On a certain algebra of higher modular forms

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Abstract

By a combined use of analytical, algebraic and computational tools we derive a description of the algebra of modular forms with respect to a certain congruence subgroup of $\mathrm{SL}_2(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z})$ of level 3.
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The aim of the present paper is to derive a complete structural description of the algebra $\mathcal{A}(\mathcal{G})$ of modular forms with respect to a certain congruence subgroup $\mathcal{G} \leq \Gamma^2$ of level 3, where $\Gamma := \mathrm{SL}_2(\mathbb{Z})$ denotes the modular group. It is in a broader research program concerned with paramodular forms where $\mathcal{A}(\mathcal{G})$ occurs very naturally. The details of this relation are beyond the scope of the present paper, and the reader is referred to [9] instead, while here we restrict ourselves to a close examination of $\mathcal{A}(\mathcal{G})$.

The strategy we obey to is as follows: We begin by giving a description of the algebra $\mathcal{A}(\Gamma[3])$, where $\Gamma[3] \leq \Gamma$ is the principal congruence subgroup of level 3. This description is derived essentially by using analytical tools. From this we obtain an algebraic description of the algebra $\mathcal{A}(\Gamma[3]^2)$, allowing us to proceed in a purely algebraic manner, as $\mathcal{A}(\mathcal{G})$ coincides with the invariant algebra $\mathcal{A}(\Gamma[3]^2)^G$ where $G := \mathcal{G}/\Gamma[3]^2$ is a certain finite group. Using tools from group representation theory and commutative algebra we thus obtain a description of the algebraic structure of $\mathcal{A}(\mathcal{G})$, in particular $\mathcal{A}(\mathcal{G})$ turns out to be a Cohen-Macaulay algebra. This brings us into a position to apply tools from computational algebra to explicitly determine a generating set of $\mathcal{A}(\mathcal{G})$.

Describing the algebraic structure of algebras of modular forms, and their connection to the algebraic theories of lattices and error-correcting codes, is a well-known topic in the literature, see e. g. [7, 6]. In the present paper, the algebraic setting prepares us to apply suitably adjusted computational tools from representation theory, commutative algebra and invariant theory. Still, we have to provide some commutative algebra machinery, which generalising the classical situation allows us to do invariant theory in non-polynomial algebras.

We assume the reader to be familiar with the basic theory of modular forms, where a general reference is [10], as well as with the basics of commutative algebra, invariant theory, and the character theory of finite groups, where general references are [3, 4, 5]. The present paper is organised as follows: In Section 1 we recall some background from the theory of modular forms, in particular to fix notation. In (1.2) we describe the structure of $\mathcal{A}(\Gamma[3])$, and in (1.4) we define the congruence subgroup $\mathcal{G}$ we are interested in. Actually, as $\mathcal{G}$ is invariant under the natural flip map, the algebra $\mathcal{A}(\mathcal{G})$ decomposes into a symmetric and a skew symmetric part, and to facilitate their description we introduce another group $\mathcal{G} \leq \mathcal{H} \leq \Gamma^2$. In (1.5) we specify the algebras and modules we are going to analyse. In Section 2 we provide the necessary machinery from commutative algebra. In (2.4) we present a generalisation of the celebrated Hochster-Eagon Theorem in polynomial invariant theory, in (2.2) we provide an appropriate characterisation of sets of primary generators of graded modules, and in (2.5) we present a technique to compute sets of secondary generators of graded modules under certain circumstances. Finally, in Section 3 we apply this machinery to the algebras and modules specified in (1.5). In (3.3) and (3.4) we obtain descriptions of $\mathcal{A}(\Gamma[3]^2)$ and of $\mathcal{A}(\mathcal{G})$ and its symmetric and skew symmetric parts, respectively. In (3.5) we explicitly compute sets of primary and secondary generators.

The computations are done using the computer algebra systems GAP [8] and MAGMA [1]. Input files containing the appropriate data and invoking the relevant commands, as well as output files containing the explicit results, are available on request from the authors. Here, we do not reproduce most of the explicit results, but are content with describing the employed techniques; a more detailed account is given in [9].

1 Modular forms

(1.1) Modular forms. Let $\Gamma := \mathrm{SL}_2(\mathbb{Z}) = \langle T, J\rangle$ be the modular group, which is generated by $T := \begin{bmatrix} 1 & 1 \\ . & 1 \end{bmatrix}$ and $J := \begin{bmatrix} . & 1 \\ -1 & . \end{bmatrix}$, and for $n \in \mathbb{N}$ let $\Gamma[n] := \{M \in \Gamma; M \equiv I \bmod n\} \trianglelefteq \Gamma$ be the principal congruence subgroup of level n, where $I \in \Gamma$ denotes the identity.

For $k \in \mathbb{Z}$ the group Γ acts from the right on the set of all holomorphic functions $f\colon \mathbb{H} \to \mathbb{C}$, where $\mathbb{H} := \{z \in \mathbb{C}; \mathrm{Im}(z) > 0\}$, by $(f|_k M)(z) := (cz+d)^{-k} \cdot f(\frac{az+b}{cz+d})$, where $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma$. The function f is called a modular form of weight k with respect to $\Gamma[n]$, if $f|_k M = f$ for all $M \in \Gamma[n]$, and if for all $M \in \Gamma$ there is a Fourier expansion of the form $(f|_k M)(z) = \sum_{m\geq 0} \alpha_f(m, M) \cdot \exp(\frac{2\pi i m z}{n})$.

Letting $\mathbb{M}_k(\Gamma[n])$ be the $\mathbb{C}$-vector space of modular forms of weight k with respect to $\Gamma[n]$, we obtain the $\mathbb{N}_0$-graded commutative $\mathbb{C}$-algebra $\mathcal{A}(\Gamma[n]) := \bigoplus_{k\geq 0} \mathbb{M}_k(\Gamma[n])$, where $\mathbb{M}_0(\Gamma[n]) \cong \mathbb{C}$. Note that $\mathcal{A}(\Gamma[n])$ by [7, Thm.II.6.11] is a finitely generated $\mathbb{C}$-algebra. The group Γ induces a group of automorphisms of $\mathcal{A}(\Gamma[n])$ as a graded $\mathbb{C}$-algebra, where $\Gamma[n]$ acts trivially, and thus we have $\mathcal{A}(\Gamma) = \mathcal{A}(\Gamma[n])^{\Gamma} = \mathcal{A}(\Gamma[n])^{\Gamma/\Gamma[n]} \subseteq \mathcal{A}(\Gamma[n])$ as graded $\mathbb{C}$-algebras, where $\mathcal{A}(\Gamma[n])^{\Gamma}$ denotes the algebra of Γ-invariants in $\mathcal{A}(\Gamma[n])$.

