The vertices and sources of the natural simple module for the alternating group in even characteristic

Susanne Danz and Jürgen Müller

Abstract

For $n \geq 5$ the natural permutation module for the alternating group $\mathfrak{A}_n$ has a unique non-trivial composition factor, being called its natural simple module. We determine the vertices and sources of the natural simple $\mathfrak{A}_n$-module over fields of characteristic 2.

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1 Introduction and result

One of the leading themes in the representation theory of finite groups is the question how far the p-modular representation theory of a given group is determined by local data, that is by its p-subgroups and their normalisers. From this viewpoint, it is immediate to ask for the vertices and sources of the building blocks of all modules, namely the simple ones. Although not too much seems to be known in general, in recent years the picture has changed considerably for the symmetric group $\mathfrak{S}_n$ on n letters, see for example the survey [3]. Now it is natural to ask how the vertices and sources of the simple $\mathfrak{S}_n$-modules and those of the constituents of their restrictions to the alternating group $\mathfrak{A}_n$ are related, and here in particular the natural $\mathfrak{S}_n$-module immediately springs to mind. The vertices and sources of the natural $\mathfrak{S}_n$-module have been determined in [13], and indeed the latter paper is the starting point of the present work. As it turns out, the case of even characteristic is by far the most interesting one.

Hence, let F be an algebraically closed field of characteristic 2, let $n \geq 3$, and let $D := D^{(n-1,1)}$ be the natural simple $F\mathfrak{S}_n$-module, that is, D is the unique non-trivial composition factor of the natural permutation $F\mathfrak{S}_n$-module $M := M^{(n-1,1)} := \mathrm{Ind}_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n}(F)$. Now $\mathrm{Res}_{\mathfrak{A}_n}^{\mathfrak{S}_n}(D)$ splits into two non-isomorphic simple modules $E_\pm^{(n-1,1)}$ if $n < 5$, and otherwise $\mathrm{Res}_{\mathfrak{A}_n}^{\mathfrak{S}_n}(D) =: E_0^{(n-1,1)}$ remains simple (cf. [1]). In consequence of Mackey's Decomposition Theorem, we deduce that $E_0^{(n-1,1)}$ is the unique non-trivial composition factor of the natural permutation $F\mathfrak{A}_n$-module $\mathrm{Ind}_{\mathfrak{A}_{n-1}}^{\mathfrak{A}_n}(F) \cong \mathrm{Res}_{\mathfrak{A}_n}^{\mathfrak{S}_n}(M)$ if $n \geq 5$, and we call $E_0^{(n-1,1)}$ the natural simple $F\mathfrak{A}_n$-module. If $n < 5$ then $\mathrm{Ind}_{\mathfrak{A}_{n-1}}^{\mathfrak{A}_n}(F)$ has two non-trivial composition factors, $E_\pm^{(2,1)}$ and $E_\pm^{(3,1)}$, respectively, also called the natural simple $F\mathfrak{A}_3$- and $F\mathfrak{A}_4$-modules, respectively.

Our main aim now is to prove the following theorem. Part of the assertions, listed for the sake of completeness, are fairly immediate, and already covered in the subsequent remark. There we also comment on the odd characteristic case, and recall the results on the vertices and sources of the module D from [13].

Theorem 1.1. *Let F be an algebraically closed field of characteristic 2, and $n \geq 3$. Let E be the natural simple $F\mathfrak{A}_n$-module, and let further Q be a vertex of E. Then the following hold:*

   (i) If n is odd then Q is conjugate to a Sylow 2-subgroup of $\mathfrak{A}_{n-3}$, and E has trivial source.

   (ii) If $n > 6$ is even then Q is a Sylow 2-subgroup of $\mathfrak{A}_n$, and $\mathrm{Res}_Q^{\mathfrak{A}_n}(E)$ is a source of E.

(iii) If $n = 4$ then Q is the Sylow 2-subgroup of $\mathfrak{A}_4$, and E has trivial source. If $n = 6$ then Q is conjugate to $(\mathfrak{S}_2 \times \mathfrak{S}_2 \times \mathfrak{S}_2) \cap \mathfrak{A}_6$, and E has a source of dimension 2.

Remark 1.2. (a) For the moment, we consider representations over an algebraically closed field of odd characteristic p, and let $p \leq n$. Then the natural simple module D is not isomorphic to its conjugate counterpart $D' := D \otimes \operatorname{sgn}$, where sgn denotes the sign representation. Hence we have the natural simple $F\mathfrak{A}_n$-module $E := \operatorname{Res}_{\mathfrak{A}_n}^{\mathfrak{S}_n}(D) \cong \operatorname{Res}_{\mathfrak{A}_n}^{\mathfrak{S}_n}(D')$. By [13, Thm. 1.2, 1.3], the vertices of D coincide with the defect groups of its block, which are contained in $\mathfrak{A}_n$ anyway and thus are the vertices of E as well. Moreover, D and E have the same sources, which are trivial if $p \nmid n$, while if $p \mid n$ then are the restrictions of D to its vertices. This settles the case of odd characteristic, hence from now on we again stick to even characteristic.

(b) In the case $n = 3$, the simple $F\mathfrak{S}_3$-module $D^{(2,1)}$ as well as the simple $F\mathfrak{A}_3$-modules $E_{\pm}^{(2,1)}$ are projective and have thus trivial sources. If $n > 3$ is odd then $D^{(n-1,1)} \cong S^{(n-1,1)}$, where $S^{(n-1,1)}$ is the natural Specht $F\mathfrak{S}_n$-module, and $E := E_0^{(n-1,1)} = \operatorname{Res}_{\mathfrak{A}_n}^{\mathfrak{S}_n}(D)$ is simple. By [13, Thm. 1.2, 1.3], the vertices of D are precisely the defect groups of its block, which are conjugate to a Sylow 2-subgroup of $\mathfrak{S}_{n-3}$, and D has trivial source. Let Q_{n-3} be a Sylow 2-subgroup of $\mathfrak{A}_{n-3}$. This is a defect group of the block of $F\mathfrak{A}_n$ containing E. Since D has trivial source, we also deduce that F is a direct summand of $\operatorname{Res}_{Q_{n-3}}^{\mathfrak{A}_n}(E)$. Therefore, also E has the defect groups of its block as vertices and trivial source.

(c) If n is even then D is contained in the principal block of $F\mathfrak{S}_n$. By [13, Thm. 1.4(b), 1.5(b)], $D^{(3,1)}$ has the Sylow 2-subgroup of $\mathfrak{A}_4$ as vertex and trivial source.

In [13, Thm. 1.4(a), 1.5(a)] it is stated that $D^{(n-1,1)}$ has the Sylow 2-subgroup of $\mathfrak{S}_n$ as vertex, and its restriction to the vertex as source, for $n \geq 6$. This statement is correct, alone the proof of [13, Prop. 4.5] used in between has a gap. It occurs in the very last line of [13], and we have detected it while writing the present paper. Actually, we are able to use the techniques developed here to close that gap, and to give a new complete proof of [13, Thm. 1.4(a), 1.5(a)]. Thus, in the present paper we only use [13, Thm. 1.2, 1.3, 1.4(b), 1.5(b)] as well as [13, L. 3.3, Prop. 4.2], which all are valid with the proofs given, at least as far as we see, and we have made sure that we do not refer to any results which have been proven elsewhere using [13].

Remark 1.3. Now the task is to determine the vertices of the natural simple $F\mathfrak{A}_n$-module $E := E_0^{(n-1,1)}$ whenever $n \geq 6$ is even. Hence from now on we suppose that $n \geq 6$ is even. Suppose further that Q_n is a Sylow 2-subgroup of $\mathfrak{A}_n$, and that $Q \leq Q_n$ is a vertex of E.

(a) Then our general strategy is to show, firstly, that $\tilde{E} := \operatorname{Res}_{Q_n}^{\mathfrak{A}_n}(E)$ is indecomposable. This implies that $Q \leq Q_n$ also is a vertex of $\tilde{E}$. Then, secondly, we assume that $\tilde{E}$ is relatively projective with respect to some maximal subgroup $R < Q_n$ containing Q. In particular, we have $\Phi(Q_n) \leq R$, where $\Phi(Q_n)$ denotes the Frattini subgroup of Q_n. Moreover, we have $\operatorname{Res}_{\mathfrak{A}_{n-1}}^{\mathfrak{A}_n}(E) \cong E_0^{(n-2,1)}$, where, by Remark 1.2, the latter has vertex Q_{n-4} and trivial source T. Thus T is a direct summand of $\operatorname{Res}_{Q_{n-4}}^{Q_n}(\tilde{E})$, hence letting S be a source of $\tilde{E}$ we infer that T is a direct summand of $\operatorname{Res}_{Q_{n-4}}^{Q_n}(\operatorname{Ind}_Q^{Q_n}(S))$. Hence Mackey's Decomposition Theorem implies that there is $g \in Q_n$ such that $Q^g \cap Q_{n-4} = Q_{n-4}$ (cf. [14, L. 4.3.4]), that is $Q_{n-4} \leq_{Q_n} Q \leq R < Q_n$. Thus, since R is normal in Q_n, we have $Q_{n-4} \leq R$ as well. In summary we have $\langle \Phi(Q_n), Q_{n-4}, Q \rangle \leq R$, which typically turns out to be a fairly restrictive condition on R.

