines a morphism $\overline{W'} \to \overline{W'}/G$ of affine varieties over $\overline{\mathbb{F}}_2$, which as G is finite is a geometric quotient, see [5, Ch.2.3]. Hence we have proved the following

(2.5) Proposition. See [10, Prop.1.3].
The moduli space $\mathcal{M}_3^{\mathrm{ord}}$ of the ordinary quartic curves is isomorphic to the open subset of the affine variety $\mathrm{Spec}(S[W'^*]^G \otimes_{\mathbb{F}_2} \overline{\mathbb{F}}_2)$ given by the non-singularity conditions $(*)$ in (2.2). ♯

3 Computing with invariant algebras

Let F be a field, let V be a finite-dimensional F-vector space, let G be a finite group acting F-linearly on V, and let $S[V]^G$ denote the algebra of G-invariants in the symmetric algebra $S[V]$ over V.

(3.1) Remark. For $d \in \mathbb{N}_0$ we have $S[V]_d^G = \bigcap_{\sigma \in G} \ker_{S[V]_d}(\sigma - 1)$, where $S[V]_d \subseteq S[V]$ denotes the the homogeneous component of $S[V]$ of degree d. Hence for a field extension $F \subseteq L$ we have $\dim_F(S[V]_d^G) = \dim_L(S[V \otimes_F L]_d^G)$, for all $d \in \mathbb{N}_0$, and thus we conclude $S[V]^G \otimes_F L \cong S[V \otimes_F L]^G$.

(3.2) To compute primary invariants in the particular situation occurring in Section 4, we exploit the following setting.

Let $\{0\} \to U' \xrightarrow{\alpha} U \xrightarrow{\beta} V \to \{0\}$ be an extension of G-modules. Hence α induces an embedding $S[U'] \subseteq S[U]$, and β induces an isomorphism $S[U]/U'S[U] \to S[V]$. Hence we have $S[U']^G \subseteq S[U]^G$ and $(S[U]/U'S[U])^G \cong S[V]^G$, where $U'S[U] \lhd S[U]$ is a G-submodule. In general, we only have an embedding $S[U]^G/(U'S[U])^G \subseteq (S[U]/U'S[U])^G$ but not an isomorphism, and in general $(U'S[U])^G \lhd S[U]^G$ is not generated by the irrelevant ideal $S[U']_+^G \lhd S[U']^G$.

Let us assume that the above extension splits, and let $\gamma\colon V \to U$ such that $\gamma\beta = \mathrm{id}_V$. Hence γ induces an embedding $S[V] \subseteq S[U]$, and we have $S[U] = S[V] \oplus U'S[U]$ as G-modules. Thus from $S[U]^G = S[V]^G \oplus (U'S[U])^G$ we conclude that β induces an isomorphism $S[U]^G/(U'S[U])^G \cong S[V]^G$.

Let us moreover assume that $U' \cong F$ is the trivial G-module, and let $0 \neq \widehat{f} \in U' \subseteq S[U]_1^G$. Then we have $(U'S[U])^G = \widehat{f} \cdot S[U]^G \lhd S[U]^G$, and thus β induces an isomorphism $S[U]^G/\widehat{f}S[U]^G \cong S[V]^G$. As $\widehat{f} \in S[U]$ is not a zero-divisor, for the corresponding Hilbert series we obtain $H_{S[V]^G}(t) = (1-t) \cdot H_{S[U]^G}(t) \in \mathbb{Q}(t)$.

Let $\widehat{\mathcal{F}} := \{\widehat{f}_0, \ldots, \widehat{f}_{n-1}\} \subseteq S[U]^G$ be a set of primary invariants, such that $\widehat{f}_0 = \widehat{f}$, and let $f_i := \widehat{f}_i\beta \in S[V]^G$, for $i \in \{1, \ldots, n-1\}$, as well as $\mathcal{F} := \{f_1, \ldots, f_{n-1}\}$. Note that by [2, Cor.1.4.6] for the Krull dimensions we have $n = \dim(S[U]^G) = \dim_F(U)$ and $n-1 = \dim(S[V]^G) = \dim_F(V)$. As $\widehat{\mathcal{F}} \subseteq S[U]^G$ is a set of primary invariants, using the Graded Nakayama Lemma, see [5, La.3.5.1], we conclude $\dim(S[U]^G/\widehat{\mathcal{F}}S[U]^G) = 0$. Hence using β we find $\dim(S[V]^G/\mathcal{F}S[V]^G) = 0$, where $|\mathcal{F}| \leq n-1$. Hence by the Graded Nakayama Lemma again we conclude that $\mathcal{F} \subseteq S[V]^G$ is set of primary invariants.

