Symmetric generation of Coxeter groups

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Abstract. We provide involutory symmetric generating sets of finitely generated Coxeter groups, fulfilling a suitable finiteness condition, which in particular is fulfilled in the finite, affine and compact hyperbolic cases.

Mathematics Subject Classification (2000). 20F55, 20F05.

Keywords. symmetric generation, symmetric presentation, Coxeter group.

Introduction

Presentations of groups having certain types of symmetry properties are being considered at least since Coxeter's work [6] in 1959. The notion of symmetric generation we are interested in here has been introduced by Curtis [8] in 1992. Since then it has gained continuing interest, in particular in the theory of sporadic finite simple groups, see Curtis's recent book [7] and the references in there, typically leading to 'good' presentations providing a practical means of computing efficiently in the groups under consideration [9].

We recall the definition of **symmetric generation** as we use it in this note: Let k^{*m} be the free product of $m \in \mathbb{N}$ copies of the cyclic groups C_k of order $k \geq 2$, where as usual the latter are often abbreviated by just writing k. Then $\mathrm{Aut}(C_k)$ acts on any of the free factors, and together with the full symmetric group $\mathcal{S}_m$ permuting the free factors, this generates the group $\mathrm{Aut}(C_k) \wr \mathcal{S}_m$ of monomial automorphisms of k^{*m}; for $k = 2$ this of course simplifies to purely permutational automorphisms. Given a **control group** $\mathcal{C} \leq \mathrm{Aut}(C_k) \wr \mathcal{S}_m$ of monomial automorphisms acting transitively on the free factors, the semidirect product $k^{*m} \colon \mathcal{C}$ is called a **progenitor**.

An epimorphic image G of a progenitor is called **(strictly) symmetrically generated** if **i)** the image of $\mathcal{C}$ in G is isomorphic to $\mathcal{C}$, **ii)** the free factors map to m cyclic subgroups of G of order k having mutually trivial intersections, and **iii)** the latter generate G. If only the first two conditions are fulfilled, but the third is not, then G is called **weakly** symmetrically generated; this less restrictive notion indeed leads to certain interesting examples [2, 3].

The aim of this note is to provide involutory symmetric generating sets of finitely generated, but not necessarily finite Coxeter groups. Given such a group, for any of its maximal parabolic subgroups fulfilling a suitable finiteness condition we obtain an associated weakly symmetric generating set, and we settle the question when it is strict. In particular, as far as irreducible Coxeter groups are concerned, the necessary conditions are fulfilled for all maximal parabolic subgroups of finite or affine Coxeter groups, i. e. the tame cases in the sense of [13, Sect.1.11], and of compact hyperbolic Coxeter groups [11, Sect.6.8], while for most non-compact hyperbolic Coxeter groups [11, Sect.6.9] there is at least one suitable maximal parabolic subgroup.

This note is motivated by [10], and gives a general explanation of the observations made there. As it turns out, Coxeter groups fit extremely nicely into the concept of symmetric generation. This to our knowledge has escaped notice so far, despite the fact that the symmetric group being generated by the adjacent transpositions is the archetypical example of symmetric generation [7, Exc.3.3(1)]. Moreover, not merely finite Coxeter groups but just finitely generated Coxeter groups are the natural appropriate class to state our results for, and hence we try to minimise our finiteness assumptions as far as possible. To our knowledge this is the first class of possibly infinite groups for which symmetric generation in the above sense is being considered at all. It remains to be seen whether the observations made in this note will be helpful to handle Coxeter groups in practice and to gain further insights into them.

This note is organised as follows: In Section 1 we introduce the progenitors, in Section 2 we prove the main result on symmetric generation, in Section 3 we examine the permutation action of the control group, and in Section 4 we consider various explicit examples. We assume the reader to be familiar with presentations of groups, a basic reference being [12], and with Coxeter groups and related concepts, as exposed in [1, 11]. Our notation for groups is as used in [4]. We consider right actions throughout.

1. The progenitor

Let W be a finitely generated, not necessarily finite Coxeter group with distinguished generators $\{s_1, \ldots, s_n\} \subseteq W$, for some $n \in \mathbb{N}$, and associated Coxeter integers $m_{ij} \in \mathbb{N} \,\dot{\cup}\, \{\infty\}$ such that $m_{ii} = 1$ and $m_{ij} = m_{ji} > 1$, for $i, j \in \{1, \ldots, n\}$ such that $i \neq j$. This gives rise to the Coxeter presentation

$$W \cong \mathcal{F}(S_1, \ldots, S_n)/\langle\langle (S_i S_j)^{m_{ij}}; i, j \in \{1, \ldots, n\} \rangle\rangle,$$