(1.2) The algebra $\mathcal{A}(\Gamma[3])$. We will need a description of the graded $\mathbb{C}$-algebra $\mathcal{A}(\Gamma[3])$, which is obtained as follows: Assume there is $\{F_1, F_2\} \subseteq \mathbb{M}_1(\Gamma[3])$ such that $\mathcal{A}(\Gamma[3]) = \mathbb{C}\langle F_1, F_2\rangle$, i. e. $\{F_1, F_2\}$ generates $\mathcal{A}(\Gamma[3])$ as a $\mathbb{C}$-algebra. Since $\mathcal{A}(\Gamma[3])^{\Gamma/\Gamma[3]} \subseteq \mathcal{A}(\Gamma[3])$ is a finite ring extension, and $\mathcal{A}(\Gamma)$ is a polynomial algebra in two indeterminates, for the Krull dimensions we have $\dim(\mathcal{A}(\Gamma[3])) = \dim(\mathcal{A}(\Gamma)) = 2$. This implies that $\{F_1, F_2\}$ is algebraically independent, and thus $\mathcal{A}(\Gamma[3]) \cong \mathbb{C}[F_1, F_2]$ is a polynomial algebra generated in degree 1.

The existence of $\{F_1, F_2\}$ as above is proven in [6, Thm.5.4] using Hilbert modular forms, while in [9] generalised Eisenstein series, see [10, Ch.7.2], are used, yielding a different generating set. Moreover, a calculation using the Fourier expansions of the elements of the chosen generating set shows that the action of Γ on $\mathbb{M}_1(\Gamma[3])$, and thus its action on $\mathcal{A}(\Gamma[3])$, is up to equivalence given by the following representation $D\colon \Gamma \to \mathrm{GL}_2(\mathbb{C})$, where $\omega := \exp(\frac{2\pi i}{3}) = \frac{-1+\sqrt{-3}}{2}$:

$$T \mapsto D_T := \begin{bmatrix} 1 & . \\ . & \omega \end{bmatrix} \quad \text{and} \quad J \mapsto D_J := \frac{1}{\sqrt{-3}} \cdot \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}.$$

(1.3) Higher modular forms. For $k \in \mathbb{Z}$ the group $\Gamma^2 := \Gamma \times \Gamma$ acts from the right on the set of all holomorphic functions $f\colon \mathbb{H} \times \mathbb{H} \to \mathbb{C}$ by

$$(f|_k(M_1, M_2))(z_1, z_2) := (c_1 z_1 + d_1)^{-k}(c_2 z_2 + d_2)^{-k} \cdot f\left(\frac{a_1 z_1 + b_1}{c_1 z_1 + d_1}, \frac{a_2 z_2 + b_2}{c_2 z_2 + d_2}\right),$$

where $M_i = \begin{bmatrix} a_i & b_i \\ c_i & d_i \end{bmatrix} \in \Gamma$. Letting $\mathcal{G} \leq \Gamma^2$ be a congruence subgroup of level $n \in \mathbb{N}$, i. e. we have $\Gamma[n]^2 := \Gamma[n] \times \Gamma[n] \leq \mathcal{G}$, the function f is called a modular form of weight k with respect to $\mathcal{G}$, if $f|_k(M_1, M_2) = f$ for all $(M_1, M_2) \in \mathcal{G}$, and if for all $(M_1, M_2) \in \Gamma^2$ there is a Fourier expansion of the form

$$f|_k(M_1, M_2)(z_1, z_2) = \sum_{m_1 \geq 0} \sum_{m_2 \geq 0} \alpha_f(m_1, m_2; M_1, M_2) \cdot \exp\left(\frac{2\pi i(m_1 z_1 + m_2 z_2)}{n}\right).$$

Let $\mathbb{M}_k(\mathcal{G})$ be the vector space of modular forms of weight k with respect to $\mathcal{G}$, and $\mathcal{A}(\mathcal{G}) := \bigoplus_{k \geq 0} \mathbb{M}_k(\mathcal{G})$ be the associated $\mathbb{N}_0$-graded commutative $\mathbb{C}$-algebra.

Assume that $\mathcal{G}$ is chosen such that $(M_1, M_2) \in \mathcal{G}$ implies $(M_2, M_1) \in \mathcal{G}$. Hence the flip map $\alpha\colon \mathcal{A}(\mathcal{G}) \to \mathcal{A}(\mathcal{G})\colon f(z_1, z_2) \mapsto f(z_2, z_1)$ is an involutory automorphism of graded $\mathbb{C}$-algebras. Thus letting $\mathcal{A}(\mathcal{G})^{\pm} := \{f \in \mathcal{A}(\mathcal{G}); f^{\alpha} = \pm f\}$ be the symmetric and skew symmetric parts of $\mathcal{A}(\mathcal{G})$, respectively, we have a decomposition $\mathcal{A}(\mathcal{G}) \cong \mathcal{A}(\mathcal{G})^+ \oplus \mathcal{A}(\mathcal{G})^-$ as graded $\mathcal{A}(\mathcal{G})^+$-modules.

(1.4) The groups $\mathcal{G}$ and $\mathcal{H}$. We are prepared to specify the congruence subgroup $\mathcal{G}$ we are interested in: Let $G := \mathrm{SL}_2(\mathbb{Z}/3\mathbb{Z})$ and let $\overline{}\colon \Gamma \to G\colon M \mapsto (M \bmod 3)$ denote the natural epimorphism. Let $\kappa\colon \Gamma \to \Gamma\colon M \mapsto M^{-\mathrm{tr}}$, which since $T^{-\mathrm{tr}} = J^{-1}TJ$ and $J^{-\mathrm{tr}} = J$ coincides with the inner automorphism induced by J. Moreover, κ induces an automorphism of $G \cong \Gamma/\Gamma[3]$, also denoted by κ, which coincides with the inner automorphism induced by $\overline{J}$. Now let

$$\mathcal{G} := \{(M_1, M_2) \in \Gamma^2; \overline{M}_1 = \overline{M}_2^{-\mathrm{tr}}\} = \{(M_1, M_2) \in \Gamma^2; M_1 \equiv M_2^{-\mathrm{tr}} \bmod 3\}$$

Table 1: The ordinary character table of $G = \mathrm{SL}_2(\mathbb{Z}/3\mathbb{Z})$.