Finally, as $\widehat{f} \in S[U]^G$ is not a zero-divisor and hence regular, see [2, Ch.4.3], we conclude that $S[U]^G$ is Cohen-Macaulay, if and only if $S[V]^G$ is. Moreover, if $S[V]^G$ is Cohen-Macaulay, then there is a maximal regular homogeneous sequence in $S[U]^G$ beginning with $\widehat{f}$, which thus is a set of primary invariants of $S[U]^G$, and the above construction indeed yields a a set of primary invariants of $S[V]^G$, which is optimal in the sense of [5, Ch.3.3.2], i. e. with respect to degree product, if the used set of primary invariants of $S[U]^G$ is.

(3.3) To compute secondary invariants in the particular situation occurring in Section 4, we use a special adaptation of the method typically used in the non-modular case, see [5, Ch.3.5].

Let us assume that $R := S[V]^G$ is known to be Cohen-Macaulay, and that the Hilbert series $H_R \in \mathbb{Q}(t)$ and a set $\mathcal{F} \subseteq R_+$ of primary invariants are known; here again $R_+ \lhd R$ denotes the irrelevant ideal. Hence by [5, Ch.3.5.1] we have $f := \prod_{i=1}^n (1-t^{d_i}) \cdot H_R \in \mathbb{Z}^{\geq 0}[t]$, and hence the cardinality of any minimal homogeneous $F[\mathcal{F}]$-module generating set $\mathcal{G}$ of R is given as $f(1)$, while the degrees of its elements can be determined from the monomials occurring in f.

Let $\mathcal{G} \subseteq R$ be a set of the appropriate cardinality, containing homogeneous elements of the appropriate degrees. By the Graded Nakayama Lemma $\mathcal{G}$ generates the $F[\mathcal{F}]$-module R if and only if $\mathcal{G}$ generates the F-vector space $R/F[\mathcal{F}]_+R$. By the assumptions made we conclude that $\mathcal{G}$ is a generating set of the $F[\mathcal{F}]$-module R if and only if $\mathcal{G} \subseteq R/F[\mathcal{F}]_+R$ is F-linearly independent.

As we are developing a method to find secondary invariants, the ring R and hence $R/F[\mathcal{F}]_+R$ are not yet known. Thus we proceed as follows. Let $H \leq G$ be a subgroup such that $\mathrm{char}(F)$ does not divide $[G\colon H]$, and let $S := S[V]^H$. As we have $F[\mathcal{F}] \subseteq R \subseteq S$, we may consider the natural map $\pi\colon R \to S \to S/(\sum_{i=1}^n f_iS)$ of F-algebras. Hence we have $F[\mathcal{F}]_+R \subseteq \ker(\pi)$. Conversely, let $h \in \ker(\pi) \trianglelefteq R$, hence we have $h = \sum_{i=1}^n f_ih_i$, where $h_i \in S$. Thus we have $h = \mathcal{R}_H^G(h) = \sum_{i=1}^n f_i \cdot \mathcal{R}_H^G(h_i) \in F[\mathcal{F}]_+R$, and hence $\ker(\pi) = F[\mathcal{F}]_+R$. Thus we have an embedding $\pi\colon R/F[\mathcal{F}]_+R \to S/(\sum_{i=1}^n f_iS)$. Hence $\mathcal{G} \subseteq R/F[\mathcal{F}]_+R$ is F-linearly independent, if and only if $\pi(\mathcal{G}) \subseteq S/(\sum_{i=1}^n f_iS)$ is.

Let us finally assume that $S = S[V]^H$ is Cohen-Macaulay, and that $\mathcal{G}' \subseteq S$ is a minimal set of secondary invariants; note that $\mathcal{F} \subseteq S$ is a set of primary invariants. As S is the free $F[\mathcal{F}]$-module generated by $\mathcal{G}'$, a description of S as a finitely presented commutative F-algebra can be derived using linear algebra techniques, see [5, Ch.3.6]. From that, a description of $S/(\sum_{i=1}^n f_i S)$ as a finitely presented commutative F-algebra is immediately derived. Hence the F-linear independence of $\pi(\mathcal{G}) \subseteq S/(\sum_{i=1}^n f_i S)$ is easily verified or falsified using Gröbner basis techniques, see [5, Ch.3.5].