where $\mathcal{F}(S_1, \ldots, S_n)$ denotes the free group with generators $S_1, \ldots, S_n$, where $\langle\langle \cdot \rangle\rangle$ denotes normal closure, and where $(S_i S_j)^{\infty} = 1$ is interpreted as being no relation at all. We have $W = W_1 \times \cdots \times W_t$, where the direct factors are the parabolic subgroups associated with the connected components of the Dynkin graph associated with W; we may assume that $s_n \in W_t$. Let $W' := \langle s_1, \ldots, s_{n-1} \rangle < W$, which is a maximal parabolic subgroup and hence by [13, Prop.9.5] has the Coxeter presentation

$$W' \cong \mathcal{F}(S_1, \ldots, S_{n-1})/\langle\langle (S_iS_j)^{m_{ij}}; i, j \in \{1, \ldots, n-1\}\rangle\rangle,$$

obtained from the presentation of W by leaving out the relations involving S_n. Letting $W'_t := W_t \cap W'$ we have $W' = W_1 \times \cdots \times W_{t-1} \times W'_t$.

Let $(s_n)^{W'} \subseteq W$ be the set of W'-conjugates of s_n, and let $C_{W'}(s_n) \leq W'$ be the centraliser of s_n in W'. **We assume that $C_{W'}(s_n) \leq W'$ has finite index.** Since $W_1 \times \cdots \times W_{t-1} \leq C_{W'}(s_n)$ this in particular is fulfilled whenever W'_t is finite. Let $m := [W' \colon C_{W'}(s_n)] = |(s_n)^{W'}| \in \mathbb{N}$, and let $\{w_1, \ldots, w_m\} \subseteq W'$ be a set of representatives of the right cosets $C_{W'}(s_n)\backslash W'$ of $C_{W'}(s_n)$ in W', where $w_1 := 1$, yielding the bijection $C_{W'}(s_n)\backslash W' \to (s_n)^{W'} \colon w_k \mapsto (s_n)^{w_k} =: t_k$ for $k \in \{1, \ldots, m\}$, where $t_1 = s_n$. Hence the action of W' on $C_{W'}(s_n)\backslash W'$ translates to the transitive permutation representation $\pi \colon W' \to \mathcal{S}_m$, where $\pi(w) \in \mathcal{S}_m$ is given by $t_{k^{\pi(w)}} = (t_k)^w$ for $k \in \{1, \ldots, m\}$.

We consider the finitely presented group

$$\begin{aligned}\mathcal{P} := \mathcal{F}(S_1, \ldots, S_{n-1}, T_1, \ldots, T_m)/\langle\langle\ & (S_iS_j)^{m_{ij}},\ T_1^2,\ (T_k)^{S_i}T^{-1}_{k^{\pi(s_i)}}; \\ & i, j \in \{1, \ldots, n-1\}, k \in \{1, \ldots, m\}\rangle\rangle.\end{aligned}$$

Since π is transitive, $\mathcal{P}$ is generated by the images of $S_1, \ldots, S_{n-1}, T_1$, and the relations $T_k^2 = 1$ for $k \in \{2, \ldots, m\}$ follow from the relations given. This implies that $\mathcal{P} \cong 2^{*m} \colon W'$, where W' acts by permuting the free factors of 2^{*m} via π. Thus $\mathcal{P}$ is a progenitor with control group W'.

2. Symmetric generation

Theorem 1. *We still assume that $m := [W' \colon C_{W'}(s_n)]$ is finite. Then W has the finite presentation*

$$\mathcal{Q} := \mathcal{P}/\langle\langle (S_iT_1)^{m_{in}}; i \in \{1, \ldots, n-1\} \text{ such that } m_{in} > 2\rangle\rangle.$$

Proof. Let $\mathcal{F} := \mathcal{F}(S_1, \ldots, S_{n-1}, T_1, \ldots, T_m)$, and let $\bar{\ } \colon \mathcal{F} \to \mathcal{Q}$ be the natural epimorphism. Letting $S_i \mapsto s_i$ for $i \in \{1, \ldots, n-1\}$, and $T_k \mapsto (s_n)^{w_k}$ for $k \in \{1, \ldots, m\}$ defines a homomorphism $\mathcal{F} \to W$. In particular we have $T_1 \mapsto s_n$, hence this is an epimorphism. Since the defining relations of $\mathcal{P}$, and the relations $(S_iT_1)^{m_{in}} = 1$ for $i \in \{1, \ldots, n-1\}$ are fulfilled by the relevant images in W, this induces epimorphisms $\mathcal{P} \to W$ and $\alpha \colon \mathcal{Q} \to W \colon \overline{S}_i \mapsto s_i, \overline{T}_k \mapsto (s_n)^{w_k} = t_k$ for $i \in \{1, \ldots, n-1\}$ and $k \in \{1, \ldots, m\}$.