M	$\overline{I}$	$-\overline{I}$	$\overline{T}$	$-\overline{T}$	$\overline{T}^2$	$-\overline{T}^2$	$\overline{J}$
$\lvert M\rvert$	1	2	3	6	3	6	4
$\lvert C_G(M)\rvert$	24	24	6	6	6	6	4
1	1	1	1	1	1	1	1
λ	1	1	ω	ω	ω^2	ω^2	1
λ'	1	1	ω^2	ω^2	ω	ω	1
τ	3	3	.	.	.	.	-1
ψ	2	-2	-1	1	-1	1	.
χ	2	-2	$-\omega^2$	ω^2	$-\omega$	ω	.
χ'	2	-2	$-\omega$	ω	$-\omega^2$	ω^2	.

be the subdirect product with respect to the epimorphisms $\overline{}$ and $\overline{} \circ \kappa$. Since $\ker(\overline{}) = \ker(\overline{} \circ \kappa) = \Gamma[3]$ we have $\Gamma[3]^2 \trianglelefteq \mathcal{G}$, yielding a group isomorphism $\iota\colon G \to \mathcal{G}/\Gamma[3]^2 \colon \overline{T} \mapsto (\overline{T}^{-\mathrm{tr}}, \overline{T})$, $\overline{J} \mapsto (\overline{J}, \overline{J})$.

Let the flip map act on Γ^2 by $\alpha\colon (M_1, M_2) \mapsto (M_2, M_1)$. As $(T^{-\mathrm{tr}}, T)^\alpha = (T^{-\mathrm{tr}}, T)^{(J,J)}$ and $(J, J)^\alpha = (J, J)$, the subgroup $\mathcal{G} \leq \Gamma^2$ is α-invariant. To facilitate a description of the symmetric and skew symmetric parts of $\mathcal{A}(\mathcal{G})$ let

$$\mathcal{H} := \langle\alpha\rangle \ltimes \mathcal{G} \quad \text{and} \quad H := \langle\kappa\rangle \ltimes G$$

be the associated semidirect products. As $\Gamma[3]^2 \trianglelefteq \mathcal{H}$ the above computation shows that ι extends to a group isomorphism $\iota\colon H \to \mathcal{H}/\Gamma[3]^2 \colon \kappa \mapsto \alpha$.

The ordinary character table of G is well-known, and e. g. available in the character table library of GAP; for convenience it is reproduced in Table 1: In the first rows we give representatives M of the conjugacy classes of G, their orders, and the orders of the centralisers $C_G(M)$. The first four characters have the centre $Z(G) = \langle -\overline{I}\rangle$ of G in their kernel, while the last three characters are faithful, where still $\omega := \exp(\frac{2\pi i}{3}) = \frac{-1+\sqrt{-3}}{2}$.

The action of $G \cong \Gamma/\Gamma[3]$ on $\mathcal{A}(\Gamma[3])$ is described by the representation $D\colon G \to \mathrm{GL}_2(\mathbb{C})\colon \overline{T} \mapsto D_T, \overline{J} \mapsto D_J$. As $\mathrm{Tr}(D_T) = 1+\omega$ and $\mathrm{Tr}(D_J) = 0$ the representation D affords the irreducible character χ. Since $\det(D_T) = \omega$ and $\det(D_J) = 1$ the determinant character associated to χ equals $\det(\chi) = \lambda$.

(1.5) Aim. Our aim now is to describe the graded commutative $\mathbb{C}$-algebras $\mathcal{A}(\mathcal{G}) = \mathcal{A}(\Gamma[3]^2)^{\mathcal{G}} = \mathcal{A}(\Gamma[3]^2)^{\mathcal{G}/\Gamma[3]^2} = \mathcal{A}(\Gamma[3]^2)^G$ and $\mathcal{A}(\mathcal{H}) = \mathcal{A}(\Gamma[3]^2)^{\mathcal{H}} = \mathcal{A}(\Gamma[3]^2)^{\mathcal{H}/\Gamma[3]^2} = \mathcal{A}(\Gamma[3]^2)^H$. Moreover, we also describe the decomposition $\mathcal{A}(\mathcal{G}) \cong \mathcal{A}(\mathcal{G})^+ \oplus \mathcal{A}(\mathcal{G})^-$, where of course $\mathcal{A}(\mathcal{G})^+ = (\mathcal{A}(\Gamma[3]^2)^{\mathcal{G}})^+ = \mathcal{A}(\mathcal{H})$.

2 Some commutative algebra

(2.1) Graded modules. Let $R = \bigoplus_{d \geq 0} R_d$ be a finitely generated $\mathbb{N}_0$-graded commutative $\mathbb{C}$-algebra, such that $R_0 \cong \mathbb{C}$, and let $R_+ := \bigoplus_{d>0} R_d$. Let $M = \bigoplus_{d \geq d_0} M_d$, for some $d_0 \in \mathbb{Z}$, be a finitely generated $\mathbb{Z}$-graded R-module, and let $H_M := \sum_{d \in \mathbb{Z}} \dim_{\mathbb{C}}(M_d) \cdot T^d \in \mathbb{Q}((T))$ be its Hilbert series. Actually, $H_M \in \mathbb{Q}(T)$, and letting $\nu(M) := -\nu_1(H_M) \in \mathbb{Z}$, where ν_1 is the valuation of $\mathbb{Q}(T)$ at the place $T = 1$, by [3, Ch.2.2] we have $\nu(M) = \dim(R/\mathrm{ann}_R(M))$, where $\mathrm{ann}_R(M) := \{f \in R; Mf = \{0\}\} \trianglelefteq R$ denotes the annihilator of M in R.