4 The invariant algebra $S[W'^*]^G$

Let again $G := GL_3(\mathbb{F}_2)$. We are prepared to analyse the structure of the invariant algebra $S[W'^*]^G$ introduced in (2.4). We begin with the following

(4.1) Proposition. The G-module W is a transitive permutation module.

Proof. If W were a transitive permutation module, the corresponding point stabiliser would be a subgroup of order 24, leading to the following sensible guess. As in (2.4) let $\mathcal{S}_4 \cong H < G$ be the subgroup permuting the set $\{x^*, y^*, z^*, (x+y+z)^*\} \subseteq M^*$. Hence H fixes $w_0 = [0,0,0,0,0,0,1] \in W$, and we are led to conjecture that $H = \mathrm{Stab}_G(w_0)$ and that $w_0 \cdot G \subseteq W$ is an $\mathbb{F}_2$-basis of W being permuted by G. To check this, let $C \in G$ be as in (2.3). Its action on W is given as

$$D_W\colon \quad C \mapsto \left[\begin{array}{ccc|ccc|c} . & . & 1 & . & . & . & . \\ 1 & . & 1 & . & . & . & . \\ . & 1 & . & . & . & . & . \\ \hline . & . & 1 & . & . & 1 & . \\ . & . & . & 1 & 1 & . & . \\ . & . & . & . & 1 & . & . \\ \hline . & . & . & . & 1 & . & 1 \end{array}\right].$$

Hence $\{C^i \in G; i = 0, \ldots, 6\} \subseteq G$ is a set of representatives of the right cosets $H|G$, and $\Omega := \{w_0 \cdot C^i \in W; i = 0, 6, 1, 2, 3, 4, 5\} \subseteq W$ is given as follows, where the rows indicate the elements of Ω in terms of the standard basis of W,

$$\left[\begin{array}{ccccccc} . & . & . & . & . & . & 1 \\ . & . & . & . & . & 1 & 1 \\ . & . & . & . & 1 & . & 1 \\ . & . & . & 1 & . & . & 1 \\ . & . & 1 & . & 1 & 1 & 1 \\ . & 1 & . & 1 & 1 & . & 1 \\ 1 & . & . & 1 & . & 1 & 1 \end{array}\right].$$

Hence Ω is an $\mathbb{F}_2$-basis of W, and it is easily checked that it is permuted by G, where in particular $A \mapsto (1,4)(2,7) \in \mathcal{S}_7$ and $B \mapsto (2,4,3)(5,7,6) \in \mathcal{S}_7$. ♯

Thus the G-module W'^* is a direct summand of the permutation module W^*. Note that direct summands of permutation modules are also called **trivial source modules**, see [5, Ch.3.10.4].

(4.2) Proposition. The Hilbert series $H_{S[W'^*]^G} \in \mathbb{Q}(t)$ of $S[W'^*]^G$ is given as

$$H_{S[W'^*]^G}(t) = \frac{1+t^4+2t^5+t^6+t^7+t^8+2t^9+2t^{10}+t^{11}+t^{12}+t^{13}+2t^{14}+t^{15}+t^{19}}{(1-t^2)\cdot(1-t^3)^2\cdot(1-t^4)\cdot(1-t^6)\cdot(1-t^7)}.$$

Proof. As the G-module W'^* is a trivial source module, by [1, Cor.3.11.4] the G-module W'^* has a unique lift to a trivial source $\mathbb{Z}_2G$-module $\widehat{W'^*}$, where $\mathbb{Z}_2 \subseteq \mathbb{Q}_2$ is the integral closure of $\mathbb{Z}$ in the 2-adic completion $\mathbb{Q}_2$ of $\mathbb{Q}$. Let $\widetilde{W'^*} := \widehat{W'^*} \otimes_{\mathbb{Z}_2} \mathbb{Q}_2$. By [5, Prop.3.10.15] we have $H_{S[W'^*]^G}(t) = H_{S[\widetilde{W'^*}]^G}(t) \in \mathbb{Q}(t)$, where by Molien's Theorem, see [2, Thm.2.5.2], the latter is given as

$$H_{S[\widetilde{W'^*}]^G}(t) = \frac{1}{|G|} \cdot \sum_{\sigma \in G} \frac{1}{\det_{\widetilde{W'^*}}(1-t\sigma)} \in \mathbb{Q}_2(t).$$