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Now, $\tilde{E}$ being relatively projective with respect to R, Green's Indecomposability Theorem implies that $\tilde{E}$ is induced from an indecomposable direct summand of $\mathrm{Res}_R^{Q_n}(\tilde{E})$, implying that $\mathrm{Res}_R^{Q_n}(\tilde{E})$ is the direct sum of two indecomposable modules of dimension $\dim(E)/2$ each. This latter conclusion is then drawn to a contradiction, implying that $\tilde{E}$, and thus E, cannot possibly be relatively projective with respect to any proper subgroup of Q_n.

(b) In the sequel, a key player will be the natural permutation $F\mathfrak{S}_n$-module M. Letting $\{\gamma_1, \ldots, \gamma_n\}$ be its permutation basis, M is uniserial with composition series $M > M' > M'' > 0$, where

$$M' := \{\sum_{i=1}^n a_i \gamma_i \in M \mid a_1, \ldots, a_n \in F, \ \sum_{i=1}^n a_i = 0\},$$

$$M'' := \{a \sum_{i=1}^n \gamma_i \in M \mid a \in F\} \cong F \cong M/M',$$

and $M'/M'' \cong D$ as $F\mathfrak{S}_n$-modules. In particular, $\dim(M') = n - 1$ and $\dim(D) = n - 2$. Moreover, $\{\gamma_i + \gamma_n \mid i \in \{1, \ldots, n-1\}\}$ is an F-basis of M', and if $^- : M \longrightarrow M/M''$ denotes the natural epimorphism then $\{\overline{\gamma}_i + \overline{\gamma}_n \mid i \in \{1, \ldots, n-2\}\}$ is an F-basis of M'/M'' (cf. [9, Ex. 5.1]). For convenience, in the following we will simply identify D with M'/M''.

This paper now is organised as follows: In Section 2 we begin by collecting some properties of the Sylow 2-subgroups of $\mathfrak{S}_n$ and $\mathfrak{A}_n$, and their subgroups. It turns out that the behaviour of E depends on the 2-adic expansion of n. Hence letting $n = \sum_{j=1}^l 2^{i_j}$, for some $l \geq 1$ and $i_1 > \ldots > i_l \geq 1$, we distinguish between several cases. In Section 3 we settle the case $l = 1$, that is, n is a 2-power. In Section 4 we begin to investigate the case $l \geq 2$, that is, n is even but not a 2-power, by first considering restrictions of E to various abelian subgroups. This leads to further case distinctions with respect to $n_l := 2^{i_l}$. In Section 5 we settle the case $n_l > 2$, while for $n_l = 2$ in Sections 6– 8 the cases $l \geq 4$, $l = 3$, and $l = 2$, are dealt with, respectively. In the very end the case $n = 6$ remains to be considered, completing the proof of Theorem 1.1. In the final Section 9, which is an appendix, we in particular use the technique employed in Section 8 to give the new proof of [13, Thm. 1.4(a), 1.5(a)].

We want to point out that, although no explicit reference to computer calculations is made in the present paper, before writing it we have dealt with various examples by way of explicit computations, which helped us to understand the different behaviour of E with respect to the 2-adic expansion of n. To do so, we have used the computer algebra systems GAP [6] and MAGMA [2], and the specially tailored computational techniques to determine vertices and sources developed in [4]. Finally, for any finite group G, an FG-module is understood to be a finitely generated right FG-module. Furthermore, the endomorphism algebra $\mathrm{End}_{FG}(M)$ is also supposed to act from the right. For an introduction to the theory of vertices and sources we refer to [14, Sec. 4.3], and for details concerning the representation theory of the symmetric groups to [9] and [10].

2 Sylow subgroups of the symmetric and alternating groups

We begin by introducing our notation for the Sylow 2-subgroups of $\mathfrak{S}_n$ and $\mathfrak{A}_n$, respectively. Moreover, we collect some properties of these Sylow subgroups and their subgroups which will be needed in the course of the subsequent sections.

Remark 2.1. (a) First of all, let $n = 2^m$, for some $m \in \mathbb{N}_0$. We set $P_1 := 1$, $P_2 := C_2 := \langle (1,2) \rangle$, and $P_{2^{i+1}} := P_{2^i} \wr C_2 = \{(x_1, x_2; \sigma) \mid x_1, x_2 \in P_{2^i},\ \sigma \in P_2\}$, for $i \geq 1$. Recall that, for $i \geq 1$ and $(x_1, x_2; \sigma),\ (y_1, y_2; \pi) \in P_{2^{i+1}}$, we have $(x_1, x_2; \sigma)(y_1, y_2; \pi) = (x_1 y_{1^\sigma}, x_2 y_{2^\sigma}; \sigma\pi)$. As usual, for $i \geq 0$, we regard P_{2^i} as a subgroup of $\mathfrak{S}_{2^i}$ in the obvious way. Then, by [10, 4.1.22, 4.1.24], $P_n = P_{2^m}$ is a Sylow 2-subgroup of $\mathfrak{S}_{2^m} = \mathfrak{S}_n$.

Moreover, by construction, P_{2^m} is generated by the following elements:

$$w_{2^i} := \prod_{k=1}^{2^{i-1}} (k, k + 2^{i-1}), \tag{1}$$

for $i = 1, \ldots, m$. For instance, P_8 is generated by $w_2 = (1,2)$, $w_4 = (1,3)(2,4)$ and $w_8 = (1,5)(2,6)(3,7)(4,8)$. Additionally let $w_1 := 1$. Since, by [8, Satz III.15.3], we have $|P_{2^m} : \Phi(P_{2^m})| = 2^m$, (1) yields in fact a minimal set of generators for $P_n = P_{2^m}$.

If $m \geq 2$, then we have $P_n = P_{2^m} = P_{2^{m-1}} \wr P_2 = \{(x_1, x_2; \sigma) \mid x_1, x_2 \in P_{2^{m-1}},\ \sigma \in P_2\}$. The base group of $P_{2^{m-1}} \wr P_2$ shall henceforth be denoted by B_n. When viewing P_{2^m} as a subgroup of $\mathfrak{S}_{2^m}$ as above, B_n corresponds to $P_{2^{m-1}} \times w_n P_{2^{m-1}} w_n$.

(b) Let now $n \geq 2$ be even with 2-adic expansion $n = \sum_{j=1}^{l} 2^{i_j}$, for appropriate $l \geq 2$ and $i_1 > \ldots > i_l \geq 1$, and let $n_j := 2^{i_j}$ for $j = 1, \ldots, l$. Then, by [10, 4.1.22, 4.1.24], $P_n = \prod_{j=1}^{l} P_{n_j}^{g_j}$, with $g_1 = 1$, and $g_j := \prod_{k=1}^{n_j} (k, k + \sum_{s=1}^{j-1} n_s)$ for $j = 2, \ldots, l$, is a Sylow 2-subgroup of $\mathfrak{S}_n$. For convenience, we will simply write $P_n = \prod_{j=1}^{l} P_{n_j}$ where P_{n_j} is understood to be acting on the subset $\Omega_j := \{1 + \sum_{s=1}^{j-1} n_s, \ldots, \sum_{s=1}^{j} n_s\}$ of $\{1, \ldots, n\}$, for $j = 1, \ldots, l$. If n is odd then we simply set $P_n := P_{n-1}$, so that again P_n is a Sylow 2-subgroup of $\mathfrak{S}_n$.

Via (1) we obtain a minimal set of generators for P_{n_j} denoted by $\mathcal{W}_j := \{w_{2,j}, \ldots, w_{n_j,j}\}$, for $j = 1, \ldots, l$. That is, $\mathcal{W} := \bigcup_{j=1}^{l} \mathcal{W}_j$ is a minimal generating set for P_n. For instance, $P_{14} = P_8 \times P_4 \times P_2$ is generated by $w_{2,1} = (1,2)$, $w_{4,1} = (1,3)(2,4)$, $w_{8,1} = (1,5)(2,6)(3,7)(4,8)$, $w_{2,2} = (9,10)$, $w_{4,2} = (9,11)(10,12)$ and $w_{2,3} = (13,14)$.

(c) Now, for any $n \geq 2$, we set $Q_n := P_n \cap \mathfrak{A}_n$ so that Q_n is a Sylow 2-subgroup of $\mathfrak{A}_n$.