An algebraically independent set $\mathcal{P} \subseteq (R/\mathrm{ann}_R(M))_+$ of homogeneous elements such that M is a finitely generated $\mathbb{C}[\mathcal{P}]$-module is called a set of primary generators of M. By [3, Thm.2.2.7] sets of primary generators always exist, and given such a set $\mathcal{P}$, a set $\mathcal{S} \subseteq M$ of homogeneous $\mathbb{C}[\mathcal{P}]$-module generators is called a set of secondary generators of M.

If there is a set of primary generators $\mathcal{P}$ such that M is a free $\mathbb{C}[\mathcal{P}]$-module, then M is called a Cohen-Macaulay R-module. In this case, by [3, Ch.4.3] M is a free $\mathbb{C}[\mathcal{P}]$-module for any set $\mathcal{P}$ of primary generators. Moreover, if $\mathcal{P} = \{p_1, \ldots, p_n\}$ is a set of primary generators, where necessarily $n = \nu(M)$, and $\mathcal{S} = \{s_1, \ldots, s_m\}$ is a minimal set of secondary generators, then we have $H_M = \frac{\sum_{j=1}^{m} T^{\deg(s_j)}}{\prod_{i=1}^{n}(1-T^{\deg(p_i)})}$; hence if a set of primary generators is fixed, then the degrees of the elements of a minimal set of secondary generators can be read off from H_M. Finally, if R is a Cohen-Macaulay R-module, then R is called a Cohen-Macaulay algebra.

(2.2) Primary generators. We will use the following characterisation of sets of primary generators: A set $\mathcal{P} \subseteq (R/\mathrm{ann}_R(M))_+$ of homogeneous elements is algebraically independent if and only if $|\mathcal{P}| = \dim(\mathbb{C}\langle\mathcal{P}\rangle)$, and by the graded Nakayama Lemma, see [5, La.3.5.1], M is a finitely generated $\mathbb{C}\langle\mathcal{P}\rangle$-module if and only if $M/M\mathcal{P}$ is a finitely generated $\mathbb{C}$-vector space. Thus $\mathcal{P}$ is a set of primary generators, if and only if $|\mathcal{P}| = \nu(M)$ and $\dim_{\mathbb{C}}(M/M\mathcal{P}) < \infty$.

(2.3) Reynolds operators. Let G be a finite group, which acts on R as a group of automorphisms of graded $\mathbb{C}$-algebras. E. g. if V is a finitely-generated $\mathbb{C}G$-module, then we may choose $R = S[V]$ to be the symmetric algebra over V, which is a polynomial algebra having any $\mathbb{C}$-basis of V as set of indeterminates.

Let $R^\lambda := \{f \in R; f^g = \lambda(g) \cdot f \text{ for all } g \in G\}$, where λ is a linear character of G. In particular, for the trivial character we get the finitely generated graded $\mathbb{C}$-subalgebra $R^G = R^1 \subseteq R$. Moreover, $R^\lambda \leq R$ is a finitely generated graded R^G-submodule, and we have the generalised Reynolds operator $\mathcal{R}^\lambda \colon R \to R^\lambda \colon f \mapsto \frac{1}{|G|} \cdot \sum_{g \in G} \lambda(g)^{-1} \cdot f^g$, which is a surjective R^G-module projection. Again we write $\mathcal{R}^G := \mathcal{R}^1$.

(2.4) The Hochster-Eagon Theorem. Assume that R is a Cohen-Macaulay algebra, and let $\mathcal{P} \subseteq R^G$ be a set of primary generators. Since $R^G \subseteq R$ is a finite ring extension, $\mathcal{P}$ also is a set of primary generators of R, and hence R is a finitely generated free graded $\mathbb{C}[\mathcal{P}]$-module. Using the Reynolds operator $\mathcal{R}^\lambda$ we conclude that R^λ is a $\mathbb{C}[\mathcal{P}]$-direct summand of R, and hence R^λ is a finitely generated projective graded $\mathbb{C}[\mathcal{P}]$-module. Thus by [3, La.4.1.1] R^λ is a finitely generated free graded $\mathbb{C}[\mathcal{P}]$-module. Hence if $R^\lambda \neq \{0\}$ we have $\mathbb{C}[\mathcal{P}] \cap \operatorname{ann}_{R^G}(R^\lambda) = \{0\}$, and thus R^λ is a Cohen-Macaulay R^G-module.

Note that the above proof shows that if $R^\lambda \neq \{0\}$, then any set of primary generators of R^G also is a set of primary generators of R^λ. Moreover, for the trivial character this shows that R^G is a Cohen-Macaulay algebra as well. In particular, since the symmetric algebra $S[V]$ is a polynomial algebra and hence a Cohen-Macaulay algebra, we deduce that $S[V]^G$ is a Cohen-Macaulay algebra as well, which is the celebrated Hochster-Eagon Theorem in polynomial invariant theory, see [3, Thm.4.3.6].

(2.5) Secondary generators. We still assume that R is a Cohen-Macaulay algebra, hence R^G is a Cohen-Macaulay algebra as well. Let the Hilbert series H_{R^G} of R^G, and a set $\mathcal{P} = \{p_1, \ldots, p_n\}$, where $n = \dim(R^G) = \dim(R)$, of primary generators of R^G be given. From H_{R^G} we deduce the cardinality of a minimal set of secondary generators of R^G and their degrees. In this situation, there is the following variant of the method in [5, Ch.3.5] to compute a minimal set of secondary generators, i. e. a minimal $\mathbb{C}[\mathcal{P}]$-module generating set of R^G:

The embedding $R^G \subseteq R$ induces a homomorphism $\sigma \colon R^G \to R/\mathcal{P}R$ of graded $\mathbb{C}$-algebras, where we have $\mathcal{P}R^G \subseteq \ker(\sigma)$. Conversely, for $f \in \ker(\sigma)$ we have $f = \sum_{i=1}^n p_i f_i$, for suitable $f_i \in R$. Applying the Reynolds operator $\mathcal{R}^G$ yields $f = \mathcal{R}^G(f) = \sum_{i=1}^n p_i \cdot \mathcal{R}^G(f_i) \in \mathcal{P}R^G$, and thus we have $\ker(\sigma) = \mathcal{P}R^G$. Hence σ induces an embedding $R^G/\mathcal{P}R^G \to R/\mathcal{P}R$ of graded $\mathbb{C}$-algebras.