As for the converse, we have an epimorphism $\mathcal{F}(S_1, \ldots, S_n) \to \mathcal{Q}$ by letting $S_i \mapsto \overline{S}_i$ for $i \in \{1, \ldots, n-1\}$, and $S_n \mapsto \overline{T}_1$. Since the defining relations of W involving S_iS_j for $i, j \in \{1, \ldots, n-1\}$, and involving S_iS_n for $i \in \{1, \ldots, n-1\}$ such that $m_{in} > 2$ are by construction fulfilled by the relevant images in $\mathcal{Q}$, only those involving S_iS_n where $m_{in} = 2$ remain to be dealt with: For $i \in \{1, \ldots, n-1\}$ such that $m_{in} = 2$ we have $s_i \in C_{W'}(s_n)$, hence $t_{1^{\pi(s_i)}} = t_1^{s_i} = (s_n)^{s_i} = s_n = t_1$, and thus $1^{\pi(s_i)} = 1$, implying that the relation $(T_1)^{S_i} = T_{1^{\pi(s_i)}} = T_1$, being equivalent to $(S_iT_1)^2 = 1$, is already included in the defining relations of $\mathcal{P}$. Hence we have an epimorphism $\beta\colon W \to \mathcal{Q}\colon s_i \mapsto \overline{S}_i, s_n \mapsto \overline{T}_1$ for $i \in \{1, \ldots, n-1\}$. Thus α and β are mutually inverse isomorphisms. □

The above proof also shows that the image of $W' \leq \mathcal{P}$ in $\mathcal{Q}$ is isomorphic to W', and the images of $T_1, \ldots, T_m$ in $\mathcal{Q}$ are pairwise distinct involutions, i. e. $\mathcal{Q}$ is weakly symmetrically generated. Hence it remains to determine when precisely $\mathcal{Q}$ is strictly symmetrically generated, i. e. when W is generated by $t_1, \ldots, t_m$. We proceed as follows:

We first describe when distinguished generators s_i and s_j are conjugate in W: Let $\sim$ be the finest equivalence relation on $\{s_1, \ldots, s_n\}$ such that $s_i \sim s_j$ whenever m_{ij} is odd. The relation $(s_is_j)^{m_{ij}} = 1$ shows that $\langle s_i, s_j\rangle \cong \mathcal{D}_{2m_{ij}}$ is isomorphic to the dihedral group of order $2m_{ij}$. Hence if m_{ij} is odd, then s_i and s_j are already conjugate in $\langle s_i, s_j\rangle$, and thus in W. This shows that if $s_i \sim s_j$, then s_i and s_j are conjugate in W. Conversely, we pick a $\sim$-equivalence class $[s]$, and let $\lambda_{[s]}\colon \{s_1, \ldots, s_n\} \to \{\pm 1\} \leq \mathbb{C}^*$ be defined by $\lambda_{[s]}(s_i) := -1$ if and only if $s_i \sim s$. Then the defining relations of W show that $\lambda_{[s]}$ extends to a linear character of W. This shows that if $s_i \not\sim s_j$, then s_i and s_j are not conjugate in W.

Let $[W, W] \leq W$ denote the derived subgroup of W. Since W is generated by involutions, $W/[W, W]$ is 2-elementary abelian. The above argument on linear characters shows that $W/[W, W] \cong 2^r$, where r is the number of $\sim$-equivalence classes, and the direct factors are generated by images of representatives of the $\sim$-equivalence classes.

Proposition 1. *W is strictly symmetrically generated, i. e. $W = \langle t_1, \ldots, t_m\rangle$, if and only if $r = 1$, i. e. if and only if the subgraph of the Dynkin graph of W consisting of the edges labelled by odd m_{ij} is connected.*

Proof. Since $t_1, \ldots, t_m$ are all conjugate in W to s_n, we conclude that $W = \langle t_1, \ldots, t_m\rangle$ implies $r = 1$. If conversely $r = 1$, then after possibly reordering $s_1, \ldots, s_{n-1}$ we may assume that for any $i \in \{1, \ldots, n-1\}$ there is $j \in \{i+1, \ldots, n\}$ such that m_{ij} is odd. We for $i \in \{n, n-1, \ldots, 1\}$ successively show that $s_i \in \langle t_1, \ldots, t_m\rangle$: We have $s_n = t_1$ anyway. For $i < n$ let j be as above. Hence there are $l \in \mathbb{N}$ and $j_k \in \{1, \ldots, m\}$ such that $s_j = t_{j_1} \cdot t_{j_2} \cdot \cdots \cdot t_{j_l}$. We have $s_j^{s_i} = t_{j_1}^{s_i} \cdot \cdots \cdot t_{j_l}^{s_i} = t_{j_1^{\pi(s_i)}} \cdot \cdots \cdot t_{j_l^{\pi(s_i)}} \in \langle t_1, \ldots, t_m\rangle$. Since m_{ij} is odd, we conclude that $s_j \neq s_j^{s_i}$ are distinct involutions in $\langle s_i, s_j\rangle \cong \mathcal{D}_{2m_{ij}}$, hence $\langle s_i, s_j\rangle = \langle s_j, s_j^{s_i}\rangle$ implies $s_i \in \langle s_j, s_j^{s_i}\rangle \leq \langle t_1, \ldots, t_m\rangle$. □