If $n = 2^m$, for some $m \geq 2$, then writing $P_n = P_{2^{m-1}} \wr P_2$ we have $Q_n = \{(x_1, x_2; \sigma) \in P_{2^{m-1}} \wr P_2 \mid x_1 x_2 \in Q_{2^{m-1}}\} \leq P_{2^{m-1}} \wr P_2$, and we define

$$w_2' := w_2 \cdot w_2^{w_{2^m}} = (1,2)(2^{m-1} + 1, 2^{m-1} + 2),\ \text{and}\ w_{2^i}' := w_{2^i}\ \text{for}\ i = 2, \ldots, m. \tag{2}$$

If $n = \sum_{j=1}^{l} 2^{i_j} = \sum_{j=1}^{l} n_j$, for some $l \geq 2$ and some $i_1 > \ldots > i_l \geq 1$, is the 2-adic expansion of n as above then we define

$$w_{2,j}' := w_{2,j} w_{2,l}\ \text{for}\ j = 1, \ldots, l-1,\ \text{and}\ w_{2^s,j}' := w_{2^s,j}\ \text{for}\ j = 1, \ldots, l\ \text{and}\ s = 2, \ldots, i_j. \tag{3}$$

Proposition 2.2. *With the notation of the previous remark we have:*

(i) *If $n = 2^m$, for some $m \geq 2$, then $Q_n = \langle w_2', \ldots, w_{2^m}' \rangle$. Furthermore, $\{w_2', \ldots, w_{2^m}'\}$ is a minimal set of generators for Q_n.*

(ii) *If $n = \sum_{j=1}^{l} 2^{i_j} = \sum_{j=1}^{l} n_j$, for some $l \geq 2$ and some $i_1 > \ldots > i_l \geq 1$, then*

$$\mathcal{W}' := \bigcup_{j=1}^{l-1} \{w_{2,j}', \ldots, w_{n_j,j}'\} \cup \{w_{4,l}', \ldots, w_{n_l,l}'\}$$

is a minimal generating set for Q_n.

Proof. Suppose first that $n = 2^m$, for some $m \geq 2$. Obviously, $Q := \langle w_2', \ldots, w_{2^m}' \rangle \leq Q_n$. Moreover, we have $w_2 w_{2^i}' w_2 = w_2' w_{2^i}' w_2' \in Q$, for $i = 1, \ldots, m-1$, and

$$w_2 w_{2^m}' w_2 = w_2 w_{2^m} w_2 \cdot w_{2^m} w_{2^m} = w_2' w_{2^m}' \in Q,$$

so that $w_2 Q w_2 = Q$. This shows that $\langle w_2 \rangle Q = Q \langle w_2 \rangle = P_n$, and thus $|P_n : Q| = 2$, that is $Q = Q_n$. It remains to show that $\{w_2', \ldots, w_{2^m}'\}$ is a minimal generating set for Q_n. For this, notice that Q_n acts transitively on $\{1, \ldots, n\}$, whereas for any $j \in \{1, \ldots, m\}$ the group $Q := \langle w_{2^i}' \mid i \in \{1, \ldots, m\} \setminus \{j\} \rangle$ does not, namely, 1 and $1 + 2^{j-1}$ then lie in different Q-orbits. Consequently, $\{w_2', \ldots, w_{2^m}'\}$ is in fact minimal.

Now suppose that $n = \sum_{j=1}^{l} 2^{i_j} = \sum_{j=1}^{l} n_j$, for some $l \geq 2$ and some $i_1 > \ldots > i_l \geq 1$. Then $Q := \langle \mathcal{W}' \rangle \leq Q_n$. Furthermore, we have

$$w_{2,l} w_{2^s,j}' w_{2,l} = w_{2^s,j}' \in Q, \text{ for } j = 1, \ldots, l-1, \ s = 1, \ldots, i_j,$$
$$w_{2,l} w_{2^s,l}' w_{2,l} = w_{2,1}' w_{2^s,l}' w_{2,1}' \in Q, \text{ for } s = 2, \ldots, i_l,$$

hence $w_{2,l} Q w_{2,l} = Q$, and clearly $\langle w_{2,l} \rangle Q = Q \langle w_{2,l} \rangle = P_n$. This shows that $|P_n : Q| = 2$ so that $Q = Q_n$. Furthermore, from Remark 2.1 we deduce that $\mathcal{W}' \cup \{w_{2,l}\}$ is a minimal generating set for P_n, and thus also $\mathcal{W}'$ has to be a minimal generating set for Q_n. $\qquad\square$

Example 2.3. In order to illustrate the rather technical notation above, consider the cases where $n = 8$ and $n = 14 = 8 + 4 + 2$, respectively: Then Q_8 is generated by

$$w_2' = (1,2)(5,6), \quad w_4' = (1,3)(2,4), \quad w_8' = (1,5)(2,6)(3,7)(4,8).$$

Analogously, Q_{14} is generated by

$$w_{2,1}' = (1,2)(13,14), \qquad w_{4,1}' = (1,3)(2,4), \qquad w_{8,1}' = (1,5)(2,6)(3,7)(4,8),$$
$$w_{2,2}' = (9,10)(13,14), \qquad w_{4,2}' = (9,11)(10,12).$$

Proposition 2.4. *For any even integer $n \geq 2$ we have $\Phi(P_n) = [P_n, P_n]$ and $\Phi(Q_n) = [Q_n, Q_n]$. If $n = 2^m$, for some $m \geq 2$, then $|\Phi(P_n) : \Phi(Q_n)| = 2$, otherwise $\Phi(P_n) = \Phi(Q_n)$.*

Proof. By [8, Satz III.3.14], $[P_n, P_n] \leq \Phi(P_n)$ and $[Q_n, Q_n] \leq \Phi(Q_n)$. As we have just seen, both P_n and Q_n are generated by elements of order 2. Consequently, the same holds true for $P_n/[P_n, P_n]$ and $Q_n/[Q_n, Q_n]$. Thus both $P_n/[P_n, P_n]$ and $Q_n/[Q_n, Q_n]$ are elementary abelian. This yields $[P_n, P_n] = \Phi(P_n)$ and $[Q_n, Q_n] = \Phi(Q_n)$. The second assertion is immediate from Proposition 2.2 and Remark 2.1. $\qquad\square$

Proposition 2.5. *Let $n = 2^m$, for some $m \geq 2$. Then we have $\Phi(P_n) = \{(x_1, x_2; 1) \mid x_1 x_2 \in \Phi(P_{2^{m-1}})\}$ and $\Phi(Q_n) = \{(x_1, x_2; 1) \mid x_1, x_2 \in Q_{2^{m-1}}, x_1 x_2 \in \Phi(P_{2^{m-1}})\}$.*

Proof. The first assertion follows from [15, L. 1.4] and the fact that $\Phi(P_n) = [P_n, P_n]$. It remains to show that $\Phi(Q_n) = H := \{(x_1, x_2; 1) \mid x_1, x_2 \in Q_{2^{m-1}}, x_1 x_2 \in \Phi(P_{2^{m-1}})\} \leq \Phi(P_n)$. For this let $x = (x_1, x_2; \sigma) \in Q_n$. Then $x_1, x_2 \in P_{2^{m-1}}$, $x_1 x_2, x_2 x_1 \in Q_{2^{m-1}}$ and $\sigma \in \{(1, 2), 1\}$. Moreover,

$$x^2 = \begin{cases} (x_1^2, x_2^2; 1), & \text{if } \sigma = 1, \\ (x_1 x_2, x_2 x_1; 1), & \text{if } \sigma \neq 1. \end{cases}$$

That is, $x^2 = (y_1, y_2; 1)$ where $y_1, y_2 \in Q_{2^{m-1}}$, and $x^2 \in \Phi(P_n)$, by [8, Satz III.3.14]. Thus $y_1 y_2 \in \Phi(P_{2^{m-1}})$. Since, again by [8, Satz III.3.14], $\Phi(Q_n)$ is generated by the squares in Q_n, this implies $\Phi(Q_n) \leq H \leq \Phi(P_n)$. But $Q_{2^{m-1}} < P_{2^{m-1}}$ so that $H \neq \Phi(P_n)$. Therefore, from Proposition 2.4 we deduce that $H = \Phi(Q_n)$. $\qquad\square$

Proposition 2.6. *Suppose that $n = 2^m$, for some $m \geq 2$. Then the following hold:*

(i) *We have $B_n = \langle w_2, \ldots, w_{2^{m-1}}, w_2^{w_n}, \ldots, w_{2^{m-1}}^{w_n} \rangle$ and*

$$B_n \cap Q_n = \langle w_2 w_2^{w_n}, w_4, \ldots, w_{2^{m-1}}, w_4^{w_n}, \ldots, w_{2^{m-1}}^{w_n} \rangle.$$

(ii) *We have $P_{2^{m-1}} \Phi(P_n) = B_n$ and $Q_{2^{m-1}} \Phi(P_n) = B_n \cap Q_n$.*

(iii) *If $m \geq 3$ then we have $Q_{2^{m-1}+2} \Phi(Q_n) = B_n \cap Q_n$.*

Proof. Since $B_n = P_{2^{m-1}} \times P_{2^{m-1}}^{w_n}$, the first statement in (i) is clear from Remark 2.1. Furthermore, $\langle w_2 w_2^{w_n}, w_4, \ldots, w_{2^{m-1}}, w_4^{w_n}, \ldots, w_{2^{m-1}}^{w_n} \rangle \leq B_n \cap \mathfrak{A}_n = B_n \cap Q_n$, and is normalized by w_2. Since $\langle w_2 \rangle \cdot \langle w_2 w_2^{w_n}, w_4, \ldots, w_{2^{m-1}}, w_4^{w_n}, \ldots, w_{2^{m-1}}^{w_n} \rangle = B_n$, also the second assertion in (i) follows.

