The Broué conjecture for the faithful 3-blocks of $4.M_{22}$

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Abstract

We verify Broué's conjecture for the faithful 3-blocks of defect 2 of the non-split central extension of the sporadic simple Mathieu group M_{22} by a cyclic group of order 4. The proof is based on a strategy due to Okuyama and Rickard, where a stable equivalence is lifted to a derived equivalence. The stable equivalence in turn is provided by exploiting a result due to Puig. To handle this particular example, next to theoretical investigations we apply a whole bunch of computational tools.
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1 Introduction

Broué's abelian defect group conjecture [2] says that the principal p-block of a finite group is derived equivalent to its Brauer correspondent, which is a p-block of the normaliser of a p-Sylow subgroup, provided the latter is abelian. One of the central techniques for proving instances of the conjecture involves finding a stable equivalence between the respective module categories, and lifting it to a derived equivalence, see e. g. [15, 21], where in many cases the appropriate stable equivalence is simply restriction. In particular, building on work of several authors it is shown in [9] that Broué's conjecture holds for principal 3-blocks in the case of elementary abelian 3-Sylow subgroups of order 9.

Broué's conjecture is believed to hold for non-principal p-blocks as well, again provided the associated defect group is abelian [2, Question 6.2]. Here too stable equivalences are important, but currently there is a dearth of methods to establish a stable equivalence in the non-principal p-block case. In the present work we prove the following

Theorem. Let $4.M_{22}$ be the non-split central extension of the sporadic simple Mathieu group M_{22} by a cyclic group of order 4. Then Broué's conjecture holds for the faithful 3-blocks of defect 2 of $4.M_{22}$.

The associated defect groups are the 3-Sylow subgroups of $4.M_{22}$, being elementary abelian of order 9. According to the catalogue [24] of 3-blocks of abelian defect group occurring in the Atlas [3] groups, there are no other 3-blocks Morita equivalent to these blocks among the Atlas groups and their subgroups. There are a few special features to this example, and it is treated here by method which has not yet, to our knowledge, been applied to the problem of verifying Broué's conjecture:

In Section 2 we consider a stable equivalence between the global and the local block, which different from those 3-blocks which have been considered to date is not simply restriction, but requires multiplication by a suitable endo-permutation module, using the construction in [17]. We construct this endo-permutation module, using a computational technique involving tensor induction. This allows us to evaluate the functor providing the stable equivalence explicitly for given modules.

Subsequently, in Section 3 we use the strategy in [21, Ch.6.3], which is a modification of the strategy invented in [15], to lift the stable equivalence to a derived equivalence: We use the results of evaluating the stable equivalence at simple modules to find a tilting complex, whose endomorphism ring hence is derived equivalent to the local block. Although the data in the catalogue [24] had indicated that there should be a very simple tilting complex, being amongst the elementary ones defined in [15], our tilting complex turns out to be a of a more general type, it is a 'mixed' elementary tilting complex as defined in [23]. Having the tilting complex in hands, we proceed to find complexes fulfilling the properties assumed in [21, La.5.2]. These are used to finally show that the global block and the endomorphism ring of the tilting complex are Morita equivalent. We remark that we could have used [21, Thm.6.1] directly, but it seemed worth-while to make the tilting complex, which works behind the scenes anyway, explicit.

To arrive at the results presented here, at crucial points we make heavy use of explicit computation, whose results are interspersed among our theoretical investigations. For group theoretical computations, e. g. finding normalisers in permutation groups, we use the facilities available in the computer algebra system GAP [6], while for computations with characters and decomposition matrices we use its character table library. For computations with matrix representations over finite fields, e. g. finding constituents, Loewy series, endomorphism rings, direct sum decompositions, or Green correspondents, we use the computer algebra system MeatAxe [22], and in particular the tools in [11, 12, 13]. Moreover, we use specially tailored GAP and MeatAxe programs, derived from the tools in [14], for induction and tensor induction.

We remark that matrix computations are carried out over the field with 9 elements. But we have to make sure that these computational results are correctly interpreted over the algebraically closed field we are using as the base field for our theoretical considerations: The field with 9 elements is a splitting field for all the relevant simple modules, thus a simple module being found by the MeatAxe actually is absolutely simple. Moreover, indecomposable modules being found by the MeatAxe are explicitly checked to be absolutely indecomposable.

We assume the reader to be familiar with the general notions of representation theory, in particular Brauer correspondence, vertex theory, pointed groups, tilting theory and derived categories; as general references see e. g. [5, 25, 8]. Unless otherwise stated, we throughout consider left modules, write module homomorphisms on the right, and use cochain complexes.

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Table 1: Decomposition matrices of B_+ and b_+.

	$56a$	$56b$	64	$160a$	$160b$
$56a$	1	.	.	.	.
$56b$	.	1	.	.	.
$160a$	.	.	.	1	.
$160b$	.	.	.	.	1
176	1	1	1	.	.
560	1	1	2	1	1

	$1a$	$1b$	2	$1c$	$1d$
$1a$	1	.	.	.	.
$1b$	.	1	.	.	.
$1c$	.	.	.	1	.
$1d$	.	.	.	.	1
2	.	.	1	.	.
8	1	1	2	1	1

2 The stable equivalence

(2.1) The faithful 3-blocks of $4.M_{22}$. Let k be an algebraically closed field of characteristic 3. Let $\zeta \in k$ be a primitive 8-th root of unity, hence $\iota := \zeta^2$ is a primitive 4-th roots of unity. For the necessary facts about the Mathieu group M_{22} we refer the reader to the Atlas. Details are easily checked computationally using GAP and a faithful permutation representation of $4.M_{22}$, e. g. the one on 4928 points available in [26].

The group M_{22} has Schur multiplier isomorphic to a cyclic group of order 12. We consider the non-split central extension $G := 4.M_{22}$ of M_{22} by a cyclic group C_4 of order 4; hence G has order $2^9 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$, and elementary abelian 3-Sylow subgroups. Let $z \in G$ be an element of order 4 generating the centre $Z := Z(G)$ of G, and let $^{-}\colon G \to G/Z \cong M_{22}$ be the natural homomorphism.

The group algebra $k[G]$ has four 3-blocks of defect 2. One is isomorphic to the principal 3-block of M_{22}, and one to the unique non-principal 3-block of defect 2 of $2.M_{22}$. There remain two conjugate faithful 3-blocks B_+ and B_-, on which z acts by scalar multiplication by ι and $-\iota$, respectively. These blocks are interchanged by the unique outer automorphism of G of order 2, and thus both the associated (mutually isoclinic) extensions $G.2$ of G have a 3-block Morita equivalent to B_+ and B_-.

The decomposition matrix of B_+ is given in Table 1, and let its block idempotent be denoted by $e_+ \in k[G]$. Let $D \cong C_3 \times C_3$ be the defect group of B_+. Picking two of its four non-trivial cyclic subgroups, D_1 and D_2 say, let $D_1 = \langle c \rangle$ and $D_2 = \langle d \rangle$, and thus $D = D_1 \times D_2 = \langle c, d \rangle$; let $D_3 := \langle cd^{-1} \rangle$ and $D_4 := \langle cd \rangle$.