Moreover, we have $\det_{\widetilde{W'^*}}(1-t\sigma) = \prod_{i=1}^{7}(1-\lambda_i(\sigma)\cdot t)$, where $\{\lambda_1(\sigma), \ldots, \lambda_7(\sigma)\}$ are the eigenvalues of the action of $\sigma \in G$ in a suitable extension field of $\mathbb{Q}_2$. Hence $\det_{\widetilde{W'^*}}(1-t\sigma)$ can be evaluated from the ordinary character table of G, see [3, p.3], and the character $\chi_{\widetilde{W'^*}}$ of $\widetilde{W'^*}$, see [2, Ch.2.5]. This method and the ordinary character table of G are available in GAP.

Hence it remains to find the character $\chi_{\widetilde{W'^*}}$. Thus we have to determine the trivial source lift $\widehat{W'^*}$. By (4.1) we have $(\mathbb{F}_2)_H^G \cong W \cong W' \oplus \mathbb{F}_2$, where $\mathbb{F}_2$ denotes the trivial G-module. Hence we have the trivial source lifts $(\mathbb{Z}_2)_H^G \cong \widehat{W} \cong \widehat{W'} \oplus \mathbb{Z}_2$, where again $\mathbb{Z}_2$ denotes the trivial G-module. Since we have $(\widehat{W})^* \cong \widehat{W^*}$ as G-modules, we obtain $((\mathbb{Z}_2)_H^G)^* \cong \widehat{W^*} \cong \widehat{W'^*} \oplus \mathbb{Z}_2$, Moreover, we conclude $((\mathbb{Q}_2)_H^G)^* \cong \widehat{W^*} \otimes_{\mathbb{Z}_2} \mathbb{Q}_2 \cong \widehat{W'^*} \oplus \mathbb{Q}_2$. By [3, p.3] the character 1_H^G of the permutation G-module $(\mathbb{Q}_2)_H^G$ is given as $1_G + \chi_6$, where χ_6 is the unique irreducible character of degree 6 and 1_G is the trivial character. As χ_6 is real-valued we have $\chi_{\widetilde{W'^*}} = 1_H^G - 1_G = \chi_6$. ♯

(4.3) Proposition. The invariant algebra $S[W'^*]^G$ is Cohen-Macaulay.

Proof. Let $D < G$ be a 2-Sylow subgroup of G, hence we have $|D| = 8$. Using the standard methods to compute primary invariants, see [5, Ch.3.3], and secondary invariants in the modular case, see [5, Ch.3.5], available in MAGMA, we find primary invariants $\{f'_1, \ldots, f'_6\} \subseteq S[W'^*]^D$ having degrees $\{1,1,2,2,2,4\}$, and a minimal set of secondary invariants $\mathcal{G}' := \{g'_0, \ldots, g'_3\} \subseteq S[W'^*]^D$ having degrees $\{0,3,3,6\}$, where of course $g'_0 = 1$. As we moreover have $|\mathcal{G}'| \cdot |D| = \prod_{i=1}^{6} \deg(f'_i)$, by [5, Thm.3.7.1] we conclude that $S[W'^*]^D$ is Cohen-Macaulay, and thus by [5, Rem.3.4.2] the algebra $S[W'^*]^G$ also is. ♯

(4.4) We are prepared to compute primary invariants of $S[W'^*]^G$. To do this, we first consider the permutation module $W^* = W'^* \oplus \mathbb{F}_2$, and compute the homogeneous components $S[W^*]^G_d$ for $d \leq 7$ as follows, see also [5, Ch.3.10]. Let $\Omega^* = \{\omega_1^*, \ldots, \omega_7^*\} \subseteq W^*$ be the $\mathbb{F}_2$-basis of W^* dual to the $\mathbb{F}_2$-basis $\Omega \subseteq W$ given in (4.1). As $S[W^*]_d$ also is a permutation module, whose $\mathbb{F}_2$-basis $(\Omega^*)^d$ consisting of the monomials of degree d in the indeterminates Ω^* is permuted by G. Hence $(\Omega^*)^d$ is partitioned into G-orbits $(\Omega^*)^d = \coprod_{i=1}^{n_d} \mathcal{O}_i$, where $n_d = \dim_{\mathbb{F}_2}(S[W^*]^G_d)$. Letting $\mathcal{O}_i^+ := \sum_{f \in \mathcal{O}_i} f \in S[W^*]^G_d$ denote the corresponding orbit sums, the set $\{\mathcal{O}_i^+; i \in \{1, \ldots, n_d\}\}$ forms an $\mathbb{F}_2$-basis of $S[W^*]^G_d$.

