On p-groups forming Brauer pairs

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November 13, 2006

Abstract

For all primes $p \geq 5$ we describe p-groups of order p^5 and exponent p^2 which form Brauer pairs, and we show that these are Brauer pairs of smallest possible p-power order. All of them arise as cyclic extensions of a certain type and we discuss general criteria under which these cyclic extensions form Brauer pairs.
Mathematics Subject Classification: 20C15, 20D15

1 Introduction

Two finite groups G and H form a *Brauer pair* if they are non-isomorphic, but their character tables including power maps are equivalent; that is, if there exists a bijection $\tau : \mathrm{Cl}(G) \to \mathrm{Cl}(H)$ on the conjugacy classes of these groups which induces a bijection on the complex characters $\sigma : \mathrm{Irr}(H) \to \mathrm{Irr}(G) : \chi \mapsto \chi \circ \tau$ and satisfies $\tau \circ \pi_n^G = \pi_n^H \circ \tau$ for every $n \in \mathbb{Z}$, where $\pi_n^G : \mathrm{Cl}(G) \to \mathrm{Cl}(G)$ is the n-th powermap induced by $G \to G : g \mapsto g^n$.

Brauer [1] asked if there exist Brauer pairs of finite groups. Dade [2] proved that there are Brauer pairs by constructing a Brauer pair of order p^7 and exponent p for all primes $p \geq 7$. Skrzipczyk [11, 8] used the computer algebra system Gap [4] and the classification [9] of all groups of order dividing 2^8 to show that the smallest 2-groups forming Brauer pairs have order 2^8. A similar approach can be used to verify that the smallest 3-groups forming Brauer pairs have order 3^6.

Here we present three new series of Brauer pairs of order p^5 and exponent p^2 for all primes $p \geq 5$. The structure of these Brauer pairs is different to the structure of the groups found by Dade [2]. For our investigation we used the classification of the groups of order dividing p^5 by Girnat [3]. This implies that for $p \geq 5$ there are 15 groups of order p^4 and $61 + 2p + 2a + b$ groups of order p^5 where $a = \gcd(p-1, 3)$ and $b = \gcd(p-1, 4)$. We prove the following.

1 Theorem: *Let $p \geq 5$. Then there exist a Brauer pair, a Brauer b-tuple and, if $a > 1$, a Brauer a-tuple of groups of order p^5. There is no Brauer pair of order p^4.*

Thus many of the groups responsible for the residue classes a and b arising in the formula $61 + 2p + 2a + b$ yield Brauer pairs. However, we note that if $a > 1$, then the second a-tuple of groups of order p^5 does not yield a Brauer pair.

The following list of groups contains the Brauer pairs for Theorem 1 as Series 1–3 and the a-tuple of groups of order p^5 which is not a Brauer pair as Series 4.

The groups are described by power commutator presentations and trivial power and commutator relations are omitted to shorten notation. Let v be a primitive root of the field $\mathbb{F}_p$.

Series 1: $\langle g_1 \dots g_5 \mid [g_2, g_1] = g_3, [g_4, g_1] = g_5, [g_3, g_2] = g_5, g_1^p = g_5^w \rangle$ with $w \in \{1, v\}$

Series 2: $\langle g_1 \dots g_5 \mid [g_2, g_1] = g_3, [g_3, g_1] = g_4, [g_4, g_1] = g_5, [g_3, g_2] = g_5, g_1^p = g_5^w \rangle$ with $w \in \{v, \dots, v^b\}$

Series 3: $\langle g_1 \dots g_5 \mid [g_2, g_1] = g_3, [g_3, g_1] = g_5, [g_3, g_2] = g_4, [g_4, g_2] = g_5, g_1^p = g_5^w \rangle$ with $w \in \{v, \dots, v^a\}$

Series 4: $\langle g_1 \dots g_5 \mid [g_2, g_1] = g_3, [g_3, g_2] = g_4, [g_4, g_2] = g_5, g_1^p = g_5^w \rangle$ with $w \in \{v, \dots, v^a\}$

All of these groups arise as a certain type of extensions of a maximal subgroup by a cyclic group of order p. We present a general criterion under which such cyclic extensions form Brauer pairs in Section 3. Then in Section 4 we show that this criterion applies to the groups in Series 1–3 and thus we obtain a proof that these form Brauer pairs.

In [3] it is shown that the groups in Series 1–4 are pairwise non-isomorphic. For completeness, we include a brief sketch for the non-isomorphism of the groups in Series 1 in Section 4.4. A proof for the other series follows with similar arguments. We remark that the groups in a fixed Series are mutually isoclinic.

In Section 5 we describe the explicit character tables of the groups in Series 1–4. Apart from being interesting in themselves, these tables can be used to observe that the groups in Series 4 are not Brauer pairs. We also comment briefly on the related notion of weak Cayley tables.

Finally, we show that there is no Brauer pairs of order p^4 for $p \geq 5$ in Section 6. This also completes our proof for Theorem 1.

Acknowledgment: We are grateful to Boris Girnat, whose computational results led us to have closer look at the groups in Series 1–4.

2 Preliminaries and Notation

Two finite groups G and H have *equivalent character tables* if there exist a bijection $\tau : \mathrm{Cl}(G) \to \mathrm{Cl}(H)$ on the conjugacy classes of these groups which induces a bijection on the complex characters $\sigma : \mathrm{Irr}(H) \to \mathrm{Irr}(G) : \chi \mapsto \chi \circ \tau$. In this case the map τ is called a *character table equivalence*.

Thus a Brauer pair is a pair of non-isomorphic groups with equivalent character tables and equivalent power maps. Lemma 2 below gives a useful criterion to check whether a pair of groups with equivalent character tables is a Brauer pair. It requires some notation. For $m \in \mathbb{N}$ let $\zeta_m := \exp(\frac{2\pi\sqrt{-1}}{m}) \in \mathbb{C}$ a primitive m-th root of unity. For $n \in \mathbb{N}$ with $\gcd(n, m) = 1$ we denote with $\gamma_n \in \mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q})$ the automorphism defined by taking n-th powers $\gamma_n : \zeta_m \mapsto \zeta_m^n$. For $\rho \in \mathbb{Q}(\zeta_m)$ we also write $\rho^{*n} := \gamma_n(\rho)$. Recall that π_n^G is the n-th power map on the conjugacy classes of G.

2 Lemma: *Let G and H be finite groups and let $\tau\colon Cl(G) \to Cl(H)$ be a character table equivalence. Then it follows that $\tau \circ \pi_n^G = \pi_n^H \circ \tau$ for every $n \in \mathbb{Z}$ if and only if $\tau \circ \pi_p^G = \pi_p^H \circ \tau$ holds for all primes $p \mid |G|$.*

Proof: First note that $\tau(1^G) = 1^H$ and hence $|G| = |H|$. For $n \in \mathbb{N}$ with $\gcd(n, |G|) = 1$ it follows that $\chi \circ (\pi_n^H \circ \tau) = (\chi \circ \pi_n^H) \circ \tau = (\gamma_n \circ \chi) \circ \tau = \gamma_n \circ (\chi \circ \tau) = (\chi \circ \tau) \circ \pi_n^G = \chi \circ (\tau \circ \pi_n^G)$ and hence $\pi_n^H \circ \tau = \tau \circ \pi_n^G$. Thus the result follows from writing $\pi_n^G = \pi_{p_1}^G \cdot \dots \cdot \pi_{p_k}^G$, where $n = p_1 \cdot \dots \cdot p_k$ is a factorization into primes. •

Next we introduce some notation for numbers which will be used in the determination of character tables later. For an odd prime p we define

$$\rho_p := (-\zeta_p)^{\frac{p^2-1}{8}} \cdot \sqrt{(-1)^{\frac{p-1}{2}}} \cdot \sqrt{p} \in \mathbb{C},$$

where we use the positive branch of the square root function. The following Lemma shows that it arises from p-th roots of unity as a Gaussian sum.