3. The representation π

To describe the permutation representation $\pi\colon W' \to \mathcal{S}_m$ we determine $C_{W'}(s_n)$. **In this section no finiteness assumption is needed.**

Theorem 2. *Let $\mathcal{X} := \{i \in \{1, \ldots, n-1\}; m_{in} = 2\}$, and let $W_{\mathcal{X}} \leq W' \leq W$ be the associated parabolic subgroup. Then we have $C_{W'}(s_n) = W_{\mathcal{X}}$.*

Proof. For $i \in \mathcal{X}$ we have $s_i \in C_{W'}(s_n)$, and thus $W_{\mathcal{X}} \leq C_{W'}(s_n)$. To show the converse, following [1, Sect.V.4] let E be an $\mathbb{R}$-vector space having $\mathbb{R}$-basis $\Pi := \{\alpha_1, \ldots, \alpha_n\}$, and carrying the symmetric $\mathbb{R}$-bilinear form $B\colon E \times E \to \mathbb{R}$ given by $b_{ij} := B(\alpha_i, \alpha_j) := -\cos(\frac{\pi}{m_{ij}})$ for $i, j \in \{1, \ldots, n\}$; we have $b_{ii} = 1$, and for $i \neq j$ we have $b_{ij} = 0$ if and only if $m_{ij} = 2$, and $b_{ij} < 0$ otherwise. Then B is invariant with respect to the geometric action ρ of W on E, which is given by $\rho(s_i)\colon E \to E\colon v \mapsto v - 2B(v, \alpha_i)\alpha_i$ for $i \in \{1, \ldots, n\}$; by [1, Cor.V.4.4.2] ρ is faithful. Hence the space $\langle \alpha_1, \ldots, \alpha_{n-1} \rangle_{\mathbb{R}} \leq E$ is W'-invariant, and W' acts trivially on the quotient space $E/\langle \alpha_1, \ldots, \alpha_{n-1} \rangle_{\mathbb{R}}$. Moreover, the eigenspaces of $\rho(s_n)$ with respect to the eigenvalues ± 1 are given as $E_-(s_n) = \langle \alpha_n \rangle_{\mathbb{R}}$ and $E_+(s_n) = \langle \alpha_n \rangle_{\mathbb{R}}^{\perp}$, respectively, where $\langle \alpha_n \rangle_{\mathbb{R}}^{\perp}$ denotes the orthogonal complement of $\langle \alpha_n \rangle_{\mathbb{R}}$ with respect to B. Since ρ is faithful, $C_W(s_n)$ is the set of all elements of W preserving the vector space decomposition $E = \langle \alpha_n \rangle_{\mathbb{R}} \oplus \langle \alpha_n \rangle_{\mathbb{R}}^{\perp}$, and we have $C_{W'}(s_n) = \{w \in W'; \alpha_n^w = \alpha_n\}$.

Let $E^* := \mathrm{Hom}_{\mathbb{R}}(E, \mathbb{R})$ be the dual space associated with E, with $\mathbb{R}$-basis $\Pi^* := \{\alpha_1^*, \ldots, \alpha_n^*\}$ dual to Π. Then W acts on E^* by the contragredient action ρ^*; in particular the pair (ρ, ρ^*) leaves the natural pairing between E and E^* invariant. Since W' acts trivially on $E/\langle \alpha_1, \ldots, \alpha_{n-1} \rangle_{\mathbb{R}}$, we infer that α_n^* is fixed by W'. Moreover, we may identify $E^*/\langle \alpha_n^* \rangle_{\mathbb{R}}$ with the contragredient geometric space E'^* of the parabolic subgroup W', the natural $\mathbb{R}$-basis of E'^* being identified with $\{\overline{\alpha}_1^*, \ldots, \overline{\alpha}_{n-1}^*\}$, where $\overline{}\colon E^* \to E^*/\langle \alpha_n^* \rangle_{\mathbb{R}}$ is the natural map. The eigenspaces of $\rho^*(s_n)$ with respect to the eigenvalues ± 1 are given as $E_+^*(s_n) = \langle \alpha_1^*, \ldots, \alpha_{n-1}^* \rangle_{\mathbb{R}}$ and $E_-^*(s_n) = \langle \beta^* \rangle_{\mathbb{R}}$ respectively, where $\beta^* := \sum_{i=1}^{n-1} b_{in}\alpha_i^* + \alpha_n^*$. Hence $C_W(s_n)$ is the set of all elements of W preserving the vector space decomposition $E^* = \langle \beta^* \rangle_{\mathbb{R}} \oplus \langle \alpha_1^*, \ldots, \alpha_{n-1}^* \rangle_{\mathbb{R}}$. Thus any element of $C_{W'}(s_n) = W' \cap C_W(s_n)$ fixes β^*, and hence $\overline{\beta}^* = \sum_{i=1}^{n-1} b_{in}\overline{\alpha}_i^* \in E'^*$ as well. Considering the cone

$$C'_{\mathcal{X}} := \left(\bigcap_{i \in \mathcal{X}} H'_i \right) \cap \left(\bigcap_{i \in \{1, \ldots, n-1\} \setminus \mathcal{X}} A'_i \right) \subseteq E'^*,$$