Now we show that $B_n \leq P_{2^{m-1}} \Phi(P_n)$. For this let $x \in B_n$, and write $x = (x_1, x_2; 1)$ with $x_1, x_2 \in P_{2^{m-1}}$. Then $x = (x_1 x_2, 1; 1)(x_2^{-1}, x_2; 1)$. Since $x_1 x_2 \in P_{2^{m-1}}$ and $(x_2^{-1}, x_2; 1) \in \Phi(P_n)$, we get $x \in P_{2^{m-1}} \Phi(P_n)$. Consequently, $P_{2^{m-1}} \Phi(P_n)$ is a proper subgroup of P_n containing the maximal subgroup B_n of P_n, and hence $B_n = P_{2^{m-1}} \Phi(P_n)$.

Next we verify that $B_n \cap Q_n = Q_{2^{m-1}} \Phi(P_n)$. By (i), we already know that $Q_{2^{m-1}} \Phi(P_n) \leq (P_{2^{m-1}} \Phi(P_n)) \cap Q_n = B_n \cap Q_n$. Now let $x \in B_n \cap Q_n$, and again write $x = (x_1, x_2; 1)$ for appropriate $x_1, x_2 \in P_{2^{m-1}}$. Since $x \in Q_n$, we have $x_1 x_2 \in Q_{2^{m-1}}$, and thus $x = (x_1 x_2, 1; 1)(x_2^{-1}, x_2; 1) \in Q_{2^{m-1}} \Phi(P_n)$. Hence $Q_{2^{m-1}} \Phi(P_n) = B_n \cap Q_n$, proving (ii).

Now let $m \geq 3$, then $Q_{2^{m-1}+2} \Phi(Q_n) \leq (P_{2^{m-1}+2} \Phi(P_n)) \cap Q_n = B_n \cap Q_n$. The last equation follows from the fact that $P_{2^{m-1}+2} < P_n$ so that $P_{2^{m-1}+2} \Phi(P_n)$ is a proper subgroup of P_n containing the maximal subgroup $B_n = P_{2^{m-1}} \Phi(P_n)$. Since

$$\Phi(Q_n) \cap Q_{2^{m-1}+2} = \{(x_1, 1; 1) \mid x_1 \in \Phi(P_{2^{m-1}})\} = Q_{2^{m-1}} \cap \Phi(P_n),$$

we now have

$$|Q_{2^{m-1}+2} \Phi(Q_n)| = \frac{|Q_{2^{m-1}+2}||\Phi(Q_n)|}{|\Phi(Q_n) \cap Q_{2^{m-1}+2}|} = \frac{|Q_{2^{m-1}+2}||\Phi(Q_n)|}{|\Phi(P_{2^{m-1}})|} = \frac{|Q_{2^{m-1}}| \cdot 2 \cdot |\Phi(Q_n)|}{|\Phi(P_{2^{m-1}})|}$$

$$= \frac{|Q_{2^{m-1}}||\Phi(P_n)|}{|\Phi(P_{2^{m-1}})|} = |Q_{2^{m-1}} \Phi(P_n)| = |B_n \cap Q_n|.$$

This finally shows that also $Q_{2^{m-1}+2} \Phi(Q_n) = B_n \cap Q_n$, and assertion (iii) follows. $\qquad\square$

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Remark 2.7. (a) Again let $n = \sum_{j=1}^{l} 2^{i_j} = \sum_{j=1}^{l} n_j$, for some $l \geq 1$ and some $i_1 > \ldots > i_l \geq 1$, be the 2-adic expansion of n. For $j = 1, \ldots, l$ define $y_{n_j} := w_{n_j,j} \cdots w_{2,j}$, where the $w_{2,j}, \ldots, w_{n_j,j}$ are as in Remark 2.1. Then, by [13, L. 3.3], y_{n_j} is an n_j-cycle, for $j = 1, \ldots, l$. Furthermore, let $Y_{n_j} := \langle y_{n_j} \rangle$, for $j = 1, \ldots, l$, and $Y_n := \prod_{j=1}^{l} Y_{n_j}$; clearly $Y_n \not\leq \mathfrak{A}_n$. Next we set $y'_{n_j} := y_{n_j} y_{n_l}$ and $Y'_{n_j} := \langle y'_{n_j} \rangle$. Then we have $Y'_n := Y_n \cap \mathfrak{A}_n = \prod_{j=1}^{l} Y'_{n_j}$, namely, we clearly have $Y := \langle y'_{n_1}, \ldots, y'_{n_l} \rangle \leq Y'_n$, and from $\langle y_{n_l} \rangle Y = Y \langle y_{n_l} \rangle = Y_n$ where $y_{n_l} \notin Y$ but $y_{n_l}^2 \in Y$ we get $|Y_n : Y| = 2$, hence $Y = Y'_n$.

Moreover, we set $x_{n_j} := y_{n_j}^2$, for $j = 1, \ldots, l$, that is, x_{n_j} has cycle type $(n_j/2, n_j/2)$, for $j = 1, \ldots, l$. We also set $X_{n_j} := \langle x_{n_j} \rangle$, for $j = 1, \ldots, l$, and $X_n := \prod_{j=1}^{l} X_{n_j}$. Note that, if $l \geq 2$ then the x_{n_j} are squares in Q_n, so that $X_n \leq \Phi(Q_n)$.

(b) When viewing the wreath product $P_{n_j} = P_{\frac{n_j}{2}} \wr C_2$ as a subgroup of $\mathfrak{S}_{n_j}$ as usual, the subgroup of P_{n_j} corresponding to the base group is isomorphic to $P_{\frac{n_j}{2}} \times P_{\frac{n_j}{2}}$ and shall be denoted by $B_{n_j} := P'_{n_j} \times P''_{n_j}$, for $j = 1, \ldots, l$. Here $P'_{n_j} \cong P_{\frac{n_j}{2}}$ is supposed to be acting on $\Omega'_j := \{n_1 + \cdots + n_{j-1} + 1, \ldots, n_1 + \cdots + n_{j-1} + n_j/2\}$, and $P''_{n_j} \cong P_{\frac{n_j}{2}}$ acting on $\Omega''_j := \{n_1 + \cdots + n_{j-1} + n_j/2 + 1, \ldots, n_1 + \cdots + n_j\}$. We also set $B_n := \prod_{j=1}^{l} B_{n_j}$, as well as $B'_{n_j} := B_{n_j} \cap \mathfrak{A}_n \leq Q_{n_j}$, for $j = 1, \ldots, l$, and $B'_n := B_n \cap \mathfrak{A}_n \leq Q_n$. Notice that $y_{n_j}^2 = w_{n_j,j} w_{n_j-1,j} \cdots w_{2,j} w_{n_j,j} \cdot w_{n_j-1,j} \cdots w_{2,j} \in B'_{n_j}$, for $j = 1, \ldots, l$.

3 The case $l = 1$

We investigate the case where $n \geq 8$ is a 2-power.

Lemma 3.1. *Let $n = 2^m$, for some $m \geq 3$. Suppose that R is a maximal subgroup of Q_n such that E is relatively R-projective. Then, if $m > 3$ we have $R = B_n \cap Q_n = B'_n$. If $m = 3$ then we have $R \in \{B'_8, (Q_4 \times Q_4)\langle w_8 \rangle, (Q_4 \times Q_4)\langle w_8^{w_2} \rangle\}$.*

Proof. By [13, Prop. 4.2], we know that $\mathrm{Res}_{Q_n}^{\mathfrak{A}_n}(E)$ is indecomposable. Hence fix some vertex $Q < Q_n$ of E, such that $Q \leq R$. Then, by Remark 1.3, we have $\langle \Phi(Q_n), Q_{n-4}, Q \rangle \leq R$. If $n = 8$ then $Q_4 \Phi(Q_8)$ is a normal subgroup of Q_8, and $Q_8/Q_4\Phi(Q_8)$ is elementary abelian of order 4. Thus there are three maximal subgroups of Q_8 containing $Q_4\Phi(Q_8)$, and these are precisely the ones listed; note that $B'_8 = (Q_4 \times Q_4)\langle w'_2 \rangle$ where $w'_2 = w_8^{w_2} \cdot w_8$. Hence we may now suppose that $m > 3$, and it suffices to prove $\Phi(Q_n)Q_{n-4} = Q_{n-4}\Phi(Q_n) = B_n \cap Q_n$. But, since $m > 3$, we have $n - 4 = 2^m - 4 > 2^{m-1} + 2$, and the latter assertion is immediate from Proposition 2.6 and the fact that $Q_{n-4} < Q_n$. $\qquad\square$

Proposition 3.2. *Let $n = 2^m$, for some $m \geq 3$. Then E has vertex Q_n and source $\mathrm{Res}_{Q_n}^{\mathfrak{A}_n}(E)$.*

Proof. As already mentioned, $\mathrm{Res}_{Q_n}^{\mathfrak{A}_n}(E)$ is indecomposable, by [13, Prop. 4.2]. We follow the strategy given in Remark 1.3, and assume that $R < Q_n$ is a maximal subgroup such that E is relatively R-projective.