3 Lemma: *Let p be an odd prime. Then $\rho_p = \sum_{s=0}^{p-1} \zeta_p^{\binom{s}{2}} \in \mathbb{Q}(\zeta_p)$.*

Proof: First we note that

$$(-\zeta_p)^{-\frac{p^2-1}{8}} \cdot \sum_{s=0}^{p-1} \zeta_p^{\frac{s(s-1)}{2}} = (-1)^{\frac{p^2-1}{8}} \cdot \sum_{s=0}^{p-1} \zeta_p^{\frac{(2s-1)^2-p^2}{8}}.$$

By the quadratic reciprocity law [10, Thm.1.3.5], it follows that $2 \in (\mathbb{F}_p)^{*2}$ if and only if $(-1)^{\frac{p^2-1}{8}} = 1$. The latter holds if and only if $p \equiv \pm 1 \bmod 8$.

Let $p \equiv \pm 1 \bmod 8$. Then with s running through $\mathbb{F}_p$ the exponent $\frac{(2s-1)^2-p^2}{8} = \frac{(2s-1)^2}{2 \cdot 2^2} \in \mathbb{F}_p$ runs through $0 \in \mathbb{F}_p$, and twice through $(\mathbb{F}_p)^{*2}$. Thus

$$\sum_{s=0}^{p-1} \zeta_p^{\frac{(2s-1)^2-p^2}{8}} = 1 + 2 \cdot \sum_{s \in (\mathbb{F}_p)^{*2}} \zeta_p^s.$$

By Gauss's Theorem [12, Cor.4.6] we have

$$\sum_{s \in (\mathbb{F}_p)^{*2}} \zeta_p^s = \frac{1}{2} \cdot \left(-1 + \sqrt{(-1)^{\frac{p-1}{2}}} \cdot \sqrt{p}\right),$$

and hence

$$\sum_{s=0}^{p-1} \zeta_p^{\frac{(2s-1)^2-p^2}{8}} = \sqrt{(-1)^{\frac{p-1}{2}}} \cdot \sqrt{p}.$$

Let $p \equiv \pm 3 \bmod 8$. Then with s running through $\mathbb{F}_p$ the exponent $\frac{(2s-1)^2-p^2}{8} \in \mathbb{F}_p$ runs through $0 \in \mathbb{F}_p$, and twice through $(\mathbb{F}_p)^* \setminus (\mathbb{F}_p)^{*2}$. Thus

$$\sum_{s=0}^{p-1} \zeta_p^{\frac{(2s-1)^2-p^2}{8}} = 1 + 2 \cdot \sum_{s \in (\mathbb{F}_p)^* \setminus (\mathbb{F}_p)^{*2}} \zeta_p^s.$$

By Gauss's Theorem we have

$$\sum_{s \in (\mathbb{F}_p)^* \setminus (\mathbb{F}_p)^{*2}} \zeta_p^s = \frac{1}{2} \cdot \left(-1 - \sqrt{(-1)^{\frac{p-1}{2}}} \cdot \sqrt{p} \right),$$

and hence

$$\sum_{s=0}^{p-1} \zeta_p^{\frac{(2s-1)^2-p^2}{8}} = -\sqrt{(-1)^{\frac{p-1}{2}}} \cdot \sqrt{p}.$$

•

3 Cyclic p-extensions and Brauer pairs

The groups in one of the Series 1–4 have a very similar structure: they are all cyclic p-extensions of a common subgroup as introduced below. The main aim of this section is a general criterion under which cyclic p-extensions form Brauer pairs.

3.1 Cyclic p-extensions

Let U be a p-group of exponent p such that $|U| \leq p^{p-1}$ and let $\alpha \in \mathrm{Aut}(U)$ have order p. Let $H = \langle h \rangle \cong C_{p^2}$ act on U via $u^h := u^\alpha$ and let $H \ltimes_\alpha U$ be the associated semidirect product.

4 Definition: For $c \in Z(U)$ of order p with $c^\alpha = c$ we define the *cyclic p-extension* G_c of U as $G_c := (H \ltimes_\alpha U)/\langle (h^p, c^{-1}) \rangle$.

The groups U and H have natural embeddings into G_c and thus they can be identified with their natural images. Then $G_c = \langle H, U \rangle$ and $U \cap H = C := \langle c \rangle \leq Z(G_c)$ has order p. Thus it follows that G_c is an extension of U by the cyclic group H/C of order p.

5 Remark: The group G_c has the following structure:

- $|G_c| = p \cdot |U| \leq p^p$ and thus G_c is a regular p-group of exponent p^2, see [5, Ch.12.4].
- The subgroup U of G_c is the unique maximal subgroup of exponent p in G_c and hence U is characteristic.
- The factor group $G_c/C \cong C_p \ltimes_\alpha (U/C)$ only depends on C, but not on the particular choice of $c \in C$.

- G_c is a central extension of G_c/C by C.

As G_c is an extension of U by H/C, it follows that every element $g \in G_c$ can be uniquely written in the form $g = h^i u$ for some $u \in U$ and some $i \in \{0, \dots, p-1\}$. This allows to describe the conjugacy structure of G_c as follows.

6 Remark: Conjugation in G_c is described by $(h^i u)^{(h^j v)} = h^i v^{-\alpha^i} u^{\alpha^j} v$, where $u, v \in U$ and $i, j \in \{0, \dots, p-1\}$. Thus conjugation in G_c is independent of c.

3.2 Isomorphism of cyclic p-extensions

Let G_d also be a cyclic p-extension of U for some suitable $d \in Z(U)$. We consider the question when there is an isomorphism $\widehat{\beta}\colon G_c \to G_d$. If such an isomorphism exists, then $U^{\widehat{\beta}} = U$ follows and thus we have $\beta := \widehat{\beta}|_U \in \mathrm{Aut}(U)$. For $v \in U$ let κ_v be the inner automorphism of U induced by the conjugation action of v on U.

7 Lemma: *$\beta \in Aut(U)$ is extendible to an isomorphism $\widehat{\beta}\colon G_c \to G_d$ if and only if there exist $r \in \{1, \dots, p-1\}$ and $v \in U$ such that $c^\beta = d^r$ and $\alpha\beta = \beta\alpha^r\kappa_v$ in $Aut(U)$. In this case $h^{\widehat{\beta}} := h^r v$ defines an isomorphism $\widehat{\beta}\colon G_c \to G_d$ extending β.*

Proof: Let $\beta \in \mathrm{Aut}(U)$ be extendible to an isomorphism $\widehat{\beta}\colon G_c \to G_d$. Then $\widehat{\beta}$ induces an automorphism of $C_p \cong G_c/U \cong G_d/U$, hence there are $r \in \{1, \dots, p-1\}$ and $v \in U$ such that $h^{\widehat{\beta}} = h^r v$. Thus we have $c^\beta = (h^p)^{\widehat{\beta}} = (h^{\widehat{\beta}})^p = (h^r v)^p$. By regularity there is $w \in \langle h^r, v\rangle' \leq U < G_d$ such that $(h^r v)^p = h^{rp} v^p w^p \in G_d$, and since U has exponent p, we conclude $c^\beta = d^r$. Moreover, for $u \in U$ we have $(u^\alpha)^\beta = (u^h)^\beta = (u^\beta)^{(h^{\widehat{\beta}})} = (u^\beta)^{(h^r v)} = ((u^\beta)^{\alpha^r})^v$, hence $\alpha\beta = \beta\alpha^r\kappa_v \in \mathrm{Aut}(U)$.