The centraliser of D in G is given as $C_G(D) = D \times Z$. Hence $C_G(D)$ has four 3-blocks, all by [25, Prop.49.11] being nilpotent, having defect group D, and being isomorphic to $k[D]$. The block idempotent f_+ of the Brauer correspondent b_0^+ of B_+ is simply the appropriate idempotent of $k[Z]$, i. e. we have $f_+ = 1/4 \cdot (1 - \iota z)(1 - z^2) \in k[C_G(D)]$.

The normaliser $N_G(D)$ of D in G is a group of order $288 = 2^5 \cdot 3^2$, being a semidirect product $N_G(D) = D\colon S$, where $S \cong C_4\colon C_8$ is a non-abelian 2-

Sylow subgroup of $N_G(D)$. Let $S = \langle a, b\rangle$, where a and b have order 8 and 4, respectively, and where $aba^{-1} = b^{-1}$. The action of S on $D = \langle c, d\rangle$ may be given as matrices in $GL_2(3)$:

$$a \mapsto \begin{bmatrix} . & -1 \\ 1 & . \end{bmatrix}, \quad b \mapsto \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}, \quad ba \mapsto \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

Note that we have ${}^aD_1 = D_2$ as well as ${}^bD_1 = D_3$ and ${}^{ba}D_1 = D_4$. For the automiser $E := N_G(D)/DC_G(D) = N_G(D)/C_G(D)$ we find

$$E = N_G(D)/(D \times Z) \cong S/Z \cong Q_8,$$

the quaternion group of order 8. Hence we have $E = \langle \overline{a}, \overline{b}\rangle$ and $Z = \langle b^2a^2\rangle \lhd S$, but note that $S \cong Z \,{}^{\cdot}\, E$ is a non-split central extension. As $E \cong Q_8$ acts faithfully on the elementary abelian group D, we conclude that E acts irreducibly and thus regularly on $D \setminus \{1\}$. Thus $N_G(D)/Z \cong D \colon E$ is a Frobenius group, see [7, Ch.V.8]. Note that in particular $\overline{b}^2 = \overline{a}^2 \in Z(E)$ acts by inverting all elements of D.

All the four 3-blocks of $C_G(D)$ are stable under the action of $N_G(D)$, thus we have $N_G(D, f_+) = N_G(D)$, see [25, Prop.40.13]. The block idempotent of the Brauer correspondent b_+ of B_+ is again $f_+ \in k[N_G(D)]$; the decomposition matrix of b_+ is given in Table 1. Since multiplication with f_+ identifies the subgroup $Z < k[N_G(D)]^*$ with the group $\langle \iota\rangle < k^*$ of 4-th roots of unity, we have

$$b_+ = f_+k[N_G(D)]f_+ \cong k \otimes_{k[Z]} k[N_G(D)] =: k_Z[D \colon S]$$

as interior $N_G(D)$-algebras, where $k[Z]$ acts on the right on k via $z \mapsto \iota$. In the computational setting it will be useful to identify b_+ with $k_Z[D \colon S]$.

Note that we have $k_Z[D \colon S] \cong k_\sharp[D \colon E]$, where the latter is a twisted group algebra in the sense of [25, Ex.10.4], and since $E \cong Q_8$ has trivial Schur multiplier $H^2(E, \mathbb{C}^*) = \{0\}$, see [7, Thm.V.25.3], which implies $H^2(E, k^*) = \{0\}$ as well, we have

$$k_Z[D \colon S] \cong k[D \colon E].$$

This just reflects the fact that the non-trivial cohomology class in $H^2(E, \langle\iota\rangle)$ belonging to the central extension $S \cong Z \,{}^{\cdot}\, E$ becomes trivial under the natural map $H^2(E, \langle\iota\rangle) \to H^2(E, k^*)$.

(2.2) The centraliser $C_G(D_1)$. In order to create an equivalence between the stable module categories of B_+ and b_+ we first analyse the centraliser $C_G(D_1)$ of D_1 in G; the same analysis of course holds for the other subgroups $D_i < D$. It turns out that

$$C_G(D_1) = D_1 \times 2.(2 \times (V_4 \colon D_2)),$$

where the right hand direct factor is a centrally amalgamated product of $Z = \langle z\rangle \cong C_4$ and the special linear group $SL_2(3) \cong Q_8 \colon C_3$. Note that $C_G(D_1)$ has a normal 2-Sylow subgroup, thus $C_G(D_1)$ is 3-nilpotent.

Thus by [25, Prop.49.13] all six 3-blocks of $k[C_G(D_1)]$ are nilpotent, and hence by [25, Thm.49.15] each possesses a unique simple module. Two of these blocks, corresponding to the 3-block of defect 0 of $SL_2(3)$, have defect 1 and defect group D_1. The other four blocks have defect 2, two of them correspond to the non-faithful 3-blocks of $SL_2(3)$ and are isomorphic to $k[D]$, the other two correspond to the faithful 3-blocks and are isomorphic to the (2×2)-matrix algebra $M_2(k[D])$ over $k[D]$, see [25, Cor.50.9]. Let b_1^+ be the faithful 3-block of $C_G(D_1)$ on which z acts by scalar multiplication by ι; note that hence B_+ is its Brauer correspondent.

To make the simple b_1^+-module V explicit, we consider the faithful natural 2-dimensional representation $\rho\colon SL_2(3) \to M_2(k)$. We may choose elements $f, g \in SL_2(3)$ of order 4 such that $\langle f, g\rangle = Q_8 \lhd SL_2(3)$ and

$$\rho(d) = \begin{bmatrix} 1 & 1 \\ . & 1 \end{bmatrix}, \quad \rho(f) = \begin{bmatrix} . & 1 \\ -1 & . \end{bmatrix}, \quad \rho(g) = \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}.$$

In order to extend ρ to all of $C_G(D_1)$ we let

$$\rho(c) = I_2 = \begin{bmatrix} 1 & . \\ . & 1 \end{bmatrix}, \quad \rho(z) = \iota \cdot I_2 = \begin{bmatrix} \iota & . \\ . & \iota \end{bmatrix}.$$

We want to extend ρ further to $N_G(D_1)$, which is an index 2 extension of $C_G(D_1)$. We consider the element $h := b^2 \in Z(S) < N_G(D)$ of order 2. Since $\overline{h} \in Z(E)$ inverts $c \in D_1$ we have $h \in N_G(D_1) \setminus C_G(D_1)$. Since $Q_8 \lhd SL_2(3)$ is the derived subgroup of $C_G(D_1)$, it is stable under conjugation with h, and since $\overline{h} \in Z(E)$ inverts $d \in D_2$ as well, we conclude that h induces a non-inner automorphism of $SL_2(3)$ of order 2, thus extending $SL_2(3)$ to $GL_2(3) = SL_2(3)\colon 2$. Since ρ is stable under outer automorphisms of $SL_2(3)$, there is are two extensions of ρ to $GL_2(3)$, differing by the determinant character. Choosing one of them and using $hdh^{-1} = d^{-1}$, we extend ρ to $N_G(D_1) = C_G(D_1)\colon 2$ by

$$\rho(h) = \begin{bmatrix} 1 & . \\ . & -1 \end{bmatrix}.$$

Since V is a simple $k[C_G(D_1)]$-module and D is abelian, it follows from Knörr's Theorem, see [25, Cor.41.8], that D is a vertex of V. As the restriction V_D is indecomposable, we conclude that V_D is a source of V. Thus by Dade's Theorem, see [25, Thm.30.5], V_D is an endo-permutation module, see [25, Ch.28]. Indeed, from

$$\rho(d) \cdot \rho(g) \cdot \rho(d)^{-1} = \rho(f) \quad \text{and} \quad \rho(d) \cdot \rho(f) \cdot \rho(d)^{-1} = \rho(f)\rho(g)$$

we conclude that conjugation by $\rho(d)$ in $E_V := \mathrm{End}_k(V)$ permutes the k-basis $\{I_2, \rho(g), \rho(f), \rho(fg)\}$. Hence E_V is an interior permutation D-algebra, actually an interior D/D_1-algebra.