Let first $m > 3$. Then, by Lemma 3.1, we have $R = B_n \cap Q_n \geq \Phi(P_n)$. In particular, $\langle x_n \rangle = X_n \leq \Phi(P_n) \leq R$ where x_n is the permutation of cycle type $(n/2, n/2)$ defined in Remark 2.7. As in the proof of [13, Prop. 4.2] we get $\mathrm{Res}_{X_n}^{\mathfrak{A}_n}(E) \cong T_{\frac{n-2}{2}} \oplus T_{\frac{n-2}{2}}$, where $T_{\frac{n-2}{2}}$ denotes the indecomposable FX_n-module of dimension $(n-2)/2$. Thus we conclude

that $\mathrm{Res}_R^{\mathfrak{A}_n}(E) = E_1 \oplus E_2$ where E_1 and E_2 are in fact uniserial with trivial heads and trivial socles. Thus, in particular, $\mathrm{Soc}(\mathrm{Res}_{X_n}^{\mathfrak{A}_n}(E)) = \mathrm{Res}_{X_n}^R(\mathrm{Soc}(\mathrm{Res}_R^{\mathfrak{A}_n}(E)))$. We set

$$s_1 := \sum_{i=1}^{n/2} \gamma_i \in M' \quad \text{and} \quad s_2 := \sum_{i=1}^{n/4} \gamma_i + \sum_{i=1+n/2}^{3n/4} \gamma_i \in M'.$$

Then $s_1(x_n + 1) = 0$ and $s_2(x_n + 1) = \sum_{i=1}^n \gamma_i \in M''$. Consequently, $\bar{s}_1 := s_1 + M''$ and $\bar{s}_2 := s_2 + M''$ are annihilated by $\mathrm{Rad}(FX_n)$, that is they are contained in $\mathrm{Soc}(\mathrm{Res}_{X_n}^{\mathfrak{A}_n}(E))$. Furthermore, $\bar{s}_1$ and $\bar{s}_2$ are linearly independent over F. Therefore $\{\bar{s}_1, \bar{s}_2\}$ is an F-basis for $\mathrm{Soc}(\mathrm{Res}_R^{\mathfrak{A}_n}(E))$. But we also have $w_{2^{m-1}} \in B_n \cap Q_n = R$, yielding the contradiction

$$\bar{s}_2 w_{2^{m-1}} = \sum_{i=1+n/4}^{3n/4} \bar{\gamma}_i \neq \bar{s}_2.$$

Let finally $m = 3$. Then R is one of the subgroups given in Lemma 3.1. To exclude the case $R_1 := B_8'$, notice that, by Remark 2.7, we have $x_8 = y_8^2 = (1,3,2,4)(5,7,6,8) \in R_1$, and hence we may argue as in the case $m > 3$. As for $R_2 := (Q_4 \times Q_4)\langle w_8 \rangle$, we have $w_8 w_4 = (1,5,3,7)(2,6,4,8) \in R_2$, and thus letting $s_1 := \gamma_1 + \gamma_3 + \gamma_5 + \gamma_7 \in M'$ and $s_2 := \gamma_1 + \gamma_3 + \gamma_2 + \gamma_4 \in M'$ instead we similarly arrive at a contradiction, using $s_1 w_4^{w_2} = \gamma_2 + \gamma_4 + \gamma_5 + \gamma_7$ where $w_4^{w_2} = (1,4)(2,3) \in Q_4$. For $R_3 := (Q_4 \times Q_4)\langle w_8^{w_2} \rangle$ again argue similarly, using $w_8^{w_2} w_4 = (1,6,3,7)(2,5,4,8) \in R_3$, and $s_1 := \gamma_1 + \gamma_3 + \gamma_6 + \gamma_7 \in M'$ and $s_2 := \gamma_1 + \gamma_3 + \gamma_2 + \gamma_4 \in M'$, where $s_1 w_4^{w_2} = \gamma_2 + \gamma_4 + \gamma_6 + \gamma_7$. $\qquad\square$

This completes the proof of Theorem 1.1 in the case where n is a 2-power.

4 The case $l \geq 2$

We now investigate the case where $n \geq 4$ is even, but not a 2-power. We aim to show that the vertices of $E = \mathrm{Res}_{\mathfrak{A}_n}^{\mathfrak{S}_n}(D)$ are then always the Sylow 2-subgroups of $\mathfrak{A}_n$, unless $n = 6$. We first need a few preparations, concerning restrictions of E to various subgroups: Recall from Remark 2.7 that Q_n contains the abelian subgroups Y_n' and X_n, where, since $l \geq 2$, we moreover have $X_n \leq \Phi(Q_n)$. We will investigate the restrictions of E to Y_n' and X_n, respectively, in detail. Thus we gain information on restrictions of E to Q_n and to maximal subgroups of Q_n. This leads to a further division into various subcases, which will be dealt with in the subsequent sections.

Remark 4.1. In Remark 1.2 we introduced the permutation basis $\{\gamma_1, \dots, \gamma_n\}$ of the natural permutation $F\mathfrak{S}_n$-module M. While working with the group Y_n', it will be convenient to re-number the elements of this basis as follows: for $j = 1, \dots, l$ and $i = n_1 + \cdots + n_{j-1} + 1, \dots, n_1 + \cdots + n_j$, we set

$$\delta_i := \gamma_{n_1 + \cdots + n_{j-1} + 1} y_{n_j}^{i - n_1 - \cdots - n_{j-1} - 1}.$$

In particular, for $j = 1, \dots, l$ and $i = n_1 + \cdots + n_{j-1} + 1, \dots, n_1 + \cdots + n_j - 1$ we have $\delta_i y_{n_j} = \delta_{i+1}$, and $\delta_{n_1 + \cdots + n_j} y_{n_j} = \delta_{n_1 + \cdots + n_{j-1} + 1}$. That is, $\{\delta_{n_1 + \cdots + n_{j-1} + 1}, \dots, \delta_{n_1 + \cdots + n_j}\}$ is the permutation basis of the natural permutation FY_{n_j}-module, for $j = 1, \dots, l$.

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For $j = 1, \ldots, l$ we now define

$$\delta_{j'}^+ := \sum_{i=0}^{n_j/2-1} \delta_{n_1+\cdots+n_{j-1}+1+2i} = \sum_{i \in \Omega_j'} \gamma_i =: \gamma_{j'}^+,$$

$$\delta_{j''}^+ := \sum_{i=0}^{n_j/2-1} \delta_{n_1+\cdots+n_{j-1}+2+2i} = \sum_{i \in \Omega_j''} \gamma_i =: \gamma_{j''}^+,$$

and

$$\delta_0^+ := \sum_{i=0}^{n/2-1} \delta_{2+2i} = \sum_{j=1}^{l} \delta_{j''}^+ = \sum_{j=1}^{l} \gamma_{j''}^+ =: \gamma_0^+,$$

as well as $\delta_j^+ := \delta_{j'}^+ + \delta_{j''}^+ = \gamma_{j'}^+ + \gamma_{j''}^+ =: \gamma_j^+$, and $\delta^+ := \sum_{j=1}^{n} \delta_j = \sum_{j=1}^{n} \gamma_j =: \gamma^+$. The sets Ω_j' and Ω_j'' are as in Remark 2.7.

Proposition 4.2. *With the above notation, for $\mathrm{Res}_{Y_n'}^{\mathfrak{A}_n}(E) = \mathrm{Res}_{Y_n'}^{\mathfrak{S}_n}(M'/M'')$, we have:*

(i) *If $l > 2$ then $\{\overline{\delta}_j^+ \mid j \in \{1, \ldots, l-1\}\}$ is an F-basis of $\mathrm{Soc}(\mathrm{Res}_{Y_n'}^{\mathfrak{A}_n}(E))$.*

(ii) *If $l = 2$ and $n_l > 2$ then $\{\overline{\delta}_0^+, \overline{\delta}_1^+\}$ is an F-basis of $\mathrm{Soc}(\mathrm{Res}_{Y_n'}^{\mathfrak{A}_n}(E))$.*

(iii) *If $l = 2$ and $n_l = 2$ then $\mathrm{Res}_{Y_n'}^{\mathfrak{A}_n}(E) \cong FY_n'$ with socle spanned by $\overline{\delta}_1^+$.*

Proof. Let $v = \sum_{i=1}^{n} c_i \delta_i \in M$, for appropriate $c_i \in F$, such that $\bar{v} \in \mathrm{Soc}(\mathrm{Res}_{Y_n'}^{\mathfrak{S}_n}(M/M''))$. Since the group Y_n' acts trivially on $\mathrm{Soc}(\mathrm{Res}_{Y_n'}^{\mathfrak{S}_n}(M/M''))$, for each $j \in \{1, \ldots, l\}$ there is some $a_j \in F$ such that $vy_{n_j}' = v + a_j \delta^+$.