As by (3.2) we have $H_{S[W^*]^G} = \frac{1}{1-t} \cdot H_{S[W'^*]^G} \in \mathbb{Q}(t)$, we look for primary invariants having degrees $\{1, 2, 3, 3, 4, 6, 7\}$. By [5, Prop.3.3.1], a set $\widehat{\mathcal{F}} = \{\widehat{f}_0, \ldots, \widehat{f}_6\} \subseteq S[W^*]^G$ of homogeneous elements is a set of primary invariants, if and only if $\dim(S[W^*]/\widehat{\mathcal{F}}S[W^*]) = 0$. Krull dimensions can be computed using Gröbner basis techniques, which are available in MAGMA, and we indeed find the following set of primary invariants of $S[W^*]^G$, consisting of orbit sums,

$$
\begin{aligned}
\widehat{f}_0 &:= \omega_1^* + \omega_2^* + \omega_3^* + \omega_4^* + \omega_5^* + \omega_6^* + \omega_7^*,\\
\widehat{f}_1 &:= \omega_1^*\omega_2^* + \omega_1^*\omega_3^* + \omega_1^*\omega_4^* + \omega_1^*\omega_5^* + \omega_1^*\omega_6^* + \omega_1^*\omega_7^* + \omega_2^*\omega_3^* +\\
&\quad \omega_2^*\omega_4^* + \omega_2^*\omega_5^* + \omega_2^*\omega_6^* + \omega_2^*\omega_7^* + \omega_3^*\omega_4^* + \omega_3^*\omega_5^* + \omega_3^*\omega_6^* +\\
&\quad \omega_3^*\omega_7^* + \omega_4^*\omega_5^* + \omega_4^*\omega_6^* + \omega_4^*\omega_7^* + \omega_5^*\omega_6^* + \omega_5^*\omega_7^* + \omega_6^*\omega_7^*,\\
\widehat{f}_2 &:= \omega_1^*\omega_2^*\omega_3^* + \omega_1^*\omega_2^*\omega_4^* + \omega_1^*\omega_2^*\omega_5^* + \omega_1^*\omega_2^*\omega_7^* + \omega_1^*\omega_3^*\omega_4^* + \omega_1^*\omega_3^*\omega_5^* +\\
&\quad \omega_1^*\omega_3^*\omega_6^* + \omega_1^*\omega_4^*\omega_6^* + \omega_1^*\omega_4^*\omega_7^* + \omega_1^*\omega_5^*\omega_6^* + \omega_1^*\omega_5^*\omega_7^* + \omega_1^*\omega_6^*\omega_7^* +\\
&\quad \omega_2^*\omega_3^*\omega_5^* + \omega_2^*\omega_3^*\omega_6^* + \omega_2^*\omega_3^*\omega_7^* + \omega_2^*\omega_4^*\omega_5^* + \omega_2^*\omega_4^*\omega_6^* + \omega_2^*\omega_4^*\omega_7^* +\\
&\quad \omega_2^*\omega_5^*\omega_6^* + \omega_2^*\omega_6^*\omega_7^* + \omega_3^*\omega_4^*\omega_5^* + \omega_3^*\omega_4^*\omega_6^* + \omega_3^*\omega_4^*\omega_7^* + \omega_3^*\omega_5^*\omega_7^* +\\
&\quad \omega_3^*\omega_6^*\omega_7^* + \omega_4^*\omega_5^*\omega_6^* + \omega_4^*\omega_5^*\omega_7^* + \omega_5^*\omega_6^*\omega_7^*,\\
\widehat{f}_3 &:= \omega_1^*\omega_2^*\omega_6^* + \omega_1^*\omega_3^*\omega_7^* + \omega_1^*\omega_4^*\omega_5^* + \omega_2^*\omega_3^*\omega_4^* +\\
&\quad \omega_2^*\omega_5^*\omega_7^* + \omega_3^*\omega_5^*\omega_6^* + \omega_4^*\omega_6^*\omega_7^*,\\
\widehat{f}_4 &:= \omega_1^*\omega_2^*\omega_3^*\omega_5^* + \omega_1^*\omega_2^*\omega_4^*\omega_7^* + \omega_1^*\omega_3^*\omega_4^*\omega_6^* + \omega_1^*\omega_5^*\omega_6^*\omega_7^* +\\
&\quad \omega_2^*\omega_3^*\omega_6^*\omega_7^* + \omega_2^*\omega_4^*\omega_5^*\omega_6^* + \omega_3^*\omega_4^*\omega_5^*\omega_7^*,\\
\widehat{f}_5 &:= \omega_1^*\omega_2^*\omega_3^*\omega_4^*\omega_5^*\omega_6^* + \omega_1^*\omega_2^*\omega_3^*\omega_4^*\omega_5^*\omega_7^* + \omega_1^*\omega_2^*\omega_3^*\omega_4^*\omega_6^*\omega_7^* +\\
&\quad \omega_1^*\omega_2^*\omega_3^*\omega_5^*\omega_6^*\omega_7^* + \omega_1^*\omega_2^*\omega_4^*\omega_5^*\omega_6^*\omega_7^* + \omega_1^*\omega_3^*\omega_4^*\omega_5^*\omega_6^*\omega_7^* +\\
&\quad \omega_2^*\omega_3^*\omega_4^*\omega_5^*\omega_6^*\omega_7^*,\\
\widehat{f}_6 &:= \omega_1^*\omega_2^*\omega_3^*\omega_4^*\omega_5^*\omega_6^*\omega_7^*.
\end{aligned}
$$