Let $j_1^+ \in b_1^+$ be a source idempotent, i. e. a representative of a source point of the block b_1^+, see [25, p.149]. Since multiplication by a source idempotent induces a Morita equivalence, see [25, Prop.18.10], then j_1^+V is again the unique simple module in $j_1^+b_1^+j_1^+$. Since $j_1^+V_D$ is a direct summand of the indecomposable V_D, we conclude that $j_1^+V = V$. Hence from Puig's Theorem [25, Thm.50.6] as interior D-algebras we get

$$b_1^+ \cong j_1^+b_1^+j_1^+ \cong E_V \otimes_k k[D],$$

where D is embedded diagonally by $D \to E_V \otimes_k k[D] \colon x \mapsto \rho(x) \otimes x$.

(2.3) An endo-permutation module. We consider the restriction of ρ to

$$T := \langle z, c, d, h\rangle = Z \times (D\colon 2) \leq N_G(D_1) \cap N_G(D).$$

Thus we have $T \lhd N_G(D)$ and $N_G(D)/T \cong E/Z(E) \cong V_4$, where $N_G(D)/T = \{1 \cdot T, a \cdot T, b \cdot T, ba \cdot T\}$. By tensor induction, see [4, Ch.13A], we obtain

$$W := (V_T)^{\otimes N_G(D)} \cong V_T \otimes_k ({}^aV_T) \otimes_k ({}^bV_T) \otimes_k ({}^{ba}V_T),$$

hence $\dim_k(W) = 2^4 = 16$. Let σ be the representation of $N_G(D)$ afforded by W. Since $T \lhd N_G(D)$, the tensor factors are stable under the action of T, and hence $t \in T$ acts on W by

$$\sigma(t) = \rho(t) \otimes \rho(a^{-1}ta) \otimes \rho(b^{-1}tb) \otimes \rho(a^{-1}b^{-1}tba).$$

This in particular yields

$$\sigma(c) = \begin{bmatrix} 1 & . \\ . & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & -1 \\ . & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & -1 \\ . & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & -1 \\ . & 1 \end{bmatrix},$$

$$\sigma(d) = \begin{bmatrix} 1 & 1 \\ . & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & . \\ . & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & -1 \\ . & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 1 \\ . & 1 \end{bmatrix},$$

$$\sigma(h) = \begin{bmatrix} 1 & . \\ . & -1 \end{bmatrix} \otimes \begin{bmatrix} 1 & . \\ . & -1 \end{bmatrix} \otimes \begin{bmatrix} 1 & . \\ . & -1 \end{bmatrix} \otimes \begin{bmatrix} 1 & . \\ . & -1 \end{bmatrix},$$

$$\sigma(z) = \begin{bmatrix} \iota & . \\ . & \iota \end{bmatrix} \otimes \begin{bmatrix} \iota & . \\ . & \iota \end{bmatrix} \otimes \begin{bmatrix} \iota & . \\ . & \iota \end{bmatrix} \otimes \begin{bmatrix} \iota & . \\ . & \iota \end{bmatrix} = I_{16}.$$

Hence W even is a module for $k[N_G(D)/Z] \cong k[D\colon E]$. Since $Z \times 2 = \langle z, h\rangle < T$ acts monomially on V, a similar computation as above shows that $a, b \in N_G(D)$, and thus $\overline{a}, \overline{b} \in E$, act monomially on W. Moreover, the MeatAxe shows that W is indecomposable and self-contragredient, i. e. we have $W \cong W^* := \mathrm{Hom}_k(W, k)$ as $k[N_G(D)]$-modules. Let $E_W := \mathrm{End}_k(W)$, which is an interior $N_G(D)$-algebra, actually an interior $(D\colon E)$-algebra.

Since the restriction W_D is a tensor product of endo-permutation $k[D]$-modules, it follows from [25, Prop.28.2] that W_D also is an endo-permutation $k[D]$-module. Indeed we have

$$E_W \cong E_V \otimes_k E_{{}^aV} \otimes_k E_{{}^bV} \otimes_k E_{{}^{ba}V}$$

as interior T-algebras. We have to check that E_W fulfils the conditions in [17, Cor.5.5]; thus we consider Brauer quotients: Since E_V is an interior permutation D-algebra, by [25, Prop.28.3] we for $D_i < D$ get the Brauer quotient

$$E_W(D_i) \cong E_V(D_i) \otimes_k E_{{}^aV}(D_i) \otimes_k E_{{}^bV}(D_i) \otimes_k E_{{}^{ba}V}(D_i).$$

Since $\rho(c) = I_2$ we have $E_V(D_1) = E_V^{D_1}/t_{\langle 1\rangle}^{D_1}(E_V) \cong E_V$ as interior D/D_1-algebras. A direct calculation using $\rho(d)$ shows

$$E_V^{D_2} = \langle I_2, \begin{bmatrix} . & 1 \\ . & . \end{bmatrix} \rangle_k \quad \text{and} \quad t_{\langle 1\rangle}^{D_2}(E_V) = \langle \begin{bmatrix} . & 1 \\ . & . \end{bmatrix} \rangle_k,$$

thus $E_V(D_2) = E_V^{D_2}/t_{\langle 1\rangle}^{D_2}(E_V) \cong k$ with trivial D/D_2-action. Using $\sigma(c)$ this yields

$$E_W(D_1) \cong E_V \otimes_k k \otimes_k k \otimes_k k \cong E_V$$

as interior D/D_1-algebras. Since $b_1^+ \cong E_V \otimes_k k[D]$ as interior D-algebras, this by [17, Pf. of Thm.5.4] verifies condition [17, 5.5.1]. Using $\sigma(d)$, $\sigma(cd^{-1})$ and $\sigma(cd)$ we similarly obtain $E_W(D_2) \cong E_{{}^aV}$ as interior D/D_2-algebras, $E_W(D_3) \cong E_{{}^bV}$ as interior D/D_3-algebras and $E_W(D_4) \cong E_{{}^{ba}V}$ as interior D/D_4-algebras, thus verifying condition [17, 5.5.1] for all subgroups $D_i < D$.