Suppose first that $l > 2$ so that $\Omega \setminus (\Omega_j \cup \Omega_l) \neq \emptyset$, for $j = 1, \ldots, l$. That is, $\delta_i y_{n_j}' = \delta_i$, for $i \in \Omega \setminus (\Omega_j \cup \Omega_l)$. In particular, for $j = 1, \ldots, l$ and $i \in \Omega \setminus (\Omega_j \cup \Omega_l)$, we get $c_i = c_i + a_j$. Thus $a_j = 0$, for $j = 1, \ldots, l$. If $j \leq l-1$ then $vy_{n_j}' = v$ holds if and only if $c_{n_1+\cdots+n_{j-1}+1} = \ldots = c_{n_1+\cdots+n_j}$ and $c_{n_1+\cdots+n_{l-1}+1} = \ldots = c_{n_1+\cdots+n_l}$. Furthermore, $vy_{n_j}' = v$ for $j = 1, \ldots, l-1$ implies also $vy_{n_l}' = v$. Consequently, we deduce:

$$\bigcap_{j=1}^{l} \{v \in M \mid vy_{n_j}' = v\} = \bigcap_{j=1}^{l-1} \{v \in M \mid vy_{n_j}' = v\} = \langle \delta_1^+, \ldots, \delta_l^+ \rangle.$$

Moreover, $\delta_1^+, \ldots, \delta_l^+$ are linearly independent, and thus form a basis for $\mathrm{Soc}(\mathrm{Res}_{Y_n'}^{\mathfrak{S}_n}(M))$. This in turn implies that $\{\overline{\delta}_j^+ \mid j \in \{1, \ldots, l-1\}\} \subset M'/M''$ is an F-basis for

$$\{\bar{v} \in M/M'' \mid v \in \langle \delta_1^+, \ldots, \delta_l^+ \rangle_F\} = \{\bar{v} \in M/M'' \mid \bar{v} y_{n_j}' = \bar{v} \text{ for } j = 1, \ldots, l\}$$
$$= \mathrm{Soc}(\mathrm{Res}_{Y_n'}^{\mathfrak{S}_n}(M/M'')).$$

Hence $\mathrm{Soc}(\mathrm{Res}_{Y_n'}^{\mathfrak{S}_n}(M'/M'')) = \mathrm{Soc}(\mathrm{Res}_{Y_n'}^{\mathfrak{S}_n}(M/M'')) = \langle \overline{\delta}_j^+ \mid j = 1, \ldots, l-1 \rangle_F$, implying (i).

Next suppose that $l = 2$. If also $j = 2$ then, since $\Omega \setminus \Omega_2 = \Omega_1 \neq \emptyset$, we get $a_2 = 0$. If $j = 1$ then, since $\Omega \setminus (\Omega_1 \cup \Omega_2) = \emptyset$, we deduce $vy'_{n_1} = \sum_{i=1}^{n} c_{i(y'^{-1}_{n_1})} \delta_i = \sum_{i=1}^{n} (c_i + a_1)\delta_i$, so that

$$c_i = \begin{cases} c_1, & \text{if } i \in \{1, \ldots, n_1\} \text{ is odd,} \\ c_1 + a_1, & \text{if } i \in \{1, \ldots, n_1\} \text{ is even,} \\ c_{n_1+1}, & \text{if } i \in \{n_1+1, \ldots, n_1+n_2\} \text{ is odd,} \\ c_{n_1+1} + a_1, & \text{if } i \in \{n_1+1, \ldots, n_1+n_2\} \text{ is even.} \end{cases}$$

Hence $\{\delta_0^+, \delta_1^+, \delta_2^+\}$ is an F-basis of $\{v \in M \mid vy'_{n_1} = v + a_1\delta^+\}$. Thus the condition $0 \neq \bar{v} \in \mathrm{Soc}(\mathrm{Res}^{\mathfrak{S}_n}_{Y'_n}(M/M''))$ is equivalent to $vy'_{n_2} = v$ where $v \in \langle \delta_0^+, \delta_1^+, \delta_2^+ \rangle_F$. But $vy'_{n_2} = v$ holds if and only if $c_{n_1+1} = c_{n_1+i}$, for $i = 1, \ldots, n_2 - 1$ odd, and $c_{n_1+2} = c_{n_1+i}$ for $i = 2, \ldots, n_2$ even. That is, $\{\delta_1, \ldots, \delta_{n_1}, \delta_{2'}^+, \delta_{2''}^+\}$ is an F-basis of $\{v \in M \mid vy'_{n_2} = v\}$. Since $\langle \delta_0^+, \delta_1^+, \delta_2^+ \rangle_F \leq \langle \delta_1, \ldots, \delta_{n_1}, \delta_{2'}^+, \delta_{2''}^+ \rangle_F$, we get $\mathrm{Soc}(\mathrm{Res}^{\mathfrak{S}_n}_{Y'_n}(M/M'')) = \langle \bar{\delta}_0^+, \bar{\delta}_1^+, \bar{\delta}_2^+ \rangle_F$. Furthermore, $\bar{\delta}_0^+$ and $\bar{\delta}_1^+$ are linearly independent over F, whereas $\bar{\delta}_2^+ = \bar{\delta}_1^+$.

Therefore, if $n_l > 2$ then $\{\bar{\delta}_0^+, \bar{\delta}_1^+\} \subset M'/M''$ is in fact a basis of $\mathrm{Soc}(\mathrm{Res}^{\mathfrak{S}_n}_{Y'_n}(M/M'')) = \mathrm{Soc}(\mathrm{Res}^{\mathfrak{S}_n}_{Y'_n}(M'/M''))$, proving (ii). If finally $n_l = 2$ then we have $\delta_0^+ \notin M'$. Hence in this case we deduce that $\mathrm{Soc}(\mathrm{Res}^{\mathfrak{S}_n}_{Y'_n}(M'/M''))$ has F-basis $\{\bar{\delta}_1^+\}$. In particular, $\mathrm{Res}^{\mathfrak{A}_n}_{Y'_n}(E)$ is indecomposable so that, comparing dimensions, we get $\mathrm{Res}^{\mathfrak{A}_n}_{Y'_n}(E) \cong FY'_n$, proving (iii). $\qquad\square$

Proposition 4.3. *Let $l \geq 2$ and $j \in \{1, \ldots, l\}$, and let $K_j := \ker_D(y'^+_{n_j})$ and $I_j := \mathrm{im}_D(y'^+_{n_j})$.*

  (i) If $j < l$ then $\dim(K_j) = \dim(D) - 1$, and $I_j = \langle \bar{\delta}_j^+ \rangle_F$.

  (ii) If $j = l$ and $n_l > 2$ then $\dim(K_j) = \dim(D) - 2$, and $I_j = \langle \bar{\delta}_{l'}^+, \bar{\delta}_{l''}^+ \rangle_F$.

  (iii) If $j = l$ and $n_l = 2$ then $y'^+_{n_j} = 1$ and thus $K_j = 0$.

Proof. For $j = 1, \ldots, l$ we define the following elements in M':

$$v_j := \delta_{n_1 + \cdots + n_j} + \delta_{n_1 + \cdots + n_j + 1}, \text{ if } j < l,$$

as well as $v'_l := \delta_n + \delta_1$ and $v''_l := \delta_{n-1} + \delta_1$. Then $v_j y'^+_{n_j} = \delta_j^+$, if $j \leq l - 1$, and

$$v'_l y'^+_{n_l} = \begin{cases} v'_l, & \text{if } n_l = 2, \\ \delta_{l''}^+, & \text{if } n_l > 2, \end{cases} \quad \text{and} \quad v''_l y'^+_{n_l} = \begin{cases} v''_l, & \text{if } n_l = 2, \\ \delta_{l'}^+, & \text{if } n_l > 2. \end{cases}$$

We first suppose that $j < l$. Then Y'_{n_j} acts regularly on $U_j := \langle \delta_{n_1 + \cdots + n_{j-1} + 1}, \ldots, \delta_{n_1 + \cdots + n_j} \rangle_F$, so that $U_j y'^+_{n_j} \cong \mathrm{Rad}^{n_j - 1}(FY'_{n_j}) \cong F$ and $\dim(\ker_{U_j}(y'^+_{n_j})) = n_j - 1$. Furthermore, the vector space $\langle \delta_{n_1 + \cdots + n_{l-1} + 1}, \ldots, \delta_{n_1 + \cdots + n_l} \rangle_F$ is an indecomposable FY'_{n_j}-module of dimension n_l. In particular, $\langle \delta_{n_1 + \cdots + n_{l-1} + 1}, \ldots, \delta_{n_1 + \cdots + n_l} \rangle_F$ is annihilated by $\mathrm{Rad}^{n_j - 1}(FY'_{n_j}) = F\langle y'^+_{n_j} \rangle = \mathrm{Soc}(FY'_{n_j})$. Furthermore, Y'_{n_j} acts trivially on the subspace

$$\langle \{\delta_1, \ldots, \delta_n\} \setminus \{\delta_{n_1 + \cdots + n_{j-1} + 1}, \ldots, \delta_{n_1 + \cdots + n_j}, \delta_{n_1 + \cdots + n_{l-1} + 1}, \ldots, \delta_{n_1 + \cdots + n_l}\} \rangle_F$$

of M which is thus annihilated by $y'^+_{n_j}$. To summarize, we have shown that $\dim(\ker_M(y'^+_{n_j})) = n - 1 = \dim(M')$. Hence $\dim(\ker_{M'}(y'^+_{n_j})) \in \{\dim(M'), \dim(M') - 1\} = \{n - 1, n - 2\}$ and

$\dim(\ker_D(y_{n_j}^{\prime+})) = \dim(K_j) \in \{\dim(D), \dim(D) - 1\} = \{n - 2, n - 3\}$. But we already know that $0 \neq \overline{\delta}_j^+ \in \operatorname{im}_D(y_{n_j}^{\prime+}) = I_j$ so that $I_j = \langle \overline{\delta}_j^+ \rangle_F$ and $\dim(K_j) = \dim(D) - 1$, proving (i).