It turns out that there is no set of primary invariants of $S[W^*]^G$ having a strictly smaller degree product, hence $\widehat{\mathcal{F}}$ is optimal in the sense of [5, Ch.3.3.2]. As $\widehat{f}_0 \in S[W^*]^G_1$, using the method described in (3.2), we find an optimal set $\mathcal{F} = \{f_1, , \ldots, f_6\} \subseteq S[W'^*]^G$ of primary invariants, having degrees $\{2, 3, 3, 4, 6, 7\}$.

(4.5) Next we compute secondary invariants of $S[W'^*]^G$. Since the algebra $S[W'^*]^G$ is Cohen-Macaulay and a set of primary invariants is known, from the Hilbert series $H_{S[W'^*]^G}$ we conclude that there is a minimal set of 18 secondary invariants, having degrees $\{0, 4, 5, 5, 6, 7, 8, 9, 9, 10, 10, 11, 12, 13, 14, 14, 15, 19\}$, see [5, Ch.3.5.1]. To find such a set of secondary invariants, we first com-

pute the homogeneous components $S[W'^*]^G_d$ for $d \leq 7$, using linear algebra techniques available in MAGMA, see [5, Ch.3.1]. We then consider products of the homogeneous invariants thus found, having appropriate degrees. Thus we successively generate homogeneous elements $\mathcal{G} := \{g_1, g_2, \ldots, g_{18}\} \in S[W'^*]^G$, repeatedly using the method described in (3.3) to make sure that we have $\dim_{\mathbb{F}_2}(\langle \pi(g_j); j \in \{1, \ldots, k\}\rangle_{\mathbb{F}_2}) = k$, for $k \in \{1, \ldots, 18\}$.

To apply the method described in (3.3), we again consider the invariant algebra $S[W'^*]^D$, see (4.3). As $S[W'^*]^D$ is Cohen-Macaulay, using linear algebra techniques available in MAGMA, we obtain the finite presentation $S[W'^*]^D \cong \langle F_1, \ldots, F_6, G_1, \ldots, G_3 | R_1, \ldots, R_3\rangle$ as commutative $\mathbb{F}_2$-algebras, where

$$\begin{array}{rcl}
R_1 & := & (F_1+F_2)^2(F_1F_2F_4 + F_3F_5 + F_4^2 + F_4F_5) + (F_3+F_4)F_5^2 + \\
 & & (F_1^3 + F_1F_2^2 + F_1F_5 + F_2F_5)\cdot G_1 + (F_1^2F_2 + F_2^3)\cdot G_2 + G_1^2, \\
R_2 & := & (F_1F_2 + F_2^2 + F_3)F_3F_4 + (F_1F_2^2 + F_1F_3 + F_2^3 + F_2F_5)\cdot G_1 + \\
 & & (F_1F_2^2 + F_1F_5 + F_2^3 + F_2F_5)\cdot G_2 + G_1G_2 + G_3, \\
R_3 & := & (F_2^2F_6 + F_3^2F_5) + F_2F_3\cdot G_2 + G_2^2,
\end{array}$$