Conversely, let $r \in \{1, \dots, p-1\}$ and $v \in U$ be as in the assertion. Then extending β by $h^{\widehat{\beta}} := h^r v$, a computation as above shows $(h^p)^{\widehat{\beta}} = c^{\widehat{\beta}}$ and $(u^h)^{\widehat{\beta}} = (u^\alpha)^{\widehat{\beta}}$, for all $u \in U$, thus defining a homomorphism $\widehat{\beta}\colon G_c \to G_d$. •

Thus verifying the criterion of Lemma 7 mainly requires an investigation of $\mathrm{Aut}(U)$. The following remark simplifies the outline of such computations and will be used later.

8 Remark: Let U be a group of exponent p with basis $u_1, \dots, u_l$. Then there exists an injective map $\mathrm{Aut}(U) \to GL_l(\mathbb{F}_p) : \alpha \mapsto (a_{ij})$ where $u_i^\alpha = u_1^{a_{1,i}} \cdots u_l^{a_{l,i}}$. This map is not a homomorphism, unless U is abelian, but it provides a convenient way of describing automorphisms of U.

3.3 Character tables of cyclic p-extensions

We describe $\mathrm{Irr}(G_c)$ in terms of the action of α on $\mathrm{Irr}(U)$. For this purpose we define $\psi^\alpha := \psi \circ \alpha^{-1}\colon u \mapsto \psi(u^{\alpha^{-1}})$ for $\psi \in \mathrm{Irr}(U)$. Then either $\psi^\alpha = \psi$ or $\psi^\alpha \neq \psi$ holds and in the latter case $\{\psi, \psi^\alpha, \dots, \psi^{\alpha^{p-1}}\}$ is an orbit of length p.

9 Remark: Clifford theory, see [6, Ch.6], yields the following.

(a) If $\psi^\alpha = \psi$ holds, then ψ has precisely p extensions to G_c. If $\widehat{\psi}_{G_c}$ is one of them, then the extensions can be described by $\lambda_s \widehat{\psi}_{G_c}$ for $s \in \{0, \ldots, p-1\}$, where $\lambda_s \in \mathrm{Irr}(G_c/U)$ is given as $\lambda_s(h) = \zeta_p^s$. In this case $\psi^{G_c} = \sum_{s \in \{0,\ldots,p-1\}} \lambda_s \widehat{\psi}_{G_c}$.

(b) If $\psi^\alpha \neq \psi$ holds, then the induced character ψ^{G_c} is irreducible, and we have $\psi^{G_c}(g) = 0$ for $g \notin U$ and $(\psi^{G_c})_U = \sum_{j=0}^{p-1} \psi^{\alpha^j}$.

Next we give a criterion to check whether two cyclic p-extensions G_c and G_d with $\langle c \rangle = \langle d \rangle$ have equivalent character tables.

10 Theorem: *Let $d := c^r$ for some $r \in \{1, \ldots, p-1\}$. Assume that for all $\psi \in Irr(U)$ with $C \not\leq \ker(\psi)$ there exists $u \in U$ and $t \in \{1, \ldots, p-1\}$ with $\psi(u) \neq 0$ and $u^\alpha = u \cdot c^t$. Then G_c and G_d have equivalent character tables.*

Proof: Recall that every element of G_c and G_d can be written uniquely in the form $h^i u$ for $u \in U$ and $i \in \{0, \ldots, p-1\}$. Let $\widehat{\tau}\colon G_c \to G_d\colon h^i u \to h^i u$. As the conjugation action in G_c and G_d does not depend on the particular choice of a generator of C, this induces a map $\tau\colon \mathrm{Cl}(G_c) \to \mathrm{Cl}(G_d)$.

The characters of G_c are either induced or extended from characters of U. Thus they are either of the form $\lambda_s \widehat{\psi}_{G_c}$ for some $\psi \in \mathrm{Irr}(U)$ with $\psi = \psi^\alpha$ and some $s \in \{0, \ldots, p-1\}$ or they are of the form ψ^{G_c} for some $\psi \in \mathrm{Irr}(U)$ with $\psi \neq \psi^\alpha$.

Let $\psi \in \mathrm{Irr}(U)$ with $C \not\leq \ker(\psi)$. Then $\psi(c)/\psi(1)$ is a primitive p-th root of unity. As there exists $u \in U$ with $\psi(u) \neq 0$ and $u^\alpha = uc^t$, it follows that $\psi^\alpha(u) = \psi(u^{\alpha^{-1}}) = \psi(u \cdot c^{-t}) = (\psi(c)/\psi(1))^{-t} \cdot \psi(u) \neq \psi(u)$. Hence $\psi^\alpha \neq \psi$ follows. Now we consider two cases.

(1) Let $\psi \in \mathrm{Irr}(U)$ with $\psi^\alpha = \psi$. Then $C \leq \ker(\psi)$ follows by the preceding argument and thus $C \leq \ker(\psi^\alpha)$ holds. Thus $C \leq \ker(\chi)$ for all extensions $\chi \in \mathrm{Irr}(G_c)$ of ψ. As G_c/C is independent of c, this yields that all such χ are independent of c. More precisely, it follows that $\lambda_s \widehat{\psi}_{G_d} \circ \tau(h^i u) = \lambda_s \widehat{\psi}_{G_d}(h^i u) = \zeta_p^{is} \cdot \widehat{\psi}_{G_d}(h^i u \cdot C) = \lambda_s \widehat{\psi}_{G_c}(h^i u \cdot C) = \lambda_s \widehat{\psi}_{G_c}(h^i u)$.

(2) Let $\psi \in \mathrm{Irr}(U)$ with $\psi^\alpha \neq \psi$. Then $\psi^{G_d} \circ \tau(h^i u) = \psi^{G_d}(h^i u) = 0 = \psi^{G_c}(h^i u)$ for $i \in \{1, \ldots, p-1\}$ and $\psi^{G_d} \circ \tau(u) = \psi^{G_d}(u) = \sum_{j=0}^{p-1} \psi^{\alpha^j}(u) = \psi^{G_c}(u)$.

In summary, (1) and (2) yield that $\sigma(\lambda_s \widehat{\psi}_{G_d}) = \lambda_s \widehat{\psi}_{G_c}$ and $\sigma(\psi^{G_d}) = \psi^{G_c}$ and hence τ is a character table equivalence. •

11 Theorem: *In addition to the assumptions of Theorem 10, assume that there is a transversal $T \subseteq U$ of the left cosets of C in U such that for all $t \in T$ and for all $\psi \in Irr(U)$ with $C \not\leq \ker(\psi)$ it follows that $\psi^{G_c}(t) \in \mathbb{Z}$. Then G_c and G_d have equivalent character tables including power maps.*

Proof: We define $\widehat{\tau}\colon G_c \to G_d\colon h^i t c^k \to h^i t c^{kr}$ for $i, k \in \{0, \ldots, p-1\}$ and $t \in T$. As $c^\alpha = c$ holds, this induces a bijection $\tau\colon \mathrm{Cl}(G_c) \to \mathrm{Cl}(G_d)$. By Lemma 2, it is sufficient to prove that τ is compatible with the p-th power map and induces a character table equivalence.