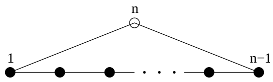
where $H'_i := \{\alpha^* \in E'^*; \alpha_i \cdot \alpha^* = 0\}$ and $A'_i := \{\alpha^* \in E'^*; \alpha_i \cdot \alpha^* > 0\}$, we from $\mathcal{X} = \{i \in \{1, \ldots, n-1\}; b_{in} = 0\}$ get $-\overline{\beta}^* \in C'_{\mathcal{X}}$. Thus the isotropy group of $\overline{\beta}^*$ in W' by [1, Cor.V.4.6] coincides with $W_{\mathcal{X}}$, hence we have $C_{W'}(s_n) \leq W_{\mathcal{X}}$. $\square$

Remark. We more closely consider the representation π in the case W affine: By factoring out $W_1 \times \cdots \times W_{t-1} \trianglelefteq C_{W'}(s_n)$ we may assume that $t = 1$, hence $W' < W$ is finite and integral. We consider the particular case where s_n is chosen such that the Dynkin graph of W is the completion, in the sense of [1, Sect.VI.4.3], of the Dynkin graph of W'. We derive another description of $C_{W'}(s_n) = W_{\mathcal{X}}$:

Let E' be the $\mathbb{R}$-vector space underlying the geometric representation of W', now having a set $\Pi' = \{\alpha'_1, \ldots, \alpha'_{n-1}\}$ of fundamental roots in the root system $\Phi' := \langle \Pi' \rangle_{\mathbb{R}} \subseteq E'$ as its natural $\mathbb{R}$-basis, and carrying the invariant symmetric $\mathbb{R}$-bilinear form B'. Let $-\alpha'_0 \in \Phi'$ be the highest positive root; if Φ' contains roots of different lengths, then α'_0 is a long root. We have $B'(\alpha'_i, \alpha'_0) = B(\alpha_i, \alpha_n) = b_{in} \leq 0$ for all $i \in \{1, \ldots, n-1\}$, implying $W_{\mathcal{X}} \leq \{w \in W'; \alpha_0'^w = \alpha'_0\}$. To show the converse, by [11, Thm.1.12] we conclude that $\{w \in W'; \alpha_0'^w = \alpha_0'\}$ is generated by the reflections it contains, and by [11, Prop.1.10, Prop.1.14] any such reflection is of the form $s_{\beta'} \in W'$ for some positive root $\beta' = \sum_{i=1}^{n-1} b_i \alpha_i' \in \Phi'$, i. e. we have $b_i \geq 0$ for all i. Thus from $B(\beta', \alpha_0') = 0$, and $B(\alpha_i', \alpha_0') \leq 0$ for all $i \in \{1, \ldots, n-1\}$, we infer $b_i = 0$ whenever $B(\alpha_i', \alpha_0') < 0$, hence $\beta' \in \langle \mathcal{X} \rangle_{\mathbb{R}} \cap \Phi'$ and thus $s_{\beta'} \in W_{\mathcal{X}}$, implying $C_{W'}(s_n) = W_{\mathcal{X}} = \{w \in W'; \alpha_0'^w = \alpha_0'\}$. In conclusion, the permutation action of W' on the cosets of $C_{W'}(s_n)$ is isomorphic to its action on $(\alpha_0')^{W'} \subseteq \Phi'$, where $(\alpha_0')^{W'}$ encompasses all of Φ' if all roots have the same length, and is the set of all long roots if Φ' contains roots of different lengths.

4. Examples

We consider the irreducible finite and affine cases of rank at least 3, and a particular irreducible infinite compact hyperbolic case.



4.1. Type A_{n-1}, $n \geq 2$.

This is the archetypical example of symmetric generation [7, Exc.3.3(1)]: We have $W(A_{n-1}) \cong \mathcal{S}_n$, the symmetric group on n points [1, Pl.I], where in the natural permutation representation the distinguished Coxeter generators are the adjacent transpositions $s_i = (i, i+1)$ for $i \in \{1, \ldots, n-1\}$. Letting $t := s_{n-1} = (n-1, n)$ we get $W' = \langle s_1, \ldots, s_{n-2} \rangle \cong W(A_{n-2}) \cong \mathcal{S}_{n-1}$ and $C_{W'}(t) = \langle s_1, \ldots, s_{n-3} \rangle \cong W(A_{n-3}) \cong \mathcal{S}_{n-2}$, where we let $W(A_0) := \mathcal{S}_1 = \{1\}$ and $W(A_{-1}) := \mathcal{S}_0 = \{1\}$. Hence $\mathcal{S}_n$ is symmetrically generated by $(t)^{\mathcal{S}_{n-1}} = \{(i, n); i \in \{1, \ldots, n-1\}\}$, where $\mathcal{S}_{n-1}$ acts on $(t)^{\mathcal{S}_{n-1}}$ by the natural action: $\mathcal{S}_n \cong (2^{*(n-1)} \colon \mathcal{S}_{n-1}) / \langle\langle (s_{n-2}t)^3 \rangle\rangle$.