Finally we get $E_V^D = E_V^{D_1} \cap E_V^{D_2} = E_V^{D_2}$ and

$$\sum_{i=1}^{4} t_{D_i}^D(E_V^{D_i}) = t_{D_1}^D(E_V^{D_1}) = t_{D_1}^D(E_V) = t_{\langle 1\rangle}^{D_2}(E_V),$$

thus $E_V(D) = E_V^D / \sum_{i=1}^4 t_{D_i}^D(E_V^{D_i}) = E_V^{D_2}/t_{\langle 1\rangle}^{D_2}(E_V) \cong k$, hence

$$E_W(D) \cong E_V(D) \otimes_k E_{{}^aV}(D) \otimes_k E_{{}^bV}(D) \otimes_k E_{{}^{ba}V}(D) \cong k.$$

Since $b_0^+ \cong k \otimes_k k[D]$ as interior D-algebras, this verifies condition [17, 5.5.1] for D. Note that a computation using the MeatAxe actually shows that W_D is indecomposable, thus $E_W^D = \mathrm{End}_{k[D]}(W)$ is a primitive algebra, i. e. $1 \in E_W^D$ is a primitive idempotent.

(2.4) The stable equivalence. Let $i_+ \in B_+^D$ be a source idempotent of B_+. By Puig's Theorem [25, Thm.45.11] the source algebra $i_+B_+i_+$ carries the structure of an interior $(D\colon E)$-algebra, hence via $k_Z[D\colon S] \cong k[D\colon E]$ also is an interior $N_G(D)$-algebra. Note that B_+ also becomes an interior $N_G(D)$-algebra by the natural homomorphism $N_G(D) \to G \to (i_+k[G]i_+)^* = B_+^*$, but the proof of Puig's Theorem [25, (45.8)] only shows that the natural map $i_+B_+i_+ \to B_+$ is an embedding of interior $C_G(D)$-algebras, while the interior $N_G(D)$-algebra structures of $i_+B_+i_+$ and B_+ might be different.

Using the embedding $E_W \otimes_k i_+B_+i_+ \to E_W \otimes_k B_+$ of interior $C_G(D)$-algebras, there is a primitive idempotent $\widetilde{i_+} \in (E_W \otimes_k B_+)^D$ such that

$$(1 \otimes i_+)\widetilde{i_+} = \widetilde{i_+} = \widetilde{i_+}(1 \otimes i_+).$$

The proof of [17, Thm.5.8] shows that $\widetilde{i_+}(E_W \otimes_k B_+)\widetilde{i_+} = \widetilde{i_+}(E_W \otimes_k i_+B_+i_+)\widetilde{i_+}$ becomes an interior $N_G(D)$-algebra by diagonal action. Hence using the associated structural homomorphism

$$b_+ \cong k_Z[D\colon S] \to \widetilde{i_+}(E_W \otimes_k i_+B_+i_+)\widetilde{i_+}$$

the $\widetilde{i_+}(E_W \otimes_k i_+B_+i_+)\widetilde{i_+}$-module $\widetilde{i_+}(W \otimes_k i_+M)$, where M is a B_+-module, becomes a b_+-module. Hence we have an exact functor

$$\mathcal{F}\colon \begin{cases} B_+\text{-mod} & \to & b_+\text{-mod} \\ M & \mapsto & \widetilde{i_+}(W \otimes_k i_+M). \end{cases}$$

between the associated categories of finitely generated modules. The functor $\mathcal{F}$ by [17, Rem.6.8] induces an equivalence B_+-$\underline{\text{mod}} \to b_+$-$\underline{\text{mod}}$ of the associated stable module categories. Moreover, $\mathcal{F}$ preserves relative projectivity with respect to subgroups of D. Thus, possibly going over to a direct summand of $\mathcal{F}$, we may assume that $\mathcal{F}$ maps non-projective indecomposable B_+-modules to non-projective indecomposable b_+-modules, preserving vertices.

(2.5) Evaluating $\mathcal{F}$. In the computational setting we have to evaluate $\mathcal{F}$ on certain B_+-modules M explicitly. Note that i_+M, considered as a $k[C_G(D)]$-module, is a direct summand of the restriction $M_{C_G(D)}$, but the $k_Z[D\colon S]$-module structure of i_+M cannot be directly read off from $M_{N_G(D)}$. We will be content with the following restricted situation:

Assumption: Let M be an indecomposable B_+-module having vertex D, and let M' be its Green correspondent with respect to $N_G(D)$. Let M' occur with multiplicity 1 in a direct sum decomposition of $M_{N_G(D)}$, while the other direct summands of $M_{N_G(D)}$ are projective, and let M'_D be indecomposable.

Thus M'_D is a source of M, and we have

$$M_{N_G(D)} \cong M' \oplus (\text{proj}) \quad \text{and} \quad M_D \cong M'_D \oplus (\text{proj}).$$

Since $\mathcal{F}(M)$ is an indecomposable $k_Z[D\colon S]$-module having vertex D, there is an indecomposable direct summand M'' of the $k_Z[D\colon S]$-module i_+M, having vertex D, such that $\mathcal{F}(M) \cong \widetilde{i_+}(W \otimes_k M'')$ as $k_Z[D\colon S]$-modules. Hence M'_D also is a source of M'', and thus M'' is a direct summand of the induced module

$$k_Z[D\colon S] \otimes_{k[C_G(D)]} M'_{C_G(D)} \cong k[N_G(D)] \otimes_{k[C_G(D)]} M'_{C_G(D)} =: {M'}_{C_G(D)}^{N_G(D)}.$$

Since M' is an extension of $M'_{C_G(D)}$ to $N_G(D)$, as $k[N_G(D)]$-modules we have

$${M'}_{C_G(D)}^{N_G(D)} \cong (k \otimes_k M_{C_G(D)})'^{N_G(D)} \cong k_{C_G(D)}^{N_G(D)} \otimes_k M' \cong k[E] \otimes_k M'.$$

This yields the following decomposition of the induced module into indecomposable summands, where $\{1a, \ldots, 1d, 2\}$ are the simple $k[E]$-modules,

$${M'}_{C_G(D)}^{N_G(D)} \cong \bigoplus_{x \in \{a,b,c,d\}} (1x \otimes_k M') \oplus (2 \otimes_k M') \oplus (2 \otimes_k M').$$

Since $(1x \otimes_k M')_D \cong M'_D$ and $(2 \otimes_k M')_D \cong M'_D \oplus M'_D$, we conclude that $M'' \cong 1x \otimes_k M'$ as $k[N_G(D)]$-modules, for some $x \in \{a, b, c, d\}$, hence $M''_D \cong M'_D$.

Let $E_M := \mathrm{End}_k(M)$, which carries an interior G-algebra structure via the structural homomorphism $B_+ \to E_M$. Up to conjugation by an element of $(E_M^D)^*$ we may assume that the submodule M' of $M_{N_G(D)}$ is $k_Z[D\colon S]$-invariant as well. Hence $E_{M'} := \mathrm{End}_k(M')$ carries both interior $N_G(D)$-algebra structures, which coincide on D, and since M'_D is indecomposable the algebra $E_{M'}^D$ is primitive. Since both interior $N_G(D)$-algebra structures yield the same homomorphism $N_G(D) \to \mathrm{Aut}(D)$, they by [25, Prop.44.2] yield the same subgroup of $N_{E_{M'}}(D)/(E_{M'}^D)^*$, and hence by [25, Cor.45.7] they differ by conjugation by an element of $(E_{M'}^D)^*$.

Thus to find $\mathcal{F}(M)$ we have to pick a suitable indecomposable direct summand of the tensor product $W \otimes_k M'$ of $k[N_G(D)]$-modules.