First, we show that this bijection is compatible with the p-th power map. Since G_c is regular, it follows that $(h^i t c^k)^p = h^{ip}(tc^k)^p w^p = c^i \in G_c$ for some $w \in$

$\langle h^i, tc^k\rangle' \leq U < G_c$. Similarly, it follows that $(h^i tc^k)^p = c^{ir} \in G_d$. This yields that $\tau \circ \pi_p^{G_c}(h^i tc^k) = \tau(c^i) = c^{ir} = \pi_p^{G_d}(h^i tc^{kr}) = \pi_p^{G_d} \circ \tau(h^i tc^k)$ and τ is compatible with the p-th power map.

Next, we prove that τ induces a character table equivalence using a similar approach as in the proof of Theorem 10. We consider the following cases.

(1) Let $\psi = \psi^\alpha \in \mathrm{Irr}(U)$. Then $C \leq \ker(\psi)$ and $C \leq \ker(\psi^\alpha)$ holds as in the proof of Theorem 10. Thus for $s \in \{0, \ldots, p-1\}$ we find that $\lambda_s \widehat{\psi}_{G_d} \circ \tau(h^i tc^k) = \lambda_s \widehat{\psi}_{G_d}(h^i tc^{kr}) = \zeta_p^{is} \cdot \widehat{\psi}_{G_d}(h^i t \cdot C) = \lambda_s \widehat{\psi}_{G_c}(h^i t \cdot C) = \lambda_s \widehat{\psi}_{G_c}(h^i tc^k)$.

(2) Let $\psi \neq \psi^\alpha \in \mathrm{Irr}(U)$. Then for $i \in \{1, \ldots, p-1\}$ it follows that $\psi^{G_d} \circ \tau(h^i tc^k) = \psi^{G_d}(h^i tc^{kr}) = 0 = \psi^{G_c}(h^i tc^k)$. Thus it remains to consider the case $i = 0$. If $C \leq \ker(\psi)$, then $\psi^{G_d} \circ \tau(tc^k) = \psi^{G_d}(tc^{kr}) = \sum_{j=0}^{p-1} \psi^{\alpha^j}(t \cdot C) = \psi^{G_c}(tc^k)$ holds. If $C \not\leq \ker(\psi)$, then denote $\zeta := \psi(c)/\psi(1)$ and use the integrality assumption to observe $\psi^{G_d} \circ \tau(tc^k) = \psi^{G_d}(tc^{kr}) = \sum_{j=0}^{p-1} \psi^{\alpha^j}(tc^{kr}) = \zeta^{kr} \cdot \psi^{G_c}(t) = \gamma_r(\zeta^k \cdot \psi^{G_c}(t)) = \gamma_r \circ \psi^{G_c}(tc^k))$.

In summary, (1) and (2) yield that $\sigma(\lambda_s \widehat{\psi}_{G_d}) = \lambda_s \widehat{\psi}_{G_c}$, while $\sigma(\psi^{G_d}) = \psi^{G_c}$ for $C \leq \ker(\psi)$, and $\sigma(\psi^{G_d}) = \gamma_r \circ \psi^{G_c}$ for $C \not\leq \ker(\psi)$. Hence τ is a character table equivalence. •

4 The groups in Series 1–3 are Brauer pairs

In this section we show that the groups in Series 1–3 form Brauer pairs. First, we show that the groups in Series 1–3 have equivalent character tables including power maps using Theorems 10 and 11. To apply these Theorems, we express the considered groups as cyclic p-extensions of a common subgroup U and we determine the character table of the relevant groups U. Secondly, we briefly discuss the non-isomorphism of the groups in the series.

For all groups G_w in the Series 1–4 we use $U := \langle g_2, \ldots, g_5\rangle < G_w$ and we let $c := g_5^w \in Z(U)$ and $h := g_1 \in G_w$. Then as $p \geq 5$ and $|U| = p^4$, it follows that all groups are cyclic p-extensions of the form G_c. The subgroups U for the groups in Series 1 and 2 and in Series 3 and 4 coincide.

4.1 The character table of U in Series 1 and 2

Here $U = U_1 \times U_2$ with $U_1 = \langle g_2, g_3, g_5\rangle \cong \langle g_2, g_3, g_5 \mid [g_3, g_2] = g_5\rangle$ extraspecial of order p^3 and $U_2 = \langle g_4\rangle$ cyclic of order p. The character theory for U_1 and U_2 is well-known and the character table of U can be obtained as product of the tables of these two subgroups, see Figure 1.

4.2 The character table of U in Series 3 and 4

Here $U \cong \langle g_2, \ldots, g_5 \mid [g_3, g_2] = g_4, [g_4, g_2] = g_5\rangle$ and $C = \langle g_5\rangle$. Thus U/C is extraspecial of order p^3 and exponent p and the character theory for U/C is well-known. In particular, the group U/C has p^2 linear irreducible characters and $p-1$ irreducible characters of degree p. These characters inflate to irreducible characters

		centralizer	p^4	p^3
		representative	$g_4^k g_5^l$	$g_2^i g_3^j g_4^k$
		parameter	$0 \leq k, l < p$	$0 \leq i, j, k < p$ $(i, j) \neq (0, 0)$
#	parameter	#	p^2	$p(p^2 - 1)$
p^3	$0 \leq a, b, c < p$	$\chi_{a,b,c}$	ζ_p^{kc}	$\zeta_p^{ia+jb+kc}$
$p(p-1)$	$0 \leq c < p$ $1 \leq d < p$	$\chi_{c,d}$	$p \cdot \zeta_p^{kc+ld}$	0

Figure 1: The character table of $U = \langle g_2, \ldots, g_5 \mid [g_3, g_2] = g_5 \rangle$

of U and it remains to determine those irreducible characters of U which do not have C in their kernel.

For this purpose we consider $V = \langle g_3, g_4, g_5 \rangle \lhd U$. Then V is elementary abelian of order p^3 and thus has p^3 linear irreducible characters $\chi_{b,c,d}$. Let $\kappa \in \mathrm{Aut}(V)$ be the automorphism of V induced by the conjugation action of $g_2 \in U$ on V. Then with respect to the basis $\{g_3, \ldots, g_5\}$ this automorphism is given as

$$\kappa \mapsto \begin{pmatrix} 1 & 1 & . \\ . & 1 & 1 \\ . & . & 1 \end{pmatrix} \in GL_3(\mathbb{F}_p).$$

The automorphism κ acts on $\mathrm{Irr}(V) \cong V^* := \mathrm{Hom}(V, \mathbb{C})$, with orbits of length 1 and p. Amongst others, there are orbits $\{\chi_{b+\binom{t}{2}d,td,d} \mid t \in \{0, \ldots, p-1\}\}$ for $0 \leq b \leq p-1$ and $1 \leq d \leq p-1$, where $\{\chi_{1,0,0}, \chi_{0,1,0}, \chi_{1,0,0}\}$ is the associated dual basis of V^*. Thus defining $\chi_{b,d} := \chi_{b,0,d}^U \in \mathrm{Irr}(U)$, we obtain that $\chi_{b,d}(g_5^l) = p \cdot \zeta_p^{ld}$ and $\chi_{b,d}(g_4^k) = \sum_{t=0}^{p-1} (\zeta_p^{kd})^t = 0$ and, using Lemma 3,

$$\chi_{b,d}(g_3^j g_5^l) = \sum_{t=0}^{p-1} \chi_{b+\binom{t}{2}d,td,d}(g_3^j g_5^l) = \zeta_p^{jb+ld} \cdot \sum_{t=0}^{p-1} (\zeta_p^{\binom{t}{2}})^{jd} = \zeta_p^{jb+ld} \cdot \rho_p^{*jd}.$$

This completes the character table of U, see Figure 2.