Now let $j = l$ and $n_l > 2$. Then Y_{n_l}' acts trivially on $\langle \delta_1, \ldots, \delta_{n_1 + \ldots + n_{l-1}} \rangle_F$. Furthermore, $\langle \delta_{n_1 + \cdots + n_{l-1} + 1}, \ldots, \delta_{n_1 + \ldots + n_l} \rangle_F$ is an FY_{n_l}'-module isomorphic to $FY_{n_l}' \oplus FY_{n_l}'$. Consequently, $\langle \delta_1, \ldots, \delta_{n_1 + \ldots + n_{l-1}} \rangle_F$ is annihilated by $y_{n_l}^{\prime+}$, and

$$\langle \delta_{n_1 + \cdots + n_{l-1} + 1}, \ldots, \delta_{n_1 + \ldots + n_l} \rangle_F \cdot y_{n_l}^{\prime+} \cong \operatorname{Rad}^{n_l - 1}(FY_{n_l}') \oplus \operatorname{Rad}^{n_l - 1}(FY_{n_l}') \cong F \oplus F.$$

Thus $\dim(\ker_M(y_{n_l}^{\prime+})) = n - 2 = \dim(M') - 1$ so that $\dim(\ker_{M'}(y_{n_l}^{\prime+})) \geq \dim(M') - 2 = n - 3$ and $\dim(K_l) = \dim(\ker_D(y_{n_l}^{\prime+})) \geq \dim(D) - 2 = n - 4$. As we have already shown above, $\langle \overline{\delta}_{l'}^+, \overline{\delta}_{l''}^+ \rangle_F \leq \operatorname{im}_D(y_{n_l}^{\prime+}) = I_l$. Since $l \geq 2$, we conclude that $\overline{\delta}_{l'}^+$ and $\overline{\delta}_{l''}^+$ are linearly independent over F, and hence $I_l = \langle \overline{\delta}_{l'}^+, \overline{\delta}_{l''}^+ \rangle_F$ and $\dim(K_l) = n - 4 = \dim(D) - 2$. This proves (ii), and assertion (iii) is obviously true, since $y_{n_l}' = 1$ for $n_l = 2$. $\qquad\square$

Remark 4.4. Let $l \geq 2$. We consider the subgroup $X_n = \prod_{j=1}^l X_{n_j} \leq \Phi(Q_n) \leq Q_n \leq P_n$. Letting $Q \leq P_n$ be any subgroup such that $X_n \leq Q$, we show when X_n can be used to prove the indecomposability of $\operatorname{Res}_Q^{\mathfrak{S}_n}(D)$. Note that this is slightly more general than needed to prove Theorem 1.1, but will be useful in the proof of Theorem 9.1. Therefore, we fix an indecomposable direct sum decomposition of $\operatorname{Res}_Q^{\mathfrak{S}_n}(D)$.

(a) Suppose that $j \in \{1, \ldots, l\}$ such that $n_j > 2$, that is, we exclude just the case $j = l$ and $n_l = 2$. Let $u := \gamma_{n_1 + \cdots + n_j} + \gamma_{n_1 + \cdots + n_j + 1} \in M'$ and $v := \gamma_{n_1 + \cdots + n_{j-1} + 1} + \gamma_{n_1 + \cdots + n_{j-1} + n_j/2 + 1} \in M'$ where the indices should be read modulo n. Then

$$u x_{n_j}^+ = \gamma_{j''}^+ = \delta_{j''}^+ \notin M'' \quad \text{and} \quad v x_{n_j}^+ = \gamma_j^+ = \delta_j^+ \notin M''. \tag{4}$$

Furthermore, $\langle \gamma_{n_1 + \cdots + n_{j-1} + 1}, \ldots, \gamma_{n_1 + \cdots + n_j} \rangle_F$ is an FX_{n_j}-module isomorphic to $FX_{n_j} \oplus FX_{n_j}$, and X_{n_j} acts trivially on $\langle \gamma_1, \ldots, \gamma_{n_1 + \cdots + n_{j-1}}, \gamma_{n_1 + \cdots + n_j + 1}, \ldots, \gamma_n \rangle_F$. Hence

$$\operatorname{im}_M(x_{n_j}^+) = \operatorname{Soc}(\langle \gamma_{n_1 + \cdots + n_{j-1} + 1}, \ldots, \gamma_{n_1 + \cdots + n_j} \rangle_F) \cong F \oplus F,$$

and $\dim(\ker_M(x_{n_j}^+)) = n - 2$. This implies $\dim(\ker_{M'}(x_{n_j}^+)) \geq \dim(M') - 2$ and thus $\dim(\ker_D(x_{n_j}^+)) \geq \dim(D) - 2$. Since, by (4), $\overline{\gamma}_{j'}^+, \overline{\gamma}_{j''}^+ \in \operatorname{im}_D(x_{n_j}^+)$, we get $\dim(\ker_D(x_{n_j}^+)) = \dim(D) - 2 = n - 4$ and $\operatorname{im}_D(x_{n_j}^+) = \langle \overline{\gamma}_{j'}^+, \overline{\gamma}_{j''}^+ \rangle_F$. Hence we have $(FX_{n_j} \oplus FX_{n_j}) | \operatorname{Res}_{X_{n_j}}^{\mathfrak{S}_n}(D)$, and either precisely one or precisely two indecomposable summands in the fixed direct sum decomposition of $\operatorname{Res}_Q^{\mathfrak{S}_n}(D)$ are not annihilated by $x_{n_j}^+$.

(b) Suppose that there are two of these, U' and U'', say. Recall further that we are still assuming $n_j > 2$. Then there are some $a', a'', b', b'' \in F$ such that $0 \neq u' := a' \overline{\gamma}_{j'}^+ + a'' \overline{\gamma}_{j''}^+ \in U'$ and $0 \neq u'' := b' \overline{\gamma}_{j'}^+ + b'' \overline{\gamma}_{j''}^+ \in U''$. In particular, u' and u'' are linearly independent. Next consider $Q^{p_j q_j}$, where

$$p_j : P_n \longrightarrow P_{n_j} \quad \text{and} \quad q_j : P_{n_j} \longrightarrow P_{n_j}/B_{n_j} \cong \langle w_{n_j, j} \rangle$$

are the natural epimorphisms. We have either $Q^{p_j q_j} = \langle w_{n_j, j} \rangle$ or $Q^{p_j q_j} = \{1\}$. Assume that $Q^{p_j q_j} = \langle w_{n_j, j} \rangle$. Then there exists some $g \in Q$ such that $\gamma_{j'}^+ g = \gamma_{j''}^+$ and $\gamma_{j''}^+ g = \gamma_{j'}^+$. If $a' \neq a''$ then $u' + u'g = (a' + a'')\overline{\gamma}_j^+ \in U'$ and thus also $\overline{\gamma}_j^+ \in U'$. If $a' = a''$ then

$0 \neq a'\overline{\gamma}_j^+ \in U'$, and again $\overline{\gamma}_j^+ \in U'$. But, analogously, we deduce $\overline{\gamma}_j^+ \in U''$, yielding the contradiction $0 \neq \overline{\gamma}_j^+ \in U' \cap U''$. Consequently, this forces $Q^{p_j q_j} = \{1\}$, that is $Q \leq \ker(p_j q_j)$.

(c) Now suppose that $n_l > 2$ and that, for each $j = 1, \ldots, l$, there is exactly one indecomposable direct summand U_j in the fixed direct sum decomposition of $\mathrm{Res}_Q^{\mathfrak{S}_n}(D)$ which is not annihilated by $x_{n_j}^+$. We thus deduce $\overline{\gamma}_j^+ \in \langle \overline{\gamma}_{j'}^+, \overline{\gamma}_{j''}^+ \rangle_F = \mathrm{im}_D(x_{n_j}^+) \leq U_j$, for $j = 1, \ldots, l$. Since $\sum_{j=1}^l \overline{\gamma}_j^+ = \overline{\gamma}^+ = 0$, this implies $U_1 = \ldots = U_l$.