Now let $\mathcal{S} \subseteq R^G$ be a set of homogeneous elements. By the graded Nakayama Lemma, see [5, La.3.5.1], $\mathcal{S}$ is a minimal $\mathbb{C}[\mathcal{P}]$-module generating set of R^G if and only if its image in $R^G/\mathcal{P}R^G$ is a $\mathbb{C}$-basis of $R^G/\mathcal{P}R^G$. Hence if $\mathcal{S}$ additionally fulfils the above numerical conditions on cardinality and degrees, then $\mathcal{S}$ is a minimal $\mathbb{C}[\mathcal{P}]$-module generating set of R^G if and only if its image in $R^G/\mathcal{P}R^G$ is $\mathbb{C}$-linearly independent, which in turn is the case if and only if $\sigma(\mathcal{S}) \subseteq R/\mathcal{P}R$ is $\mathbb{C}$-linearly independent. Since $\dim_{\mathbb{C}}(R/\mathcal{P}R) < \infty$ the latter condition can be verified computationally using Gröbner basis techniques.

(2.6) Molien's formula. For the case of the symmetric algebra $S[V]$, representation theory provides a tool to compute the Hilbert series of $S[V]^\lambda$ as

$$H_{S[V]^\lambda} = \frac{1}{|G|} \cdot \sum_{g \in G} \lambda(g)^{-1} \cdot \left(\prod_{i=1}^{n} \frac{1}{1 - \epsilon_i(g) \cdot T} \right) \in \mathbb{C}(T),$$

where $n = \dim_{\mathbb{C}}(V)$ and $\epsilon_1(g), \ldots, \epsilon_n(g) \in \mathbb{C}$ are the eigenvalues of the action of $g \in G$ on V. Hence if the character afforded by V is known, this formula can be evaluated from the character table of G, see [3, Ch.2.5]. This method and tools to compute character tables are available as built-in facilities in GAP.

3 The algebras $\mathcal{A}(\Gamma[3]^2)$, $\mathcal{A}(\mathcal{G})$ and $\mathcal{A}(\mathcal{H})$

In the sequel let $R := \mathbb{C}[X_{11}, X_{12}, X_{21}, X_{22}]$ and $\delta := X_{11}X_{22} - X_{12}X_{21} \in R$. Moreover, let $\{F_1, F_2\} \subseteq \mathbb{M}_1(\Gamma[3])$ such that $\mathcal{A}(\Gamma[3]) = \mathbb{C}\langle F_1, F_2\rangle$ as in (1.2).

(3.1) Theorem. For $i, j \in \{1, 2\}$ let $F_{ij} := F_i(z_1)F_j(z_2) \in \mathbb{M}_1(\Gamma[3]^2)$. Then the algebra homomorphism $R \to \mathbb{C}\langle F_{11}, F_{12}, F_{21}, F_{22}\rangle \colon X_{ij} \mapsto F_{ij}$ induces an isomorphism of graded algebras $R/\delta R \cong \mathbb{C}\langle F_{11}, F_{12}, F_{21}, F_{22}\rangle = \mathcal{A}(\Gamma[3]^2)$. In particular, $\mathcal{A}(\Gamma[3]^2)$ is a Cohen-Macaulay algebra.

Proof. Any $f(z_1, z_2) \in \mathbb{M}_k(\Gamma[3]^2)$, where $k \geq 0$, is a modular form in either variable, if the other variable is fixed. Hence f is a $\mathbb{C}$-linear combination of $\{f_i(z_1)f_j(z_2); 1 \leq i, j \leq l\}$, where $\{f_1, \ldots, f_l\}$ is a $\mathbb{C}$-basis of $\mathbb{M}_k(\Gamma[3])$. Hence we have $\mathcal{A}(\Gamma[3]^2) = \mathbb{C}\langle F_{11}, F_{12}, F_{21}, F_{22}\rangle$. As $\{F_1, F_2\}$ is algebraically independent, $\{F_1(z_1), F_2(z_1), F_1(z_2), F_2(z_2)\}$ is algebraically independent as well, hence $\mathbb{C}[X_1', X_2', X_1'', X_2''] \to \mathbb{C}[F_1(z_1), F_2(z_1), F_1(z_2), F_2(z_2)] \colon X_i' \mapsto F_i(z_1)$, $X_i'' \mapsto F_i(z_2)$ is an isomorphism of $\mathbb{C}$-algebras. Hence by restriction the inclusion $\mathbb{C}\langle F_{11}, F_{12}, F_{21}, F_{22}\rangle \subseteq \mathbb{C}[F_1(z_1), F_2(z_1), F_1(z_2), F_2(z_2)]$ yields the isomorphism of $\mathbb{C}$-algebras $\mathbb{C}\langle X_1'X_1'', X_1'X_2'', X_2'X_1'', X_2'X_2''\rangle \to \mathbb{C}\langle F_{11}, F_{12}, F_{21}, F_{22}\rangle$.

The Jacobian matrix of partial derivatives of $\{X_1'X_1'', X_1'X_2'', X_2'X_1'', X_2'X_2''\}$ with respect to $\{X_1', X_2', X_1'', X_2''\}$ is by a straightforward calculation seen to have vanishing determinant, but non-vanishing (3×3)-minors. Thus by the Jacobian Criterion, see [3, Prop.5.4.2], we have $\dim(\mathbb{C}\langle X_1'X_1'', X_1'X_2'', X_2'X_1'', X_2'X_2''\rangle) = 3$.

Letting $\pi \colon R \to \mathbb{C}\langle X_1'X_1'', X_1'X_2'', X_2'X_1'', X_2'X_2''\rangle \colon X_{ij} \mapsto X_i'X_j''$, we conclude that $\ker(\pi) \trianglelefteq R$ is a prime ideal such that $\dim(R/\ker(\pi)) = 3$. We have $\delta \in \ker(\pi)$. Since δ is irreducible, $\delta R \trianglelefteq R$ is a prime ideal, where by Krull's principal ideal theorem, see [3, Cor.2.3.3], we have $\dim(R/\delta R) = 3$, thus $\ker(\pi) = \delta R$. (Note that $\ker(\pi) \trianglelefteq R$ is an ideal of syzygies, which hence can also be computed explicitly as an elimination ideal using Gröbner basis techniques, see [2, Ch.6.1].)