4.3 The equivalence of character tables including power maps

Now we can read off that the groups in Series 1–3 have equivalent character tables including power maps. We give an outline in the following lemmas.

12 Lemma: *The assumptions of Theorem 10 are satisfied for the groups in Series 1–3, but not for the groups in Series 4.*

Proof: This can be verified by direct inspection for each individual case using the character tables in Figures 1 and 2. For the groups in Series 1 and 2, one can use $u = g_4$ and $t = 1$ to obtain the desired result. For the groups in Series 3, possible

		centralizer	p^4	p^3	p^3	p^2
		representative	g_5^l	g_4^k	$g_3^j g_5^l$	$g_2^i g_3^j$
		parameter	$0 \le l < p$	$1 \le k < p$	$1 \le j < p$ $0 \le l < p$	$1 \le i < p$ $0 \le j < p$
#	parameter	#	p	$p-1$	$p(p-1)$	$p(p-1)$
p^2	$0 \le a, b < p$	$\chi_{a,b}$	1	1	ζ_p^{jb}	ζ_p^{ia+jb}
$p-1$	$1 \le c < p$	χ_c	p	$p \cdot \zeta_p^{kc}$	0	0
$p(p-1)$	$0 \le b < p$ $1 \le d < p$	$\chi_{b,d}$	$p \cdot \zeta_p^{ld}$	0	$\zeta_p^{jb+ld} \cdot \rho_p^{*jd}$	0

Figure 2: The character table of $U = \langle g_2, \ldots, g_5 \mid [g_3, g_2] = g_4, [g_4, g_2] = g_5 \rangle$

choices are $u = g_3$ and $t = 1$. For the groups in Series 4, note that u has to be of the form $u = g_3^j g_5^l$ for some j and l. But then $u^\alpha = u$ holds and hence the assumptions of Theorem 10 are not satisfied. •

13 Lemma: *The assumptions of Theorem 11 hold for the groups in Series 1–3.*

Proof: By Lemma 12 it is sufficient to check the additional assumption in Theorem 11. Again, this can be verified by direct inspection for each individual case using the character tables in Figures 1 and 2 and the conjugation action of $g_1 \in G_w$ on U. As transversal T for the left cosets of C in U we use $T = \{g_2^i g_3^j g_4^k \mid 0 \le i, j, k \le p-1\}$ in all cases. Then it is straightforward to determine for every $\psi \in \mathrm{Irr}(U)$ with $C \not\le \ker(\psi)$ that $\psi^{G_c}(t) = 0$ holds and hence the additional assumption of Theorem 11 is satisfied. •

4.4 Non-Isomorphism of the groups in Series 1

We include a brief sketch for a proof that the groups described in Series 1 are pairwise non-isomorphic. The non-isomorphism for the groups in Series 2-4 can be proved with a similar approach.

Let $G_w = \langle g_1 \ldots g_5 \mid [g_2, g_1] = g_3, [g_4, g_1] = g_5, [g_3, g_2] = g_5, g_1^p = g_5^w \rangle$ for some $w \in \mathbb{F}_p^*$ and let $U = \langle g_2, \ldots, g_5 \mid [g_3, g_2] = g_5 \rangle$. We want to apply Lemma 7 to this setup. For this purpose we first investigate $\mathrm{Aut}(U)$ using Remark 8 to describe automorphisms with respect to the basis $g_2, \ldots, g_5$ of U.

First, note that $Z(U) = \langle g_4, g_5 \rangle$ and $U' = \langle g_5 \rangle \le Z(U)$ are $\mathrm{Aut}(U)$-invariant subgroups. Hence any $\beta \in \mathrm{Aut}(U)$ has the form

$$\beta \mapsto \begin{pmatrix} a & b & c & d \\ e & f & g & h \\ 0 & 0 & i & j \\ 0 & 0 & 0 & k \end{pmatrix} \in GL_4(\mathbb{F}_p),$$

where $i, k \neq 0$. Further, as $[g_3, g_2] = g_5$, it follows that $k = af - be$ holds. The inner automorphisms $\mathrm{Inn}(U) = \langle \kappa_{g_3}^{-1}, \kappa_{g_2} \rangle$ are given by

$$\kappa_{g_3}^{-1} \mapsto \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } \kappa_{g_2} \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The concatenation $\beta\beta' \in \mathrm{Aut}(U)$ is described as

$$\beta\beta' \mapsto \begin{pmatrix} aa' + be' & ab' + bf' & ac' + bg' + ci' & * \\ ea' + fe' & eb' + ff' & ec' + fg' + gi' & * \\ 0 & 0 & ii' & ij' + jk' \\ 0 & 0 & 0 & kk' \end{pmatrix},$$

where $a', \ldots, k' \in \mathbb{F}_p$ are the matrix entries associated to $\beta' \in \mathrm{Aut}(U)$. We note that for the application of Lemma 7 it is sufficient to compute modulo $\mathrm{Inn}(U)$ and hence it is not necessary to determine the matrix entries '$*$' explicitly.

Let α be the automorphism induced by the conjugation action of g_1 on U. Then for $r \in \{1, \ldots, p-1\}$ it follows that

$$\alpha^r \mapsto \begin{pmatrix} 1 & r & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & r \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and hence

$$\alpha\beta \mapsto \begin{pmatrix} a+e & b+f & c+g & * \\ e & f & g & * \\ 0 & 0 & i & j+k \\ 0 & 0 & 0 & k \end{pmatrix} \text{ and } \beta\alpha^r \mapsto \begin{pmatrix} a & ra+b & c & * \\ e & re+f & g & * \\ 0 & 0 & i & ri+j \\ 0 & 0 & 0 & k \end{pmatrix}.$$

Thus, by Lemma 7, two groups G_w and G_v are isomorphic if and only if $wk = vr$ and $e = g = 0$ and $ra = f$ and $ri = k$ holds. Since $k = af - be$ holds, this implies that $ri = af = ra^2$ and hence $i = a^2$ and $ra^2 = k$, thus $w^{-1}v = a^2 \neq 0$ follows. If conversely $w^{-1}v = a^2 \neq 0$, then we can choose an arbitrary $r \neq 0$ and use $f := ra$ and $i := a^2$ as well as $k := ra^2$ to find that the above conditions are satisfied.

5 Explicit character tables

We determine the explicit character tables for the groups in Series 1–4. We use the same general strategy in all four cases: we first determine the characters of G_w/C and inflate them to G_w and then we construct the remaining irreducible characters from characters of normal subgroups of G_w. In particular, the irreducible characters of the maximal subgroups U as described in Figures 1 and 2 are useful for this purpose. As a preliminary step, we describe a method to determine the conjugacy classes of G_w.