Let $E_{W \otimes_k M'} := \mathrm{End}_k(W \otimes_k M')$. Since M'_D is indecomposable and W_D is an endo-permutation $k[D]$-module, by [16, Thm.5.6] we have

$$E_{W \otimes_k M'}(D) \cong E_W(D) \otimes_k E_{M'}(D) \cong k \otimes_k k \cong k.$$

Thus $(W \otimes_k M')_D$ has a unique indecomposable direct summand M''' having vertex D, and M''' occurs with multiplicity 1. Thus $\mathcal{F}(M)$ is the unique indecomposable direct summand of the $k[N_G(D)]$-module $W \otimes_k M'$ having vertex D, and it also occurs with multiplicity 1.

Actually, as M''' is a source of $\mathcal{F}(M)$, we conclude that $\mathcal{F}(M)$ is a direct summand of the induced module $M'''^{N_G(D)}$. Since

$$(M'''^{N_G(D)})_D \cong \bigoplus_{g \in N_G(D)/D} ({}^g M''')$$

we conclude that $\mathcal{F}(M)_D \cong M'''$ holds, i. e. M''' is extendible to $N_G(D)$.

3 The derived equivalence.

(3.1) Evaluating $\mathcal{F}$ at simple modules. We now apply the functor $\mathcal{F}$ to the simple B_+-modules $M \in \{56a, 56b, 64, 160a, 160b\}$; explicit matrix representations are available in [26]. We have to check the assumptions made in (2.5): By Knörr's Theorem, see [25, Cor.41.8], D is a vertex of M. Using GAP we compute the restriction $M_{N_G(D)}$, and using the MeatAxe we find the direct sum decomposition of $M_{N_G(D)}$; note that projectivity of an indecomposable module is easily verified by considering its dimension and Loewy series.

The results are shown in Table 2, where the indecomposable direct summands M' are indicated by their dimension, P_S denotes the projective-indecomposable b_+-module corresponding to the simple b_+-module $S \in \{1a, 1b, 2, 1c, 1d\}$, and the superscripts indicate multiplicities. For all simple B_+-modules M we indeed have $M_{N_G(D)} \cong M' \oplus (\mathrm{proj})$, where hence $M' \in \{11a, 11b, 10, 16a, 16b\}$ is the associated Green correspondent with respect to $N_G(D)$, and restricting further the MeatAxe shows that in all cases M'_D is indecomposable as well.

Table 2: Direct sum decomposition of $M_{N_G(D)}$.

M	M'	$\oplus$	(proj)								
$56a$	$11a$	$\oplus$	P_{1a}			$\oplus$	P_{1c}	$\oplus$	P_{1d}	$\oplus$	P_2
$56b$	$11b$			$\oplus$	P_{1b}	$\oplus$	P_{1c}	$\oplus$	P_{1d}	$\oplus$	P_2
64	10	$\oplus$	P_{1a}	$\oplus$	P_{1b}					$\oplus$	P_2^2
$160a$	$16a$	$\oplus$	P_{1a}^2	$\oplus$	P_{1b}^2	$\oplus$	P_{1c}^2	$\oplus$	P_{1d}^2	$\oplus$	P_2^4
$160b$	$16b$	$\oplus$	P_{1a}^2	$\oplus$	P_{1b}^2	$\oplus$	P_{1c}^2	$\oplus$	P_{1d}^2	$\oplus$	P_2^4

Table 3: Direct sum decomposition of $W \otimes_k M'$.

M	M'	$\mathcal{F}(M)$	$\oplus$	(D_i)	$\oplus$	(proj)
$56a$	$11a$	$8a$	$\oplus$	$24a$	$\oplus$	(proj)
$56b$	$11b$	$8b$	$\oplus$	$24b$	$\oplus$	(proj)
64	10	4	$\oplus$	$12c$	$\oplus$	(proj)
$160a$	$16a$	$1c$	$\oplus$	$12a$	$\oplus$	(proj)
$160b$	$16b$	$1d$	$\oplus$	$12b$	$\oplus$	(proj)

For later use we note the Loewy and socle series of the projective indecomposable b_+-modules: For $x \in \{a, b, c, d\}$, and where $\{y, y', y''\} = \{a, b, c, d\} \setminus \{x\}$, we have

$$P_{1x} \cong \begin{bmatrix} 1x \\ 2 \\ 1y \oplus 1y' \oplus 1y'' \\ 2 \\ 1x \end{bmatrix} \quad \text{and} \quad P_2 \cong \begin{bmatrix} 2 \\ 1a \oplus 1b \oplus 1c \oplus 1d \\ 2 \oplus 2 \oplus 2 \\ 1a \oplus 1b \oplus 1c \oplus 1d \\ 2 \end{bmatrix}.$$

The direct sum decomposition of the tensor products $W \otimes_k M'$ of $k[N_G(D)]$-modules, found by the MeatAxe, is given in Table 3. Again the projective indecomposable direct summands are easily detected. By the analysis in (2.5) and using [4, Thm.19.26], $\mathcal{F}(M)$ is the unique indecomposable direct summand whose dimension is not divisible by 3, while the remaining non-projective direct summand has the subgroups $D_i < D$ as its vertices.

The structure of the indecomposable b_+-modules $\mathcal{F}(M)$ is found by the MeatAxe as follows: The Loewy and socle series of $\mathcal{F}(56a)$, $\mathcal{F}(56b)$ and $\mathcal{F}(64)$ are given as follows:

$$\mathcal{F}(56a) \cong \begin{bmatrix} 1a \\ 2 \\ 1b \oplus 1c \oplus 1d \\ 2 \end{bmatrix}, \quad \mathcal{F}(56b) \cong \begin{bmatrix} 2 \\ 1a \oplus 1c \oplus 1d \\ 2 \\ 1b \end{bmatrix}, \quad \mathcal{F}(64) \cong \begin{bmatrix} 1b \\ 2 \\ 1a \end{bmatrix}.$$

Hence we have $\mathcal{F}(56a) \cong \Omega^{-1}(1a)$ and $\mathcal{F}(56b) \cong \Omega(1b)$, where $\Omega \colon b_+\text{-}\underline{\mathrm{mod}} \to$

b_+-$\underline{\text{mod}}$ denotes the Heller operator. As already indicated in Table 3 we have $\mathcal{F}(160a) \cong 1c$ and $\mathcal{F}(160b) \cong 1d$.