Since R is a polynomial algebra, $\delta \in R$ is a regular element, and thus by [3, Prop.4.3.4] $R/\delta R$ is a Cohen-Macaulay algebra. ♯

(3.2) Lifts. The Γ^2-action on $\mathcal{A}(\Gamma[3]^2)$ is determined by its action on $\mathbb{M}_1(\Gamma[3]^2)$, which with respect to the $\mathbb{C}$-basis $\{F_{11}, F_{12}, F_{21}, F_{22}\}$ is given by the representation $D \otimes D \colon \Gamma^2 \to \mathrm{GL}_4(\mathbb{C}) \colon (M_1, M_2) \mapsto D_{M_1} \otimes D_{M_2}$, where '$\otimes$' denotes the Kronecker product of matrices; note that $\Gamma[3]^2 \leq \ker(D \otimes D)$. We lift this action along π to an action of Γ^2 on R:

Letting Γ^2 act on the degree 1 component $R_1 \leq R$, where the representing matrices with respect to the $\mathbb{C}$-basis $\{X_{11}, X_{12}, X_{21}, X_{22}\} \subseteq R_1$ are given by

$D \otimes D$, defines an action of Γ^2 as a group of automorphisms of the graded $\mathbb{C}$-algebra R. For $M \in \Gamma$ we by a straightforward calculation get $\delta^{(1,M)} = \delta^{(M,1)} = \det(D_M) \cdot \delta = \lambda(\overline{M}) \cdot \delta$, where λ is the linear character of G as in (1.4). Hence $\langle \delta \rangle_{\mathbb{C}} \leq R_2$ is a $\mathbb{C}\Gamma^2$-submodule, $\delta R \trianglelefteq R$ is a Γ^2-invariant ideal, and thus this Γ^2-action on R lifts the Γ^2-action on $\mathcal{A}(\Gamma[3]^2)$.

This also lifts the $\mathcal{G}$-action and hence the G-action, yielding the representation $\Delta \colon G \to \mathrm{GL}_4(\mathbb{C})$ given by $\overline{T} \mapsto \Delta_T := (D_{T^{-\mathrm{tr}}} \otimes D_T) = (D_{J^{-1}} D_T D_J) \otimes D_T$ and $\overline{J} \mapsto \Delta_J := D_J \otimes D_J$. As $\mathrm{Tr}(\Delta_T) = \mathrm{Tr}(D_T)^2$ and $\mathrm{Tr}(\Delta_J) = \mathrm{Tr}(D_J)^2$, the representation Δ affords the character $\chi^2 = \lambda + \tau$. Moreover, by the above computation we have $\delta^{\overline{M}} := \lambda^2(\overline{M}) \cdot \delta$ for $\overline{M} \in G$, hence the $\mathbb{C}G$-submodule $\langle \delta \rangle_{\mathbb{C}} \leq R_2$ affords the linear character $\lambda^2 = \lambda' = \lambda^{-1}$.

Similarly, we lift the flip map α on $\mathcal{A}(\Gamma[3]^2)$ to R: The action of α on $\mathcal{A}(\Gamma[3]^2)$ is given as $\alpha \colon F_{ij} \mapsto F_{ji}$. Letting α act on $R_1 \leq R$, with respect to the $\mathbb{C}$-basis $\{X_{11}, X_{12}, X_{21}, X_{22}\} \subseteq R_1$, by

$$\alpha \mapsto \Delta_\kappa := \begin{bmatrix} 1 & . & . & . \\ . & . & 1 & . \\ . & 1 & . & . \\ . & . & . & 1 \end{bmatrix} \in \mathrm{GL}_4(\mathbb{C}),$$

defines an action of α as an automorphism of the graded $\mathbb{C}$-algebra R, such that $\delta^\alpha = \delta$. Thus this lifts the action of α on $\mathcal{A}(\Gamma[3]^2)$, and also lifts the $\mathcal{H}$-action and hence the H-action, extending the representation Δ to H. Since $Z(G) \leq \ker(\Delta)$ this is a representation of $H/Z(G) \cong \langle \kappa \rangle \ltimes G/Z(G)$, and since $\kappa\overline{J} \in H/Z(G)$ centralises $G/Z(G)$ we conclude $\langle \kappa \rangle \ltimes G/Z(G) = \langle \kappa \overline{J} \rangle \times G/Z(G) \cong C_2 \times G/Z(G)$.

It turns out that $\mathrm{Tr}(\Delta_{\kappa\overline{J}}) = -2$, hence the eigenspaces of $\Delta_{\kappa\overline{J}}$ with respect to the eigenvalues 1 and -1 have $\mathbb{C}$-dimensions 1 and 3, respectively. Thus by Schur's Lemma we conclude that Δ affords the character $(1 \otimes \lambda) + (1^- \otimes \tau)$ of $C_2 \times G/Z(G)$, where 1 and 1^- denote the trivial and the non-trivial character of the cyclic group C_2 of order 2, respectively. Moreover, the $\mathbb{C}H$-submodule $\langle \delta \rangle_{\mathbb{C}} \leq R$ affords the linear character $1 \otimes \lambda^2 = 1 \otimes \lambda'$ of $C_2 \times G/Z(G)$.

(3.3) Theorem. The $\mathbb{C}$-algebra $\mathcal{A}(\Gamma[3]^2)^+ = \mathbb{C}[F_{11}, F_{22}, F_{12} + F_{21}]$ is a polynomial algebra generated in degree 1, and $\mathcal{A}(\Gamma[3]^2)^- = (F_{12} - F_{21}) \cdot \mathcal{A}(\Gamma[3]^2)^+$ is a free $\mathcal{A}(\Gamma[3]^2)^+$-module generated in degree 1.