5.1 The conjugacy classes of G_w

As G_w is a central extension of G_w/C by the cyclic group $C = \langle c \rangle$ of order p, the conjugacy classes of G_w can be described in terms of the conjugacy classes of G_w/C: The preimage of a conjugacy class of G_w/C either is a single conjugacy class, or splits into p pairwise distinct conjugacy classes of the same cardinality, whose representatives differ by a power of c. Hence in the non-split case the corresponding centralizer orders in G_w/C and G_w are the same, while in the split case these differ by a factor of p.

The factor group G_w/C is a semidirect product of the form $G_w/C \cong C_p \ltimes_\alpha (U/C)$, where $\alpha \in \mathrm{Aut}(U/C)$ induced by the conjugacy action of g_1C on U/C. Conjugation in G_w/C is described by

$$(\alpha^m, u)^{(\alpha^n, v)} = (\alpha^m, v^{-\alpha^m} u^{\alpha^n} v) = (\alpha^m, (\widetilde{v}^{-\alpha^m} u \widetilde{v})^{\alpha^n}),$$

where $\widetilde{v} := v^{\alpha^{-n}} \in U/C$. We denote the action of U/C on itself by $u \mapsto v^{-\alpha} uv$ as *α-conjugation* action. Then the conjugacy classes of G_w/C fall into cohorts parameterized by $m \in \{0, \ldots, p-1\}$. The m-th cohort is in bijection with the set of α^m-conjugacy classes of α-orbits on G_w/C which equals the set of α-orbits of α^m-conjugacy classes of G_w/C. Thus the 0-th cohort is the set of the α-orbits on the conjugacy classes of U/C and the classes in this cohort are called the *inner conjugacy classes*. The other classes are called *outer conjugacy classes*.

5.2 The character table of G_w in Series 1

In this case the character table of G_w can be determined quite easily. From the power commutator presentation of G_w one can read off that $G_w/C \cong U$ holds and hence the character tables of G_w/C and of U are given by Figure 1. The irreducible characters of G_w/C inflate to irreducible characters of G_w. Hence it remains to determine the irreducible characters of G_w not having C in their kernels.

For this purpose we consider the characters $\chi_{0,d} \in \mathrm{Irr}(U)$ with $d \in \{1, \ldots, p-1\}$ in the notation of Figure 1. Inducing these characters to G_w yields $p-1$ characters $\chi_d \in \mathrm{Irr}(G_w)$ with $\chi_d(g_5^l) = p^2 \cdot \zeta_p^{ld}$ for $l \in \{0, \ldots, p-1\}$ and $\chi_d(g) = 0$ whenever $g \notin C$; see Figure 3.

5.3 The character table of G_w in Series 2

The character tables of the groups in this Series can be determined with the same approach as in Section 5.2. In this case, one can read off from the power commutator presentation of G_w that the factor G_w/C is isomorphic to the maximal subgroup U in Series 3. Hence the character table of G_w/C is available in Figure 2. The remaining irreducible characters of G_w can be found by inducing characters of U to G_w as in Section 5.2. The resulting character table of G_w is displayed in Figure 4.

		centralizer	p^5	p^4	p^3
		representative	g_5^l	$g_3^j g_4^k$	$g_1^m g_2^i g_4^k$
		parameter	$0 \leq l < p$	$0 \leq j,k < p$ $(j,k) \neq (0,0)$	$0 \leq m,i,k < p$ $(i,m) \neq (0,0)$
#	parameter	#	p	p^2-1	$p(p^2-1)$
p^3	$0 \leq a,c,e < p$	$\chi_{a,c,e}$	1	ζ_p^{kc}	$\zeta_p^{me+ia+kc}$
$p(p-1)$	$1 \leq b < p$ $0 \leq c < p$	$\chi_{b,c}$	p	$p \cdot \zeta_p^{jb+kc}$	0
$p-1$	$1 \leq d < p$	χ_d	$p^2 \cdot \zeta_p^{ld}$	0	0

Figure 3: The character table of G_w in Series 1.

		centralizer	p^5	p^4	p^3	p^3	p^2
		representative	g_5^l	g_4^k	g_3^j	$g_2^i g_4^k$	$g_1^m g_2^i$
		parameter	$0 \leq l < p$	$1 \leq k < p$	$1 \leq j < p$	$1 \leq i < p$ $0 \leq k < p$	$1 \leq m < p$ $0 \leq i < p$
#	parameter	#	p	$p-1$	$p-1$	$p(p-1)$	$p(p-1)$
p^2	$0 \leq a < p$ $0 \leq e < p$	$\chi_{a,e}$	1	1	1	ζ_p^{ia}	ζ_p^{me+ia}
$p-1$	$1 \leq b < p$	χ_b	p	p	$p \cdot \zeta_p^{jb}$	0	0
$p(p-1)$	$0 \leq a < p$ $1 \leq c < p$	$\chi_{a,c}$	p	$p \cdot \zeta_p^{kc}$	0	$\zeta_p^{ia+kc} \cdot \rho_p^{*ic}$	0
$p-1$	$1 \leq d < p$	χ_d	$p^2 \cdot \zeta_p^{ld}$	0	0	0	0

Figure 4: The character table of G_w in Series 2.

5.4 The character table of G_w in Series 3

In this case we have $G_w/C = \langle h_1, \ldots, h_4 \mid [h_2, h_1] = h_3, [h_3, h_2] = h_4 \rangle$ and $U = \langle g_2, \ldots, g_5 \mid [g_3, g_2] = g_4, [g_4, g_2] = g_5 \rangle$. Thus $\varphi : G_w/C \to U$ defined by

$$\varphi(h_1) = g_3, \quad \varphi(h_2) = g_2, \quad \varphi(h_3) = g_4^{-1}, \quad \varphi(h_4) = g_5^{-1}$$

defines an isomorphism between G_w/C and U. Hence the character tables for G_w/C and U are given in Figure 2. The irreducible characters of G_w/C inflate to irreducible characters of G_w. The remaining irreducible characters of G_w can be found by inducing characters of U to G_w as in Section 5.2. The resulting character table of G_w is given in Figure 5.

5.5 The character table of G_w in Series 4

As in Section 5.4, we have $G_w/C = \langle h_1, \ldots, h_4 \mid [h_2, h_1] = h_3, [h_3, h_2] = h_4 \rangle$ and $U = \langle g_2, \ldots, g_5 \mid [g_3, g_2] = g_4, [g_4, g_2] = g_5 \rangle$ and thus $U \cong G_w/C$. The character

		centralizer	p^5	p^4	p^3	p^3	p^2
		representative	g_5^l	g_4^k	g_3^j	$g_1^m g_4^k$	$g_1^m g_2^i$
		parameter	$0 \leq l < p$	$1 \leq k < p$	$1 \leq j < p$	$1 \leq m < p$ $0 \leq k < p$	$0 \leq m < p$ $1 \leq i < p$
#	parameter	#	p	$p-1$	$p-1$	$p(p-1)$	$p(p-1)$
p^2	$0 \leq a < p$ $0 \leq e < p$	$\chi_{a,e}$	1	1	1	ζ_p^{me}	ζ_p^{me+ia}
$p-1$	$1 \leq b < p$	χ_b	p	p	$p \cdot \zeta_p^{jb}$	0	0
$p(p-1)$	$1 \leq c < p$ $0 \leq e < p$	$\chi_{c,e}$	p	$p \cdot \zeta_p^{kc}$	0	$\zeta_p^{me+kc} \cdot$ $\rho_p^{*(-mc)}$	0
$p-1$	$1 \leq d < p$	χ_d	$p^2 \cdot \zeta_p^{ld}$	0	0	0	0

Figure 5: The character table of G_w in Series 3.

tables of G_w/C and U are displayed in Figure 2. The irreducible characters of G_w/C inflate to irreducible characters of G_w and it remains to determine the remaining irreducible characters of G_w.