(3.2) A tilting complex. According to the heuristics in [23], in view of the appearance of the Heller operator and its inverse above, we are led to the following sensible guess: We partition the set of simple b_+-modules into $\{1a, 1b, 2, 1c, 1d\} = \mathcal{I}' \,\dot\cup\, \mathcal{I}'' \,\dot\cup\, \mathcal{I}_0$, where $\mathcal{I}' := \{1a\}$ and $\mathcal{I}'' := \{1b\}$ as well as $\mathcal{I}_0 = \{1c, 1d, 2\}$. Moreover, in the homotopy category $K^b(b_+\text{-proj})$ of bounded complexes of finitely generated projective b_+-modules let

$$\mathcal{T} := \mathcal{T}_{1a} \oplus \mathcal{T}_{1b} \oplus \mathcal{T}_2 \oplus \mathcal{T}_{1c} \oplus \mathcal{T}_{1d},$$

where

$$\begin{array}{lccccccccc} \mathcal{T}_{1a}\colon & 0 & \longrightarrow & P_{1a} & \xrightarrow{\alpha} & P_2 & \longrightarrow & 0, & & \\ \mathcal{T}_{1b}\colon & & & 0 & \longrightarrow & P_2 & \xrightarrow{\beta} & P_{1b} & \longrightarrow & 0, \\ \mathcal{T}_2\ : & & & 0 & \longrightarrow & P_2 & \longrightarrow & 0, & & \\ \mathcal{T}_{1c}\colon & & & 0 & \longrightarrow & P_{1c} & \longrightarrow & 0, & & \\ \mathcal{T}_{1d}\colon & & & 0 & \longrightarrow & P_{1d} & \longrightarrow & 0, & & \end{array}$$

where $\mathcal{T}_2$ as well as $\mathcal{T}_{1c}$ and $\mathcal{T}_{1d}$ are concentrated in degree 0, and where

$$\ker(\alpha) = \mathrm{soc}_{b_+}(P_{1a}) \quad \text{and} \quad \mathrm{im}\,(\beta) = \mathrm{rad}_{b_+}(P_{1b}).$$

Note that $\alpha \in \mathrm{Hom}_{b_+}(P_{1a}, P_2)$ and $\beta \in \mathrm{Hom}_{b_+}(P_2, P_{1b})$ are not uniquely defined by these conditions, hence we choose α and β suitably and keep them fixed.

We show that $\mathcal{T}$ indeed is a tilting complex, see [18]: Thus we firstly have to show that $\mathrm{add}(\mathcal{T})$, i. e. the full subcategory of $K^b(b_+\text{-proj})$ consisting of all direct summands of finite sums of copies of $\mathcal{T}$, generates $K^b(b_+\text{-proj})$ as a triangulated category: Since $\mathcal{T}_2$ as well as $\mathcal{T}_{1c}$ and $\mathcal{T}_{1d}$ already are direct summands of $\mathcal{T}$, it follows from [8, Ex.2.3.1], using $\mathcal{T}_{1a}$ and $\mathcal{T}_{1b}$, that the triangulated subcategory generated by $\mathrm{add}(\mathcal{T})$ contains $0 \to P_{1a} \to 0$ and $0 \to P_{1b} \to 0$ as well.

Secondly we have to show that $\mathrm{Hom}_{K^b(b_+\text{-proj})}(\mathcal{T}, \mathcal{T}[i]) = \{0\}$ for all $0 \neq i \in \mathbb{Z}$, which amounts to showing that $\mathrm{Hom}_{K^b(b_+\text{-proj})}(\mathcal{T}_S, \mathcal{T}_T[i]) = \{0\}$ for all $0 \neq i \in \mathbb{Z}$ and $S, T \in \{1a, 1b, 2, 1c, 1d\}$: Most of the non-trivial cases easily follow from the properties of the maps α and β and from the Loewy series of the projective indecomposable b_+-modules. We discuss the two cases needing closer analysis; note that this essentially amounts to checking [23, Cond.2]:

Firstly, let $S = 1a$ and $T = 1b$ as well as $i = 1$. We consider the total chain complex of the homomorphism double complex of k-vector spaces associated to $\mathcal{T}_{1a}$ and $\mathcal{T}_{1b}[1]$, see [1, Ch.2.7]:

$$0 \to \mathrm{End}_{b_+}(P_2) \xrightarrow{\partial_1} \mathrm{Hom}_{b_+}(P_{1a}, P_2) \oplus \mathrm{Hom}_{b_+}(P_2, P_{1b}) \xrightarrow{\partial_0} \mathrm{Hom}_{b_+}(P_{1a}, P_{1b}) \to 0,$$

where

$$\partial_1 \colon \eta \mapsto [\alpha\eta, \eta\beta] \quad \text{and} \quad \partial_0 \colon [\gamma, \delta] \mapsto \alpha\delta - \gamma\beta.$$

Hence showing that all chain maps between $\mathcal{T}_{1a}$ and $\mathcal{T}_{1b}[1]$ are homotopic to the zero map amounts to showing that the above complex has vanishing homology in degree 0, i. e. that $\mathrm{im}\,(\partial_1) = \ker(\partial_0)$ holds: For any $\epsilon \in \mathrm{Hom}_{b_+}(P_{1a}, P_{1b})$ we have $\mathrm{im}\,(\epsilon) \leq \mathrm{rad}_{b_+}(P_{1b})$, thus ϵ factors through β, implying that ∂_0 is surjective. Since $\dim_k(\mathrm{Hom}_{b_+}(P_{1a}, P_{1b})) = 1$ as well as $\dim_k(\mathrm{Hom}_{b_+}(P_{1a}, P_2)) = 2$ and $\dim_k(\mathrm{Hom}_{b_+}(P_2, P_{1b})) = 2$, we conclude that $\dim_k(\ker(\partial_0)) = 3$. Moreover, we use the MeatAxe to compute k-bases of $\mathrm{End}_{b_+}(P_2)$ as well as of $\mathrm{Hom}_{b_+}(P_{1a}, P_2)$ and $\mathrm{Hom}_{b_+}(P_2, P_{1b})$, and to determine the matrix of the k-linear map ∂_1. It turns out that $\dim_k(\mathrm{im}\,(\partial_1)) = 3$ holds as well.

Secondly, let $S = 1b$ and $T = 1a$ and $i = -1$. We consider the chain complex

$$0 \to \mathrm{Hom}_{b_+}(P_{1b}, P_{1a}) \xrightarrow{\partial_1} \mathrm{Hom}_{b_+}(P_2, P_{1a}) \oplus \mathrm{Hom}_{b_+}(P_{1b}, P_2) \xrightarrow{\partial_0} \mathrm{End}_{b_+}(P_2) \to 0,$$

where

$$\partial_1 \colon \eta \mapsto [\beta\eta, \eta\alpha] \quad \text{and} \quad \partial_0 \colon [\gamma, \delta] \mapsto \gamma\alpha - \beta\delta.$$

We again have to show that the above complex has vanishing homology in degree 0: For any $0 \neq \eta \in \mathrm{Hom}_{b_+}(P_{1b}, P_{1a})$ we have $\beta\eta \neq 0$ and $\eta\alpha \neq 0$. Thus from $\dim_k(\mathrm{Hom}_{b_+}(P_{1b}, P_{1a})) = 1$ we conclude that $\dim_k(\mathrm{im}\,(\partial_1)) = 1$. Moreover, we use the MeatAxe to compute k-bases of $\mathrm{Hom}_{b_+}(P_2, P_{1a})$ and $\mathrm{Hom}_{b_+}(P_{1b}, P_2)$, and to determine the matrix of the k-linear map ∂_0. It turns out that $\dim_k(\mathrm{im}\,(\partial_0)) = 3$, and since we have $\dim_k(\mathrm{Hom}_{b_+}(P_2, P_{1a})) = 2$ and $\dim_k(\mathrm{Hom}_{b_+}(P_{1b}, P_2)) = 2$, we conclude $\dim_k(\ker(\partial_0)) = 1$ as well.