Proof. Let $R^+, R^- \leq R$ be the eigenspaces of α with respect to the eigenvalues 1 and -1, respectively. Then $R \cong R^+ \oplus R^-$ as graded R^+-modules, and since $\delta \in R^+$ we have $\delta R = (\delta R)^+ \oplus (\delta R)^-$, where $(\delta R)^\pm := (\delta R) \cap R^\pm = \delta R^\pm$. Hence we have $R/\delta R \cong R^+/\delta R^+ \oplus R^-/\delta R^-$, and thus $\mathcal{A}(\Gamma[3]^2)^+ \cong R^+/\delta R^+$ as graded $\mathbb{C}$-algebras, and $\mathcal{A}(\Gamma[3]^2)^- \cong R^-/\delta R^-$ as graded $(R^+/\delta R^+)$-modules.

The eigenspaces of $\Delta_\kappa \in \mathrm{GL}_4(\mathbb{C})$ with respect to the eigenvalues 1 and -1 have $\mathbb{C}$-dimensions 3 and 1, respectively. Hence evaluating Molien's formula with respect to the character $3 \cdot 1 + 1 \cdot 1^-$ of $\langle \alpha \rangle \cong C_2$, using the facilities available in

GAP, we find the Hilbert series of $R^+ = R^{\langle\alpha\rangle}$ as $H_{R^+} = \frac{1}{(1-T)^3\cdot(1-T^2)}$. As Δ_κ is a reflection, by the Shephard-Todd-Chevalley Theorem, see [3, Thm.7.2.1], R^+ is a polynomial algebra. We have $\{X_{11}, X_{22}, X_{12}+X_{21}, \delta\} \subseteq R^+$, and as the corresponding Jacobian matrix of partial derivatives with respect to $\{X_{11}, X_{12}, X_{21}, X_{22}\}$ is by a straightforward calculation seen to have determinant equal to $X_{21} - X_{12} \neq 0$, this set is algebraically independent, implying $R^+ = \mathbb{C}[X_{11}, X_{22}, X_{12}+X_{21}, \delta]$. Hence $R^+/\delta R^+ \cong \mathbb{C}[X_{11}, X_{22}, X_{12}+X_{21}]$ is a polynomial algebra.

We have $(X_{12}-X_{21})R^+ \subseteq R^-$, hence the Hilbert series of $R^+ \oplus (X_{12}-X_{21})R^+ \leq R$ is given as $(1+T)\cdot H_{R^+} = \frac{1}{(1-T)^4} = H_R$, hence $R^- = (X_{12}-X_{21})R^+$. ♯

(3.4) Theorem. a) The $\mathbb{C}$-algebra $\mathcal{A}(\mathcal{G})$ is a Cohen-Macaulay algebra, and its Hilbert series is $H_{\mathcal{A}(\mathcal{G})} = \frac{1+T^4+T^5}{(1-T^2)\cdot(1-T^3)^2} \in \mathbb{Q}(T)$.

b) The $\mathbb{C}$-algebra $\mathcal{A}(\mathcal{G})^+ = \mathcal{A}(\mathcal{H})$ is a Cohen-Macaulay algebra, and $\mathcal{A}(\mathcal{G})^- = \pi(p_3)\mathcal{A}(\mathcal{G})^+$ is a Cohen-Macaulay $\mathcal{A}(\mathcal{G})^+$-module, where the Hilbert series are $H_{\mathcal{A}(\mathcal{G})^+} = \frac{1+T^4+T^5}{(1-T^2)\cdot(1-T^3)\cdot(1-T^6)} \in \mathbb{Q}(T)$ and $H_{\mathcal{A}(\mathcal{G})^-} = T^3 \cdot H_{\mathcal{A}(\mathcal{G})^+} \in \mathbb{Q}(T)$.

Proof. a) Since $\mathcal{A}(\Gamma[3]^2)$ is a Cohen-Macaulay algebra, $\mathcal{A}(\mathcal{G}) = \mathcal{A}(\Gamma[3]^2)^G \cong (R/\delta R)^G$ is a Cohen-Macaulay algebra as well. Moreover, using the Reynolds operator $\mathcal{R}^G$ we conclude that $(R/\delta R)^G \cong R^G/(\delta R)^G$. Since for $d \geq 0$ and $f \in R_d$ we have $\delta f \in R^G_{d+2}$ if and only if $f \in R^\lambda_d$, we get $H_{\mathcal{A}(\mathcal{G})} = H_{R^G} - T^2 \cdot H_{R^\lambda}$. As Δ affords the character $\lambda + \tau$ of G, evaluating Molien's formula using GAP we obtain $H_{R^G} = \frac{1+T^3+T^4+T^5+2T^6}{(1-T^2)\cdot(1-T^3)^2\cdot(1-T^4)}$ and $H_{R^\lambda} = \frac{T+T^2+2T^4+T^6+T^7}{(1-T^2)\cdot(1-T^3)^2\cdot(1-T^4)}$.

b) Similarly, $\mathcal{A}(\mathcal{G})^+ = \mathcal{A}(\Gamma[3]^2)^H$ is a Cohen-Macaulay algebra, and $\mathcal{A}(\mathcal{G})^-$ is a Cohen-Macaulay $\mathcal{A}(\mathcal{G})^+$-module. We have $H_{\mathcal{A}(\mathcal{G})^+} = H_{R^H} - T^2 \cdot H_{R^{1\otimes\lambda}}$ and $H_{\mathcal{A}(\mathcal{G})^-} = H_{R^{1^-\otimes 1}} - T^2 \cdot H_{R^{1^-\otimes\lambda}}$. Since Δ affords the character $(1\otimes\lambda)+(1^-\otimes\tau)$ of H, Molien's formula yields $H_{R^H} = \frac{1+T^3+T^4+T^5+2T^6}{(1-T^2)\cdot(1-T^3)\cdot(1-T^4)\cdot(1-T^6)}$ and $H_{R^{1\otimes\lambda}} = \frac{T+T^2+2T^4+T^6+T^7}{(1-T^2)\cdot(1-T^3)\cdot(1-T^4)\cdot(1-T^6)}$, as well as $H_{R^{1^-\otimes 1}} = \frac{T^3+T^6+T^7+T^8+2T^9}{(1-T^2)\cdot(1-T^3)\cdot(1-T^4)\cdot(1-T^6)}$ and $H_{R^{1^-\otimes\lambda}} = \frac{T^4+T^5+2T^7+T^9+T^{10}}{(1-T^2)\cdot(1-T^3)\cdot(1-T^4)\cdot(1-T^6)}$. ♯

(3.5) Primary and secondary generators of $\mathcal{A}(\mathcal{G})$ and $\mathcal{A}(\mathcal{H})$. To compute a set of primary generators of $\mathcal{A}(\mathcal{G})$, we first do so for R^G. Since R is a polynomial algebra, there are well-known methods to compute a set of primary generators of R^G, based on Gröbner basis techniques, see [5, Ch.3.3], and being available as built-in facilities in MAGMA. We find a set $\{p_1, \ldots, p_4\} \subseteq R^G$ of primary generators, having degrees $2, 3, 3, 4$, respectively, and being optimal with respect to degree product; p_1, p_2, p_3 are shown in Table 2.