For this purpose we consider $V := \langle g_1, g_3, g_4 \rangle \lhd G_w$ and we note that $V \cong C_{p^2} \times C_p \times C_p$. Let $\kappa \in \mathrm{Aut}(V)$ be the automorphism of V induced by the conjugation action of $g_2 \in G$ and let $w' \in \mathbb{F}_p^*$ be such that $ww' = 1 \in \mathbb{F}_p^*$. Then it follows that $g_5 = g_1^{pw'} \in V$. Identifying V with $\mathbb{Z}/p^2\mathbb{Z} \times \mathbb{F}_p \times \mathbb{F}_p$ using the basis $\{g_1, g_3, g_4\}$ shows that κ is given as

$$\kappa \mapsto \begin{pmatrix} 1 & -1 & . \\ . & 1 & 1 \\ pw' & . & 1 \end{pmatrix} \in GL_3(\mathbb{Z}/p^2\mathbb{Z}).$$

Identifying V^* with $\mathbb{Z}/p^2\mathbb{Z} \times \mathbb{F}_p \times \mathbb{F}_p$ and using the associated dual basis of V^* shows that κ acts on V^* by

$$\kappa \mapsto \begin{pmatrix} 1 & . & w' \\ -p & 1 & . \\ . & 1 & 1 \end{pmatrix} \in GL_3(\mathbb{Z}/p^2\mathbb{Z}).$$

For $f \in \{0, \ldots, p^2-1\}$ such that $p \nmid f$ and $b, c \in \{0, \ldots, p^2-1\}$ in particular there are the following κ-orbits of length p on V^*:

$$\{\chi_{f-ptb-p\binom{t}{3}w'f,\, b+\binom{t}{2}w'f,\, tw'f}; t \in \{0, \ldots, p-1\}\}$$

(Note that here we explicitly need $p \geq 5$.) Thus we obtain $\chi_{f,b,0}^{G_w}(g_5) = p \cdot \zeta_p^{w'f}$ and for $j \in \{1, \ldots, p-1\}$

$$\chi_{f,b,0}^{G_w}(g_3^j) = \zeta_p^{jb} \cdot \sum_{t=0}^{p-1} (\zeta_p^{\binom{t}{2}})^{jw'f} = \zeta_p^{jb} \cdot \rho_p^{*jw'f}.$$

Now let $d \in \{1, \ldots, p-1\}$ and $e \in \{0, \ldots, p-1\}$ such that $f \equiv wd + pe \bmod p^2$. Then $w'f \equiv d \bmod p$ and this yields the characters $\chi_{b,d,e}$ of G_w of degree p, where for $m \in \{1, \ldots, p-1\}$ and $k, l \in \{0, \ldots, p-1\}$ we get

$$\begin{aligned}
\chi_{b,d,e}(g_1^m g_4^k g_5^l) &= \chi_{f,b,0}^{G_w}(g_1^m g_4^k g_5^l) \\
&= \sum_{t=0}^{p-1} \chi_{f-ptb-p\binom{t}{3}w'f,\, b+\binom{t}{2}w'f,\, tw'f}(g_1^m g_4^k g_5^l) \\
&= \zeta_{p^2}^{mf} \cdot \zeta_p^{lw'f} \cdot \sum_{t=0}^{p-1} \zeta_p^{-m(tb+\binom{t}{3}w'f)+ktw'f} \\
&= \zeta_{p^2}^{mwd} \cdot \zeta_p^{me+ld} \cdot \sum_{t=0}^{p-1} \zeta_p^{-m(tb+\binom{t}{3}d)+tkd}.
\end{aligned}$$

Note that the irreducible characters of G_w which are not inflated from G_w/C are not induced from U in this example, so the standard strategy of the examples in Sections 5.2–5.4 does not work here.

		centralizer	p^5	p^4	p^4	p^4	p^2
		representative	g_5^l	g_4^k	$g_3^j g_5^l$	$g_1^m g_4^k g_5^l$	$g_1^m g_2^i$
		parameter	$0 \le l < p$	$1 \le k < p$	$1 \le j < p$ $0 \le l < p$	$1 \le m < p$ $0 \le k, l < p$	$0 \le m < p$ $1 \le i < p$
#	param.	#	p	$p-1$	$p(p-1)$	$p^2(p-1)$	$p(p-1)$
p^2	$0 \le a < p$ $0 \le e < p$	$\chi_{a,e}$	1	1	1	ζ_p^{me}	ζ_p^{me+ia}
$p-1$	$1 \le b < p$	χ_b	p	p	$p \cdot \zeta_p^{jb}$	0	0
$p(p-1)$	$1 \le c < p$ $0 \le e < p$	$\chi_{c,e}$	p	$p \cdot \zeta_p^{kc}$	0	$\zeta_p^{me+kc} \cdot \rho_p^{*(-mc)}$	0
$p^2(p-1)$	$0 \le b, e < p$ $1 \le d < p$	$\chi_{b,d,e}$	$p \cdot \zeta_p^{ld}$	0	$\zeta_p^{jb+ld} \cdot$ ρ_p^{*jd}	$\zeta_{p^2}^{mwd} \cdot \zeta_p^{me+ld} \cdot$ $\sum_{t=0}^{p-1} \zeta_p^{-m(tb+\binom{t}{3}d)+tkd}$	0

Figure 6: The character table of G_w in Series 4.

5.6 The groups in Series 4 are not Brauer pairs - some comments

The character tables for the groups in Series 4 are the only character tables determined in this Section which depend on the parameter w. This yields directly that the groups in Series 1–3 have equivalent character tables and it indicates that the groups in Series 4 are different from the Series 1–3 in this respect.

Based on the character table determined in Figure 6 one can now show that G_w and G_v in Series 4 have equivalent character tables if and only if $G_w \cong G_v$ holds. However, the proof for this fact is rather technical and lengthy and we omit the explicit arguments here. A similar, but much simpler proof of the same type is given in Section 6 to show that there is no Brauer pair of order p^4.

We finally comment briefly on *weak Cayley tables*, for more details see e. g. [7]: The rows and columns of the weak Cayley table of a finite group G are indexed by the elements of G, and the (g,h)-entry is the conjugacy class of $gh \in G$. Two finite groups G and H have *equivalent weak Cayley tables* if there exist a bijection $\tau : G \to H$ respecting conjugacy classes, such that $\tau(gh)$ and $\tau(g)\tau(h)$ are conjugate in H, for all $g, h \in G$. Thus in this case G and H have equivalent character tables.

Hence groups G_w and G_v in Series 4 have equivalent weak Cayley tables if and only if $G_w \cong G_v$. For groups G_w and G_v in one of Series 1–3, by the parametrization of the conjugacy classes given in Figures 3–5, the assumptions of [7, Thm.3.1] are fulfilled with respect to the normal subgroup C. Hence G_w and G_v also have equivalent weak Cayley tables.