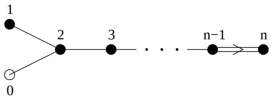
More generally, letting $t := s_{n-k} = (n-k, n-k+1)$ for $k \in \{1, \ldots, \lfloor \frac{n}{2} \rfloor\}$, we get $W' = \langle s_1, \ldots, s_{n-k-1}, s_{n-k+1}, \ldots, s_{n-1} \rangle \cong W(A_{n-k-1}) \times W(A_{k-1}) \cong \mathcal{S}_{n-k} \times \mathcal{S}_k$ and $C_{W'}(t) = \langle s_1, \ldots, s_{n-k-2}, s_{n-k+2}, \ldots, s_{n-1} \rangle \cong W(A_{n-k-2}) \times W(A_{k-2}) \cong \mathcal{S}_{n-k-1} \times \mathcal{S}_{k-1}$. Thus $\mathcal{S}_n$ is symmetrically generated by $(t)^{\mathcal{S}_{n-k} \times \mathcal{S}_k} = \{(i, j); i \in \{1, \ldots, n-k\}, j \in \{n-k+1, \ldots, n\}\}$, where $\mathcal{S}_{n-k} \times \mathcal{S}_k$ acts on $(t)^{\mathcal{S}_{n-k} \times \mathcal{S}_k}$ by the natural action on $(n-k) \cdot k$ points:

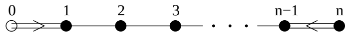
$$\mathcal{S}_n \cong (2^{*(n-k)k} \colon (\mathcal{S}_{n-k} \times \mathcal{S}_k)) / \langle\langle (s_{n-k-1}t)^3, \ \ (s_{n-k+1}t)^3 \rangle\rangle$$

t	W'	$C_{W'}(t)$	m	relations
s_{n-1}	A_{n-2}	A_{n-3}	$n-1$	$(s_{n-2}t)^3$
s_{n-k}	$A_{n-k-1} \times A_{k-1}$	$A_{n-k-2} \times A_{k-2}$	$k(n-k)$	$(s_{n-k-1}t)^3, (s_{n-k+1}t)^3$

4.2. Type $\widetilde{A}_{n-1}$, $n \geq 3$. Letting $t := s_n$, we get that $W(\widetilde{A}_{n-1})$ is symmetrically generated by $(t)^{\mathcal{S}_n}$, where $W' = W(A_{n-1}) \cong \mathcal{S}_n$ acts on $(t)^{\mathcal{S}_n}$ by the natural action on ordered pairs of distinct points in $\{1, \ldots, n\}$, which coincides with its action on the roots in the root system of type A_{n-1}:

t	W'	$C_{W'}(t)$	m	relations
s_n	A_{n-1}	A_{n-3}	$n(n-1)$	$(s_1t)^3, (s_{n-1}t)^3$

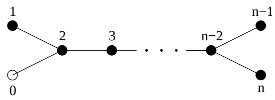




4.3. Types B_n, $n \geq 3$, and C_n, $n \geq 2$. We have $W(B_n) \cong W(C_n) \cong 2^n : \mathcal{S}_n$ [1, Pl.II, III], where the geometric representation restricts to the natural permutation representation of $\langle s_1, \ldots, s_{n-1} \rangle \cong \mathcal{S}_n$, while s_n with respect to the associated permutation basis acts by the diagonal matrix diag$[1, \ldots, 1, -1]$. Hence $W(B_n)$ is isomorphic to the group of all monomial $\{\pm 1\}$-matrices of size n.

We get the following presentations, where $k \in \{2, \ldots, n-2\}$, but we only have weak symmetric generation. To describe the various actions of W' on $(t)^{W'}$ it suffices to note that $W(B_n) \cong W(C_n)$ acts on the cosets of $W(B_{n-1}) \cong W(C_{n-1})$ as it acts on the short roots in the root system of type B_n, or equivalently on the long roots in the root system of type C_n:

t	W'	$C_{W'}(t)$	m	relations
s_n	A_{n-1}	A_{n-2}	n	$(s_{n-1}t)^4$
s_{n-1}	$A_{n-2} \times A_1$	A_{n-3}	$2(n-1)$	$(s_{n-2}t)^3, (s_n t)^4$
s_{n-k}	$A_{n-k-1} \times B_k$	$A_{n-k-2} \times B_{k-1}$	$2k(n-k)$	$(s_{n-k-1}t)^3, (s_{n-k+1}t)^3$
s_1	B_{n-1}	B_{n-2}	$2(n-1)$	$(s_2 t)^3$