(3.3) The Rickard-Okuyama method. Let $D^b(b_+\text{-mod})$ be the bounded derived category of the underlying module category. Recall that by [19] the natural embedding $b_+\text{-mod} \to D^b(b_+\text{-mod})$ induces an equivalence of triangulated categories

$$b_+\text{-}\underline{\mathrm{mod}} \to D^b(b_+\text{-mod})/K^b(b_+\text{-proj}),$$

where the translation functor of $b_+\text{-}\underline{\mathrm{mod}}$ is given by the inverse Ω^{-1} of the Heller operator. Thus complexes in $D^b(b_+\text{-mod})$ which become isomorphic in $D^b(b_+\text{-mod})/K^b(\text{proj-}b_+)$ are called stably isomorphic.

Due to the heuristics in [21, Ch.6.3] we look for complexes $\mathcal{X}_S$ in $D^b(b_+\text{-mod})$, where $S \in \{1a, 1b, 2, 1c, 1d\}$, having homology concentrated in one degree, and being stably isomorphic to $\mathcal{F}(M)$ for $M \in \{56a, 56b, 64, 160a, 160b\}$: Let

$$\begin{array}{llllllllllll}
\mathcal{X}_{1a} & := & 1a[1] & : & 0 & \longrightarrow & 1a & \longrightarrow & 0, & & & \\
\mathcal{X}_{1b} & := & 1b[-1] & : & & & & & 0 & \longrightarrow & 1b & \longrightarrow \; 0, \\
\mathcal{X}_2 & := & \mathcal{F}(64) & : & & & 0 & \longrightarrow & \begin{bmatrix} 1b \\ 2 \\ 1a \end{bmatrix} & \longrightarrow & 0, & \\
\mathcal{X}_{1c} & := & 1c & : & & & 0 & \longrightarrow & 1c & \longrightarrow & 0, & \\
\mathcal{X}_{1d} & := & 1d & : & & & 0 & \longrightarrow & 1d & \longrightarrow & 0, &
\end{array}$$

where $\mathcal{X}_2$ and $\mathcal{X}_{1c}$ as well as $\mathcal{X}_{1d}$ are concentrated in degree 0, while $\mathcal{X}_{1a}$ and $\mathcal{X}_{1b}$ are concentrated in degree -1 and 1, respectively.

Recall that we have $\mathcal{F}(160a) \cong 1c$ and $\mathcal{F}(160b) \cong 1d$ as b_+-modules. Moreover, we have $\mathcal{F}(56a) \cong \Omega^{-1}(1a)$ as b_+-modules, where $\Omega^{-1}(1a)$ and $1a[1]$ are stably isomorphic, as well as $\mathcal{F}(56b) \cong \Omega(1b)$ as b_+-modules, where similarly $\Omega(1b)$ and $1b[-1]$ are stably isomorphic.

(3.4) As we are going to apply [21, La.5.2] to $\{\mathcal{X}_{1a}, \mathcal{X}_{1b}, \mathcal{X}_2, \mathcal{X}_{1c}, \mathcal{X}_{1d}\}$, we have to check the properties listed in [21, Ch.5]: Firstly, the triangulated category generated by $\{\mathcal{X}_{1a}, \mathcal{X}_{1b}, \mathcal{X}_2, \mathcal{X}_{1c}, \mathcal{X}_{1d}\}$ contains the 1-dimensional simple b_+-modules anyway, and it follows from [8, Ex.2.3.1], using non-zero homomorphisms $1a \to \mathcal{F}(64)$ and $\mathcal{F}(64) \to 1b$ of b_+-modules, that it also contains the simple 2-dimensional b_+-module. Thus $\{\mathcal{X}_{1a}, \mathcal{X}_{1b}, \mathcal{X}_2, \mathcal{X}_{1c}, \mathcal{X}_{1d}\}$ generates $D^b(b_+\text{-mod})$ as a triangulated category.

Secondly we have to show that for all $S, T \in \{1a, 1b, 2, 1c, 1d\}$ and for all $i \in \mathbb{N}$ we have $\mathrm{Hom}_{D^b(b_+\text{-mod})}(\mathcal{X}_S, \mathcal{X}_T[-i]) = \{0\}$ and

$$\mathrm{Hom}_{D^b(b_+\text{-mod})}(\mathcal{X}_S, \mathcal{X}_T) \cong \begin{cases} k, & \text{if } S \cong T, \\ \{0\}, & \text{if } S \not\cong T. \end{cases}$$

Considering the $\mathcal{X}_S$ as complexes in $K^b(b_+\text{-mod})$, it is immediate that we even have $\mathrm{Hom}_{K^b(b_+\text{-mod})}(\mathcal{X}_S, \mathcal{X}_T[-i]) = \{0\}$ and $\mathrm{Hom}_{K^b(b_+\text{-mod})}(\mathcal{X}_S, \mathcal{X}_T) = \{0\}$, whenever $S \not\cong T$. Finally, we have $\dim_k(\mathrm{End}_{b_+}(\mathcal{X}_S)) = 1$, and since $\mathcal{X}_S \not\cong 0$ in $D^b(b_+\text{-mod})$ we conclude $\dim_k(\mathrm{End}_{D^b(b_+\text{-mod})}(\mathcal{X}_S)) = 1$.

(3.5) To relate the tilting complex $\mathcal{T}$ to the complexes $\{\mathcal{X}_{1a}, \mathcal{X}_{1b}, \mathcal{X}_2, \mathcal{X}_{1c}, \mathcal{X}_{1d}\}$, we moreover have to check the assumptions of [21, La.5.2]: We have to show that for all $S, T \in \{1a, 1b, 2, 1c, 1d\}$ and for all $i \in \mathbb{Z}$ we have

$$\mathrm{Hom}_{D^b(b_+\text{-mod})}(\mathcal{T}_S, \mathcal{X}_T[i]) \cong \begin{cases} k, & \text{if } S \cong T \text{ and } i = 0, \\ \{0\}, & \text{otherwise}. \end{cases}$$

It is immediate that we even have $\mathrm{Hom}_{K^b(b_+\text{-mod})}(\mathcal{T}_S, \mathcal{X}_T[i]) = \{0\}$ unless possibly $S \cong T$ and $i = 0$, or $T = 2$ and $[S, i] \in \{[1a, 0], [1a, 1], [1b, 0], [1b, -1]\}$. We exclude the last four possibilities:

Firstly, let $S = 1a$. For $i = 0$ let $\gamma \in \mathrm{Hom}_{b_+}(P_2, \mathcal{F}(64))$ such that $\alpha\gamma = 0$, and assume that $\gamma \neq 0$. Then we have $\mathrm{im}\,(\alpha) \leq \ker(\gamma) < P_2$ and $P_2/\ker(\gamma) \cong \mathrm{im}\,(\gamma) \cong \begin{bmatrix} 2 \\ 1a \end{bmatrix}$. Since $\mathrm{im}\,(\alpha)$ has Loewy length 4, while P_2 has Loewy length 5, we conclude that the simple b_+-module $1a$ occurs with multiplicity at least 2 as a constituent of the second Loewy layer $\mathrm{rad}_{b_+}(P_2)/\mathrm{rad}^2_{b_+}(P_2)$ of P_2, a contradiction. Hence we already have $\gamma = 0$.