the isomorphism from the finitely presented algebra to $S[W'^*]^D$ being given by $F_i \mapsto f'_i$ and $G_j \mapsto g'_j$, where $\{f'_1, \ldots, f'_6\} \subseteq S[W'^*]^D$ and $\{g'_1, \ldots, g'_3\} \subseteq S[W'^*]^D$ are as in (4.3). Decomposing the set of primary invariants $\mathcal{F} \subseteq S[W'^*]^G \subseteq S[W'^*]^D$ into the $\mathbb{F}_2$-algebra generators $\{f'_1, \ldots, f'_6\} \cup \{g'_1, \ldots, g'_3\}$ of $S[W'^*]^D$, again using linear algebra techniques available in MAGMA, yields the finite presentation

$$S[W'^*]^D/(\sum_{i=1}^{6} f_i S[W'^*]^D) \cong \langle F_1, \ldots, F_6, G_1, \ldots, G_3 | R_1, , \ldots, R_3, R'_1, \ldots, R'_6\rangle$$

as commutative $\mathbb{F}_2$-algebras, where the additional relations are given as

$$\begin{array}{rcl}
R'_1 & := & F_2^2 + F_4 + F_5, \\
R'_2 & := & F_1F_4 + F_2F_5 + G_1 + G_2, \\
R'_3 & := & F_2F_4 + G_1, \\
R'_4 & := & F_1^2(F_1+F_2)^2 + F_1(F_1+F_2)(F_4+F_5) + F_3(F_3+F_4) + F_6 + F_1\cdot G_1, \\
R'_5 & := & F_1^2(F_3^2 + F_3F_4 + F_6) + F_1F_2(F_2^2F_4F_3F_4 + F_4F_5 + F_6) + \\
 & & F_2^2(F_3F_4 + F_3F_5 + F_4^2) + (F_3+F_4)(F_5^2 + F_6) + F_4^2F_5 + \\
 & & (F_1F_3 + F_1F_5 + F_2^3 + F_2F_4)\cdot G_1 + \\
 & & (F_1F_2^2 + F_1F_3 + F_1F_4 + F_1F_5)\cdot G_2 + G_3, \\
R'_6 & := & F_1F_6(F_3+F_4).
\end{array}$$

We end up with secondary invariants $\mathcal{G} := \{g_1, \ldots, g_6\} \cup \{g_7, \ldots, g_{18}\}$, where $g_1 = 1$ and $\{g_1, \ldots, g_6\}$ have degrees $\{0, 4, 5, 5, 6, 7\}$, while

$$\begin{array}{rcrl|rcrl|rcrl}
g_7 & := & g_2^2 & (8), & g_{11} & := & g_3g_4 & (10), & g_{15} & := & g_2^2g_5 & (14), \\
g_8 & := & g_2g_3 & (9), & g_{12} & := & g_2g_6 & (11), & g_{16} & := & g_2g_3g_4 & (14), \\
g_9 & := & g_2g_4 & (9), & g_{13} & := & g_2^3 & (12), & g_{17} & := & g_2^2g_6 & (15), \\
g_{10} & := & g_2g_5 & (10), & g_{14} & := & g_2^2g_3 & (13), & g_{18} & := & g_2^3g_6 & (19),
\end{array}$$

where the bracketed numbers indicate the degrees. Hence in particular we conclude that $\{f_1, \ldots, f_6\} \cup \{g_1, \ldots, g_6\} \subseteq S[W'^*]^G$ is a minimal $\mathbb{F}_2$-algebra generating set of $S[W'^*]^G$, and in particular $S[W'^*]^G$ is as an $\mathbb{F}_2$-algebra generated by invariants of degree at most 7.