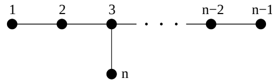


4.4. Type D_n, $n \geq 4$.

We have $W(D_n) \cong 2^{n-1} : \mathcal{S}_n$ [1, Pl.IV], where again the geometric representation restricts to the the natural permutation representation of $\langle s_1, \ldots, s_{n-1} \rangle \cong W(A_{n-1}) \cong \mathcal{S}_n$. Moreover, viewing $W(B_n)$ as the group of all monomial $\{\pm 1\}$-matrices of size n, we have an embedding $W(D_n) < W(B_n)$ such that $W(D_n)$ consists of all such matrices having an even number of entries -1.

We get symmetric generation subject to the following presentations, where $k \in \{3, \ldots, n-2\}$ and we let $D_3 := A_3$ and $D_2 := A_1 \times A_1$. To describe the various actions of W' on $(t)^{W'}$ it suffices to note that $\mathcal{S}_n$ acts on the cosets of $\mathcal{S}_{n-2} \times \mathcal{S}_2$ by the natural action on unordered pairs of distinct points in $\{1, \ldots, n\}$, and that $W(D_n)$ acts on the cosets of $W(D_{n-1})$ as on the short roots in the root system of type B_n:

t	W'	$C_{W'}(t)$	m	relations
s_n	A_{n-1}	$A_{n-3} \times A_1$	$\binom{n}{2}$	$(s_{n-2}t)^3$
s_{n-2}	$A_{n-3} \times A_1 \times A_1$	A_{n-4}	$4(n-2)$	$(s_{n-3}t)^3, (s_{n-1}t)^3, (s_n t)^3$
s_{n-k}	$A_{n-k-1} \times D_k$	$A_{n-k-2} \times D_{k-1}$	$2k(n-k)$	$(s_{n-k-1}t)^3, (s_{n-k+1}t)^3$
s_1	D_{n-1}	D_{n-2}	$2(n-1)$	$(s_2 t)^3$

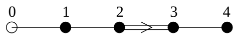


4.5. Type E_n, $n \in \{6,7,8\}$.

We have $W(E_6) \cong GO_6^-(2) = O_6^-(2).2$, $W(E_7) \cong 2 \times GO_7(2) = 2 \times O_7(2)$ and $W(E_8) \cong 2.GO_8^+(2) = 2.O_8^+(2).2$, [1, Pl.V–VII] and [4, p.26, 46, 85]. We get symmetric generation subject to the following presentations, where we still let $W(A_0) = W(A_{-1}) = \{1\}$.

To describe the various actions of W' on $(t)^{W'}$ we note that $\mathcal{S}_n$ acts on the cosets of $\mathcal{S}_2 \times \mathcal{S}_{n-2}$ and of $\mathcal{S}_3 \times \mathcal{S}_{n-3}$ by the natural action on unordered pairs and triples of pairwise distinct points in $\{1, \ldots, n\}$, respectively, and that $W(D_n) \cong 2^{n-1} \colon \mathcal{S}_n$ acts on the cosets of $W(A_{n-1}) \cong \mathcal{S}_n$ by its affine action on the regular normal subgroup 2^{n-1}, i. e. $\mathcal{S}_n$ and 2^{n-1} act by conjugation and translation, respectively. Moreover, $W(E_6) \cong GO_6^-(2)$ acts on the cosets of $W(D_5)$ as on the isotropic vectors in the associated quadratic space, and $W(E_7)$ acts on the cosets of $W(E_6)$ as on a certain set of 56 vectors in the root system of type E_8:

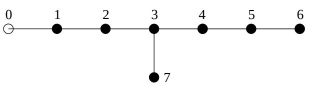
t	W'	$C_{W'}(t)$	m	relations
s_n	A_{n-1}	$A_2 \times A_{n-4}$	$\binom{n}{3}$	$(s_3t)^3$
$s_7, n=8$	E_7	E_6	56	$(s_6t)^3$
$s_6, n=7$	E_6	D_5	27	$(s_5t)^3$
$s_6, n=8$	$E_6 \times A_1$	D_5	54	$(s_5t)^3, (s_7t)^3$
$s_5, n=6$	D_5	A_4	16	$(s_4t)^3$
$s_5, n\geq 7$	$D_5 \times A_{n-6}$	$A_4 \times A_{n-7}$	$16(n-5)$	$(s_4t)^3, (s_6t)^3$
s_4	$A_4 \times A_{n-5}$	$A_2 \times A_1 \times A_{n-6}$	$10(n-4)$	$(s_3t)^3, (s_5t)^3$
s_3	$A_2 \times A_{n-4} \times A_1$	$A_1 \times A_{n-5}$	$6(n-3)$	$(s_2t)^3, (s_4t)^3, (s_nt)^3$
s_2	$A_1 \times A_{n-2}$	$A_1 \times A_{n-4}$	$2\binom{n-1}{2}$	$(s_1t)^3, (s_3t)^3$
s_1	D_{n-1}	A_{n-2}	2^{n-2}	$(s_2t)^3$

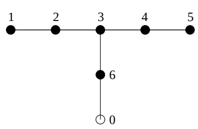


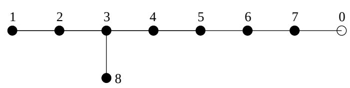
4.6. Type F_4.