As we expect to find a set of primary generators of $\mathcal{A}(\mathcal{G})$ having degrees $2, 3, 3$, respectively, we show that $\mathcal{P} := \{\pi(p_1), \pi(p_2), \pi(p_3)\} \subseteq (R/\delta R)^G \cong \mathcal{A}(\mathcal{G})$ is a set of primary generators: Since $(R/\delta R)^G \subseteq R/\delta R$ is a finite ring extension, $\mathcal{P}$ is a set of primary generators of $(R/\delta R)^G$ if and only if $\mathcal{P}$ is a set of primary generators of $R/\delta R$. By (2.2) this in turn is the case if and only if $\dim_{\mathbb{C}}(R/(\delta R+\mathcal{P}R)) < \infty$. The latter condition can be verified computationally using Gröbner basis techniques, see [2, Ch.6.3], and actually is easily checked using MAGMA.

To compute a set of secondary generators of $\mathcal{A}(\mathcal{G})$ we proceed similarly: As R is a polynomial algebra, R^G is a Cohen-Macaulay algebra, and there are well-known methods to compute a set of secondary generators of R^G, see [5, Ch.3.5], and being available as built-in facilities in MAGMA. We find a minimal set $\{s_1, \ldots, s_6\} \subseteq R^G$ of secondary generators, according to H_{R^G} having degrees $0, 3, 4, 5, 6, 6$, respectively; they are not reproduced here.

Since there is a minimal set of secondary generators of $\mathcal{A}(\mathcal{G})$ having degrees $0, 4, 5$, respectively, we show that $\mathcal{S} := \{\pi(s_1), \pi(s_3), \pi(s_4)\} \subseteq (R/\delta R)^G \cong \mathcal{A}(\mathcal{G})$ is a minimal set of secondary generators: By (2.5) $\mathcal{S}$ is a minimal set of secondary generators of $(R/\delta R)^G$ if and only if its image in $(R/\delta R)/\mathcal{P}(R/\delta R) \cong R/(\delta R+\mathcal{P}R)$ is $\mathbb{C}$-linearly independent. As $\dim_{\mathbb{C}}(R/(\delta R+\mathcal{P}R)) < \infty$, the latter condition can be verified computationally using Gröbner basis techniques, and actually is easily checked using MAGMA.

For the $\mathbb{C}$-algebra $\mathcal{A}(\mathcal{H})$ we proceed similarly: We find a set $\{p'_1, \ldots, p'_4\} \subseteq R^H$ of primary generators, having degrees $2, 3, 4, 6$, respectively, and being optimal with respect to degree product. Then $\mathcal{P}' := \{\pi(p'_1), \pi(p'_2), \pi(p'_4)\} \subseteq (R/\delta R)^H \cong \mathcal{A}(\mathcal{H})$ turns out to be a set of primary generators. Moreover, we find a minimal set $\{s'_1, \ldots, s'_6\} \subseteq R^H$ of secondary generators, having degrees $0, 3, 4, 5, 6, 6$, respectively, and $\mathcal{S}' := \{\pi(s'_1), \pi(s'_3), \pi(s'_4)\} \subseteq (R/\delta R)^H \cong \mathcal{A}(\mathcal{H})$ turns out to be a minimal set of secondary generators, having degrees $0, 4, 5$. Finally, as $\mathcal{A}(\mathcal{G})^-$ is a Cohen-Macaulay $\mathcal{A}(\mathcal{G})^+$-module, from $H_{\mathcal{A}(\mathcal{G})^-}$ we conclude that $\mathcal{A}(\mathcal{G})^-$ is a free $\mathcal{A}(\mathcal{G})^+$-module being generated by an element of degree 3, and actually it turns out that $\pi(p_3) \in \mathcal{A}(\mathcal{G})^-$, see Table 2.

In conclusion, we remark that it is possible to compute the algebraic relations between the chosen generators of $\mathcal{A}(\mathcal{G})$ and $\mathcal{A}(\mathcal{H})$, using Gröbner basis and linear algebra techniques; we spare the details. This eventually allows us to write $\mathcal{A}(\mathcal{G})$ and $\mathcal{A}(\mathcal{H})$ as finitely presented commutative algebras, completing the structural algebraic description of these function algebras.

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Table 2: Primary generators.

$$
\begin{aligned}
p_1 &:= X_{11}^2 + 4\cdot(X_{12}^2 + X_{21}^2 - 2X_{22}^2) \\
&\quad +4\cdot(X_{11}+X_{22})(X_{12}+X_{21}) - 4\cdot(X_{11}X_{22}+X_{12}X_{21}) \\
p_2 &:= X_{11}^3 - X_{12}^3 - X_{21}^3 - 8X_{22}^3 + 3\cdot(2X_{11}X_{22}+X_{12}X_{21})(X_{12}+X_{21}) \\
&\quad -3\cdot(X_{11}^2+4X_{22}^2)(X_{12}+X_{21}) + 3\cdot(X_{11}-2X_{22})(X_{12}^2+X_{21}^2) \\
p_3 &:= X_{12}^3 - X_{21}^3 + (2X_{11}X_{22}+X_{12}X_{21})(X_{12}-X_{21}) \\
&\quad +(X_{11}^2+4X_{22}^2)(X_{12}-X_{21}) + (X_{11}-2X_{22})(X_{12}^2-X_{21}^2)
\end{aligned}
$$

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