(4.6) Remark. In conclusion we note the following observations.
a) As W is a transitive permutation module of odd degree in characteristic 2, there is a non-zero G-invariant quadratic form on W. Indeed, this form coincides with the primary invariant $\widehat{f_1} \in S[W^*]^G$ of degree 2. Moreover, we have

$$f_1 = a^*e^* + b^*f^* + c^*d^* + d^*e^* + d^*f^* + e^*f^* + d^{*2} + e^{*2} + f^{*2} \in S[W'^*]^G.$$

Hence this shows that $D_{W'}$ is an embedding $D_{W'}\colon G \to SO_6^+(\mathbb{F}_2)$, where $SO_6^+(\mathbb{F}_2)$ denotes the special orthogonal group of degree 6 over $\mathbb{F}_2$ of maximal Witt index type, see [3, p.22].

b) We have $\{0\} \to (W'/W'')^* \to W'^* \to W''^* \to \{0\}$, see (2.3). Assume that this extension splits. Then we also have $W' \cong W'' \oplus W'/W''$, and hence $(\mathbb{F}_2)_H^G \cong W \cong W'' \oplus W'/W'' \oplus \mathbb{F}_2$ is semisimple, and thus $\dim_{\mathbb{F}_2}(\mathrm{End}_G((\mathbb{F}_2)_H^G)) = 3$. By (4.2) we have $\dim_{\mathbb{Q}_2}(\mathrm{End}_G((\mathbb{Q}_2)_H^G)) = 2$, a contradiction to [1, Thm.3.11.3]. Hence this extension does not split.

Still we are tempted to apply the method described in (3.2), yielding an embedding $S[W'^*]^G/((W'/W'')^*S[W'^*])^G \to S[W''^*]^G$. The question arises whether this map is still surjective. While the authors do not see a structural reason why this should be the case, from the primary invariants $\{f_4, f_5, f_6\} \subseteq S[W'^*]^G$ of degrees $\{4, 6, 7\}$ we obtain $\{c_0, c_1, c_2\} \subseteq S[W''^*]^G$, where

$$\begin{aligned}
c_2 &:= a^{*4} + b^{*4} + c^{*4} + a^{*2}b^{*2} + a^{*2}c^{*2} + b^{*2}c^{*2} + \\
&\quad a^{*2}b^*c^* + a^*b^{*2}c^* + a^*b^*c^{*2}, \\
c_1 &:= a^{*4}b^{*2} + a^{*2}b^{*4} + a^{*4}c^{*2} + a^{*2}c^{*4} + b^{*4}c^{*2} + b^{*2}c^{*4} + \\
&\quad a^{*4}b^*c^* + a^*b^{*4}c^* + a^*b^*c^{*4} + a^{*2}b^{*2}c^{*2}, \\
c_0 &:= a^{*4}b^{*2}c^* + a^{*4}b^*c^{*2} + a^{*2}b^{*4}c^* + a^*b^{*4}c^{*2} + a^{*2}b^*c^{*4} + a^*b^{*2}c^{*4}.
\end{aligned}$$

It turns out that these are the Dickson invariants of $S[W''^*]^G$, see [2, Ch.8.1]. By Dickson's Theorem, see [2, Thm.8.1.1], the set $\{c_0, c_1, c_2\} \subseteq S[W''^*]^G$ is algebraically independent, and we have $S[W''^*]^G = \mathbb{F}_2[c_0, c_1, c_2]$. Hence the above map $S[W'^*]^G/((W'/W'')^*S[W'^*])^G \to S[W''^*]^G$ indeed is surjective.

c) Using the method described in (3.2), we may also find an optimal set of primary invariants of the invariant algebra $S[\widetilde{W'^*}]^G$, which by the Hochster-Eagon Theorem, see [2, Thm.4.3.6], is Cohen-Macaulay. These optimal primary invariants turn out to have degrees $\{2, 3, 3, 4, 4, 7\}$. Indeed, the Hilbert series $H_{S[\widetilde{W'^*}]^G}(t) = H_{S[W'^*]^G}(t)$ can be rewritten as

$$H_{S[\widetilde{W'^*}]^G}(t) = \frac{1 + 2t^5 + 2t^6 + t^7 + t^{10} + 2t^{11} + 2t^{12} + t^{17}}{(1-t^2)\cdot(1-t^3)^2\cdot(1-t^4)^2\cdot(1-t^7)}.$$

Since the optimal set $\mathcal{F}$ of primary invariants has degrees $\{2, 3, 3, 4, 6, 7\}$, there is no set of primary invariants of $S[W'^*]^G$ having degrees $\{2, 3, 3, 4, 4, 7\}$. This

shows, although $S[\widetilde{W'^*}]^G$ and $S[W'^*]^G$ do have the same Hilbert series, that their algebra structures are different. Still, as the underlying modules W'^* and $\widetilde{W'^*}$ are closely related, the corresponding invariant algebras should be closely related as well. But how this relationship might look like, for the time being remains mysterious to the authors.

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