We have $W(F_4) \cong W(D_4) \colon \mathcal{S}_3 \cong (2^3 \colon \mathcal{S}_4) \colon \mathcal{S}_3$ [1, Pl.VIII]. We get the following presentations, but only weak symmetric generation. To describe the various actions of W' on $(t)^{W'}$ it suffices to note that $W(B_3) \cong 2^3 \colon \mathcal{S}_3$ acts on the cosets of $W(A_2) \cong \mathcal{S}_3$ by its affine action on the regular normal subgroup 2^3:

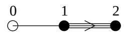
t	W'	$C_{W'}(t)$	m	relations
s_4	B_3	A_2	8	$(s_3t)^3$
s_3	$A_2 \times A_1$	A_1	6	$(s_2t)^4, (s_4t)^3$







4.7. Types $\widetilde{B}_n$, $\widetilde{C}_n$, $\widetilde{D}_n$, $\widetilde{E}_n$, $\widetilde{F}_4$ and $\widetilde{G}_2$.



Letting $t := s_0$, we get symmetric generation if and only if the Dynkin graph associated with W is simply-laced, and W' acts on $(t)^{W'}$ as on the long roots in its associated root system:

t	W'	$C_{W'}(t)$	m	relations
s_0	B_n	$A_1 \times B_{n-2}$	$2n(n-1)$	$(s_2t)^3$
s_0	C_n	C_{n-1}	$2n$	$(s_1t)^4$
s_0	D_n	$A_1 \times D_{n-2}$	$2n(n-1)$	$(s_2t)^3$
s_0	E_6	A_5	72	$(s_6t)^3$
s_0	E_7	D_6	126	$(s_1t)^3$
s_0	E_8	E_7	240	$(s_2t)^3$
s_0	F_4	C_3	24	$(s_1t)^3$
s_0	G_2	A_1	6	$(s_1t)^3$

4.8. Type H_n, $n \in \{3, 4, 5\}$.

We have $W(H_3) \cong 2 \times \mathcal{A}_5$ and $W(H_4) \cong 2.(\mathcal{A}_5 \times \mathcal{A}_5).2$ [1, Exc.VI.4.11, VI.4.12], where $\mathcal{A}_5$ denotes the alternating group on 5 points. Recall that the geometric representation of the parabolic subgroup $W(I_2(5)) \cong \mathcal{D}_{10}$ is the group of Euclidean symmetries of the regular pentagon. The geometric representation of $W(H_3)$ is the group of Euclidean symmetries of the regular dodecahedron, and hence also of its reciprocal, the regular icosahedron; recall that the dodecahedron has pentagonal faces. The geometric representation of $W(H_4)$ is the group of Euclidean symmetries of a 4-dimensional regular polytope having 120 dodecahedral faces [5, Ch.7.8]. Finally $W(H_5)$ is an infinite compact hyperbolic group, [1, Exc.V.4.15] and [11, Ch.6.9].

We get symmetric generation subject to the following presentations. To describe the various actions of W' on $(t)^{W'}$ we note that $W(I_2(5))$ acts on the cosets of $W(A_1)$ by the action on the 5 faces of the regular pentagon, that $W(H_3)$ acts on the cosets of $W(I_2(5))$ by the action on the 12 faces of the regular dodecahedron, and that $W(H_4)$ acts on the cosets of $W(H_3)$ by the action on the 120 faces of the 4-dimensional regular polytope mentioned above:

t	W'	$C_{W'}(t)$	m	relations
s_n	A_{n-1}	A_{n-2}	n	$(s_{n-1}t)^5$
s_{n-1}	$A_{n-2} \times A_1$	A_{n-3}	$2(n-1)$	$(s_{n-2}t)^3, (s_nt)^5$
$s_3, n=5$	$A_2 \times I_2(5)$	$A_1 \times A_1$	15	$(s_2t)^3, (s_4t)^3$
$s_2, n=4$	$A_1 \times I_2(5)$	A_1	10	$(s_1t)^3, (s_3t)^3$
$s_2, n=5$	$A_1 \times H_3$	$I_2(5)$	24	$(s_1t)^3, (s_3t)^3$
$s_1, n=3$	$I_2(5)$	A_1	5	$(s_2t)^3$
$s_1, n=4$	H_3	$I_2(5)$	12	$(s_2t)^3$
$s_1, n=5$	H_4	H_3	120	$(s_2t)^3$

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