For $i = 1$ let $\gamma \in \mathrm{Hom}_{b_+}(P_{1a}, \mathcal{F}(64))$, and let $\epsilon\colon \mathcal{F}(64) \to P_{1a}$ be an embedding of $\mathcal{F}(64)$ into its injective hull P_{1a}. Then we have $\mathrm{soc}_{b_+}(P_{1a}) \leq \ker(\gamma\epsilon)$, hence there is $\delta\colon P_2 \to P_{1a}$ such that $\gamma\epsilon = \alpha\delta$. Using the Loewy series of P_{1a} we conclude that $\mathrm{im}\,(\delta) \leq \mathrm{im}\,(\epsilon)$, implying that γ factors through α as well, thus γ is homotopic to the zero map.

Secondly, let $S = 1b$. For $i = 0$ let $\gamma \in \mathrm{Hom}_{b_+}(P_2, \mathcal{F}(64))$. Since the simple b_+-module $1a$ occurs with multiplicity 1 as a constituent of the second Loewy layer both of $\mathrm{im}\,(\beta)$ and P_2, we conclude that $\ker(\beta) \leq \ker(\gamma)$. Letting $\epsilon\colon \mathcal{F}(64) \to P_{1a}$ be as above, we conclude that there is $\delta\colon P_{1b} \to P_{1a}$ such that $\gamma\epsilon = \beta\delta$. Using the Loewy series of P_{1a} we conclude that $\mathrm{im}\,(\delta) = \mathrm{im}\,(\epsilon)$, implying that γ factors through δ as well, thus γ is homotopic to the zero map. Finally, for $i = -1$ let $\gamma \in \mathrm{Hom}_{b_+}(P_{1b}, \mathcal{F}(64))$ such that $\beta\gamma = 0$. Then we have $\gamma = 0$.

It remains to consider $\mathrm{Hom}_{D^b(b_+\text{-mod})}(\mathcal{T}_S, \mathcal{X}_S)$ for $S \in \{1a, 1b, 2, 1c, 1d\}$: For $S = 1a$ we have $\dim_k(\mathrm{Hom}_{D^b(b_+\text{-mod})}(\mathcal{T}_{1a}, \mathcal{X}_{1a})) \leq 1$. Hence it remains to show that the homomorphism $\mathcal{T}_{1a} \to \mathcal{X}_{1a}$ determined by a b_+-epimorphism $P_{1a} \to 1a$ is not the zero map in $D^b(b_+\text{-mod})$. Replacing $\mathcal{X}_{1a} = 1a[1]$ by a projective resolution

$$\mathcal{P}_{1a}[1]\colon\ \cdots \longrightarrow P_2 \longrightarrow P_{1a} \longrightarrow 0$$

in $K^-(b_+\text{-proj})$, with homology concentrated in degree -1, this amounts to show that the homomorphism $\mathcal{T}_{1a} \to \mathcal{P}_{1a}[1]$ determined by the identity map on P_{1a} is not homotopic to the zero map. Since there is no b_+-epimorphism from P_2 to P_{1a}, this is immediate.

For $S = 1b$ we have $\dim_k(\mathrm{Hom}_{D^b(b_+\text{-mod})}(\mathcal{T}_{1b}, \mathcal{X}_{1b})) \leq 1$. Hence it remains to show that the homomorphism $\mathcal{T}_{1b} \to \mathcal{X}_{1b}$ determined by a b_+-epimorphism $P_{1b} \to 1b$ is not the zero map in $D^b(b_+\text{-mod})$. Replacing $\mathcal{X}_{1b} = 1b[-1]$ by an injective resolution

$$\mathcal{I}_{1b}[-1]\colon 0 \longrightarrow P_{1b} \longrightarrow P_2 \longrightarrow \cdots .$$

in $K^+(b_+\text{-proj})$, with homology concentrated in degree 1, this amounts to show that the homomorphism $\mathcal{T}_{1b} \to \mathcal{I}_{1b}[-1]$ determined by a non-zero b_+-homomorphism $P_{1b} \to \mathrm{soc}_{b_+}(P_{1b})$ is not homotopic to the zero map, which is immediate.

For $S = 2$ as well as $S = 1c$ and $S = 1d$ we argue similarly, using injective resolutions of $\mathcal{F}(64)$ and $1c$ as well as $1d$, respectively, with homology concentrated in degree 0.

(3.6) Conclusion. Hence letting $E_{\mathcal{T}} := \mathrm{End}^{\circ}_{b_+\text{-mod}}(\mathcal{T})$, the proof of [21, La.5.2] implies that there is an equivalence $D^b(b_+\text{-mod}) \to D^b(E_{\mathcal{T}}\text{-mod})$, mapping the complexes $\mathcal{X}_S$, for $S \in \{1a, 1b, 2, 1c, 1d\}$, to the simple $E_{\mathcal{T}}$-modules.

From $E_{\mathcal{T}}$ being derived equivalent to the symmetric k-algebra b_+, we conclude that $E_{\mathcal{T}}$ also a symmetric k-algebra. Moreover, by [20] there is an $E_{\mathcal{T}}$-b_+-bimodule Y which is both a finitely generated projective $E_{\mathcal{T}}$-module and a finitely generated projective b_+-right module, such that the tensor functor

$$\mathcal{G} := Y \otimes_{b_+} ?\colon b_+\text{-mod} \to E_{\mathcal{T}}\text{-mod},$$

which hence is exact and maps projective b_+-modules to projective $E_{\mathcal{T}}$-modules, induces an equivalence b_+-$\underline{\mathrm{mod}} \to E_{\mathcal{T}}$-$\underline{\mathrm{mod}}$. Moreover, $\mathcal{G}$ maps the images of the complexes $\mathcal{X}_S$ in b_+-$\underline{\mathrm{mod}} \cong D^b(b_+\text{-mod})/K^b(b_+\text{-proj})$ to the simple $E_{\mathcal{T}}$-modules in $E_{\mathcal{T}}$-$\underline{\mathrm{mod}}$.

Hence the exact functor $\mathcal{G} \circ \mathcal{F} \colon B_+\text{-mod} \to E_{\mathcal{T}}\text{-mod}$ induces an equivalence $B_+\text{-}\underline{\text{mod}} \to E_{\mathcal{T}}\text{-}\underline{\text{mod}}$. Moreover, the b_+-modules $\mathcal{F}(M)$, for the simple B_+-modules $M \in \{56a, 56b, 64, 160a, 160b\}$, are stably isomorphic to the complexes $\mathcal{X}_S$, where $S \in \{1a, 1b, 2, 1c, 1d\}$, respectively. Thus, possibly going over to a direct summand of $\mathcal{G} \circ \mathcal{F}$, we may assume that $\mathcal{G} \circ \mathcal{F}$ maps the simple B_+-modules to the simple $E_{\mathcal{T}}$-modules. Hence by [10, Prop.2.5] the functor $\mathcal{G} \circ \mathcal{F} \colon B_+\text{-mod} \to E_{\mathcal{T}}\text{-mod}$ is an equivalence.

Thus we have shown that B_+ and $E_{\mathcal{T}}$ are Morita equivalent, while b_+ and $E_{\mathcal{T}}$ are derived equivalent, thus proving Broué's conjecture for B_+ and b_+.

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