6 The groups of order p^4

There are 15 groups of order p^4 for $p \geq 5$. Power commutator presentations for the non-abelian groups of order p^4 are given as follows. Let v be a primitive root of $\mathbb{F}_p$:

$$\begin{aligned}
G_1 &= \langle g_1, \ldots, g_4 \mid [g_2, g_1] = g_4, g_1^p = g_3 \rangle \\
G_2 &= \langle g_1, \ldots, g_4 \mid [g_2, g_1] = g_4, g_1^p = g_3, g_2^p = g_4 \rangle \\
G_3 &= \langle g_1, \ldots, g_4 \mid [g_2, g_1] = g_4, g_1^p = g_3, g_3^p = g_4 \rangle \\
G_4 &= \langle g_1, \ldots, g_4 \mid [g_2, g_1] = g_3, [g_3, g_1] = g_4 \rangle \\
G_5 &= \langle g_1, \ldots, g_4 \mid [g_2, g_1] = g_3, [g_3, g_1] = g_4, g_1^p = g_4 \rangle \\
G_6 &= \langle g_1, \ldots, g_4 \mid [g_2, g_1] = g_3, [g_3, g_1] = g_4, g_2^p = g_4 \rangle \\
G_7 &= \langle g_1, \ldots, g_4 \mid [g_2, g_1] = g_3, [g_3, g_1] = g_4, g_2^p = g_4^v \rangle \\
G_8 &= \langle g_1, \ldots, g_4 \mid [g_2, g_1] = g_4 \rangle \\
G_9 &= \langle g_1, \ldots, g_4 \mid [g_2, g_1] = g_4, g_2^p = g_4 \rangle \\
G_{10} &= \langle g_1, \ldots, g_4 \mid [g_2, g_1] = g_4, g_3^p = g_4 \rangle
\end{aligned}$$

There are pairs of groups in this list having equivalent character tables, but none of them form a Brauer pair as we prove in the following.

14 Theorem: *Let $p \geq 5$. Then there exists no Brauer pair of order p^4.*

Proof: Let G and H two non-isomorphic groups of order p^4 and assume that G and H form a Brauer pair. Then there exists a bijection $\tau : \mathrm{Cl}(G) \to \mathrm{Cl}(H)$ such that $\sigma : \mathrm{Irr}(H) \to \mathrm{Irr}(G) : \chi \mapsto \chi \circ \tau$ is also a bijection and $\tau \circ \pi_n^G = \pi_n^H \circ \tau$ holds for $n \in \mathbb{Z}$.

Then the linear characters of G and H are in bijection with each other, as σ respects degrees of characters. Hence $G/G' \cong H/H'$ follows. Further the number of conjugacy classes of elements of order p coincides in G and in H, since $\tau \circ \pi_p^G = \pi_p^H \circ \tau$ holds. Also, as τ respects centralizer orders, the number of elements of order p

coincides in G and in H, and $Z(G) \cong Z(H)$ holds. In summary, this yields that the only remaining candidates for a Brauer pair are G_6 and G_7.

We prove that G_6 and G_7 have non-equivalent character tables. For this purpose we write $G_6 = G_{v^0}$ and $G_7 = G_v$ where v is a primitive root of $\mathbb{F}_p$ and

$$G_w := \langle g_1, \ldots, g_4 \mid [g_2, g_1] = g_3, [g_3, g_1] = g_4, g_2^p = g_4^w \rangle \text{ for } w \in \mathbb{F}_p^*.$$

Thus G_w is a cyclic p-extension with a common maximal subgroup $U := \langle g_1, g_3, g_4 \rangle$ and central subgroup $C = \langle g_4 \rangle$. The character table of G_w can be determined as in Section 5. First, note that G_w has p^2 linear characters. The remaining irreducible characters of G_w can be induced from the subgroup $V = \langle g_2, g_3 \rangle \cong C_{p^2} \times C_p$ of G_w. It is straightforward to determine that there are p characters in $\mathrm{Irr}(V)$ which are invariant under the action of G_w and the remaining $p(p^2-1)$ characters in $\mathrm{Irr}(V)$ fall into p^2-1 orbits of length p. Thus G_w has p^2-1 characters of degree p; see Figure 7.

Now suppose that there exists a bijection $\tau : \mathrm{Cl}(G_w) \to \mathrm{Cl}(G_v)$ which induces a bijection $\sigma : \mathrm{Irr}(G_w) \to \mathrm{Irr}(G_v)$. Then τ respects the conjugacy class types displayed in Figure 7 and hence, using Galois automorphisms, we may assume that $\tau(g_2^{G_w}) = g_2^{G_v}$. Moreover, we have $\tau(g_4^{G_w}) = (g_4^\lambda)^{G_v}$ for some $\lambda \in \{1, \ldots, p-1\}$. Thus for the faithful character $\chi_{1,0}^v$ of G_v there exists $e \in \{0, \ldots, p\}$ such that $(\chi_{1,0}^v)^\tau = \chi_{\lambda,e}^w$. Hence we have that $\zeta_{p^2}^{w\lambda} \cdot \zeta_p^e \cdot \rho_p^{*\lambda} = \zeta_{p^2}^v \cdot \rho_p$ and this implies $\zeta_{p^2}^{v-w\lambda} \in \mathbb{Q}(\zeta_p)$ and thus $\lambda = v/w \in \mathbb{F}_p^*$. We denote

$$\sigma_p := \sqrt{(-1)^{\frac{p-1}{2}} \cdot p} \in \mathbb{Q}(\zeta_p).$$

Then it follows that

$$(\zeta_p^{\frac{(p^2-1)\lambda}{8}} \cdot \sigma_p^{*\lambda})/(\zeta_p^{\frac{(p^2-1)}{8}} \cdot \sigma_p) = \rho_p^{*\lambda}/\rho_p = \zeta_{p^2}^{v-w\lambda} \cdot \zeta_p^{-e}$$

and hence $\sigma_p^{*\lambda}/\sigma_p$ is a root of unity. Since σ_p is a quadratic irrationality, it follows that $\sigma_p^{*\lambda} = \sigma_p$ and hence $v/w = \lambda \in \mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})^2 \cong (\mathbb{F}_p^*)^2$. But then G_v and G_w are isomorphic. •

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		centralizer	p^4	p^3	p^3	p^2
		representative	g_4^l	g_3^k	$g_2^j g_4^l$	$g_2^j g_1^i$
		parameter	$0 \leq l < p$	$1 \leq k < p$	$1 \leq j < p$ $0 \leq l < p$	$1 \leq i < p$ $0 \leq j < p$
#	parameter	#	p	$p-1$	$p(p-1)$	$p(p-1)$
p^2	$0 \leq a, b < p$	$\chi_{a,b}$	1	1	ζ_p^{jb}	ζ_p^{jb+ia}
$p-1$	$1 \leq c < p$	χ_c	p	$p \cdot \zeta_p^{kc}$	0	0
$p(p-1)$	$1 \leq d < p$ $0 \leq e < p$	$\chi_{d,e}$	$p \cdot \zeta_p^{ld}$	0	$\zeta_{p^2}^{jwd} \cdot \zeta_p^{je+ld} \cdot \rho_p^{*jd}$	0

Figure 7: The character table of G_w of order p^4.

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