Next we show that $\mathrm{Soc}(\mathrm{Res}_Q^{\mathfrak{S}_n}(D)) \leq \mathrm{Res}_{X_n}^Q(U_1) + \cdots + \mathrm{Res}_{X_n}^Q(U_l) = \mathrm{Res}_{X_n}^Q(U_1)$, which then implies that $\mathrm{Res}_Q^{\mathfrak{S}_n}(D) = U_1$ is indecomposable. For this, let $v \in M'$. Then $\overline{v} \in \mathrm{Soc}(\mathrm{Res}_{X_n}^{\mathfrak{S}_n}(D))$ if and only if $\overline{v} x_{n_j} = \overline{v}$, for all $j = 1, \ldots, l$, that is, if and only if for each $j \in \{1, \ldots, l\}$ there is some $a_j \in F$ such that $v x_{n_j} = v + a_j \gamma^+$. Let $j \in \{1, \ldots, l\}$. Since X_{n_j} acts trivially on $\Omega \setminus \Omega_j \neq \emptyset$, this condition is actually equivalent to $v x_{n_j} = v$. That is, $\overline{v} \in D$ is fixed by X_{n_j} if and only if $v \in M'$ is. Since $\langle \gamma_{n_1+\cdots+n_{j-1}+1}, \ldots, \gamma_{n_1+\cdots+n_j} \rangle_F$ is isomorphic to $FX_{n_j} \oplus FX_{n_j}$ as an FX_{n_j}-module, we get

$$\mathrm{Soc}(\mathrm{Res}_{X_n}^{\mathfrak{S}_n}(M)) = \{v \in M \mid v x_{n_j} = v \text{ for all } j = 1, \ldots, l\} = \bigoplus_{j=1}^l \langle \gamma_{j'}^+, \gamma_{j''}^+ \rangle_F \leq \mathrm{Res}_{X_n}^{\mathfrak{S}_n}(M').$$

Consequently, $\mathrm{Soc}(\mathrm{Res}_{X_n}^{\mathfrak{S}_n}(M)) = \mathrm{Soc}(\mathrm{Res}_{X_n}^{\mathfrak{S}_n}(M')) = \bigoplus_{j=1}^l \langle \gamma_{j'}^+, \gamma_{j''}^+ \rangle_F$, and thus indeed

$$\mathrm{Soc}(\mathrm{Res}_{X_n}^{\mathfrak{S}_n}(D)) = \langle \overline{\gamma}_{1'}^+, \overline{\gamma}_{1''}^+, \ldots, \overline{\gamma}_{l'}^+, \overline{\gamma}_{l''}^+ \rangle_F \leq \mathrm{Res}_{X_n}^Q(U_1) + \cdots + \mathrm{Res}_{X_n}^Q(U_l).$$

5   The case $l \geq 2$ and $n_l > 2$

The behaviour of the natural $F\mathfrak{A}_n$-module E upon restriction to the abelian subgroup Y_n' of Q_n depends on whether $n_l > 2$ or $n_l = 2$. Therefore, we will now distinguish between these two cases, and start off with the case $n_l > 2$.

Proposition 5.1. *Let $l \geq 2$ and $n_l > 2$. Then E has vertex Q_n and source $\mathrm{Res}_{Q_n}^{\mathfrak{A}_n}(E)$.*

Proof. We follow the strategy given in Remark 1.3, and assume that $R < Q_n$ is a maximal subgroup such that E is relatively R-projective, hence $X_n = \prod_{j=1}^l X_{n_j} \leq \Phi(Q_n) \leq R$ and $Q_{n-4} \leq R$.

First of all, we fix an indecomposable direct sum decomposition of $\mathrm{Res}_{Q_n}^{\mathfrak{A}_n}(E)$. Then, for each $j = 1, \ldots, l$, we have $Q_n^{p_j q_j} = \langle w_{n_j, j} \rangle \neq 1$; note that here we need $n_l \geq 4$. Therefore, in consequence of Remark 4.4, part (b), we obtain that, for each $j = 1, \ldots, l$, there is precisely one indecomposable direct summand U_j in the decomposition which is not annihilated by $x_{n_j}^+$. Hence, Remark 4.4, part (c), shows that $\mathrm{Res}_{Q_n}^{\mathfrak{A}_n}(E)$ is, in fact, indecomposable.

Now we fix an indecomposable direct sum decomposition of $\mathrm{Res}_R^{\mathfrak{A}_n}(E)$. Hence, by Remark 4.4, part (a), there is either one indecomposable direct summand U or there are exactly two indecomposable direct summands U' and U'' in the decomposition which are not annihilated by $x_{n_1}^+$. In the first case we get $(FX_{n_1} \oplus FX_{n_1}) \mid \mathrm{Res}_{X_{n_1}}^R(U)$, hence $\dim(U) \geq 2\dim(FX_{n_1}) = n_1 \geq (n+2)/2 > (n-2)/2 = \dim(E)/2$, a contradiction. In the second case Remark 4.4, part (b), implies that $R \leq \ker(p_1 q_1)$. But, since $l \geq 2$ and $n_l \geq 4$, we have $w_{n_1,1} \in Q_{n_1} \leq Q_{n-4} \leq R$ and $w_{n_1,1} \notin \ker(p_1 q_1)$, a contradiction. □

It remains to deal with the case $n_l = 2$. It turns out that E behaves differently upon restriction to Y_n', depending on the value of $l \geq 2$. In the following section we begin with the generic case $l \geq 4$, while the exceptional cases $l \leq 3$ will be settled in subsequent sections.

6 The case $n_l = 2$ and $l \geq 4$

Lemma 6.1. *Let $n_l = 2$ and $l \geq 4$. Then $\mathrm{Res}_{Y_n'}^{\mathfrak{A}_n}(E)$ is indecomposable.*

Proof. We fix an indecomposable direct sum decomposition of $\mathrm{Res}_{Y_n'}^{\mathfrak{A}_n}(E)$. For $j \in \{1, \ldots, l-1\}$, by Proposition 4.3, there is precisely one indecomposable direct summand V_j in this decomposition which is not annihilated by $y_{n_j}'^+$. We denote the preimage of V_j under the natural epimorphism $M' \longrightarrow M'/M''$ by $\hat{V}_j$. From $\overline{\gamma}_j^+ \in V_j$, for $j = 1, \ldots, l-1$, we by Proposition 4.2 conclude that $\mathrm{Soc}(\mathrm{Res}_{Y_n'}^{\mathfrak{A}_n}(E)) \leq V_1 + \cdots + V_{l-1}$, so that $\mathrm{Res}_{Y_n'}^{\mathfrak{A}_n}(E) = V_1 + \cdots + V_{l-1}$. Hence it suffices to show that $V_1 = \ldots = V_{l-1}$. Suppose, for a contradiction, that this is not the case. After appropriate re-numbering, we may suppose that $V_1 = \ldots = V_k$ and $V_{k+1} \neq V_1, \ldots, V_{l-1} \neq V_1$. Furthermore, since $l \geq 4$, we may also suppose that $k < l-2$.

For $i = 1, \ldots, l$, we have the FP_n-epimorphism

$$\pi_i : M \longrightarrow M_{n_i} := \langle \gamma_{n_1 + \cdots + n_{i-1}+1}, \ldots, \gamma_{n_1 + \cdots + n_i} \rangle_F,$$

and we consider $\pi := \bigoplus_{j=1}^k \pi_j \oplus \pi_l : M \longrightarrow \bigoplus_{j=1}^k M_{n_j} \oplus M_{n_l}$. For $j = 1, \ldots, l$ and $v \in M_{n_j}$, set $h_j(v) := \min\{e \in \mathbb{N}_0 \mid v \in \ker_{M_{n_j}}((y_{n_j}+1)^e)\}$; note that $h_j(v) = \min\{e \in \mathbb{N}_0 \mid v \in \ker_{M_{n_j}}((y_{n_j}'+1)^e)\}$ whenever $j \leq l-1$.

Now assume that there is some $0 \neq \tilde{v} \in \hat{V}_1 \cap \ker(\pi)$. If $h_j(\tilde{v}^{\pi_j}) \leq 1$ for all $j \in \{k+1, \ldots, l-1\}$, then $\tilde{v}^{\pi_j} = a_j \gamma_j^+$, for some $a_j \in F$, and hence $0 \neq \tilde{v} \in \hat{V}_1 \cap (\hat{V}_{k+1} + \cdots + \hat{V}_{l-1}) = \langle \gamma^+ \rangle_F$. If there is some $j \in \{k+1, \ldots, l-1\}$ such that $h_j(\tilde{v}^{\pi_j}) > 1$, then let $v := \tilde{v}(y_{n_j}'+1)^{h_j(\tilde{v}^{\pi_j})-1} \in (\hat{V}_1 \cap \ker(\pi)) \setminus \{0\}$, hence $v^{\pi_r} = 0$ for $r \in \{k+1, \ldots, l-1\} \setminus \{j\}$, and $v^{\pi_j} = a\gamma_j^+$, for some $0 \neq a \in F$, thus $0 \neq v \in \hat{V}_1 \cap \hat{V}_j = \langle \gamma^+ \rangle_F$. In both cases, $\gamma^+ \notin \ker(\pi)$ yields a contradiction. Consequently, $\pi_{|\hat{V}_1}$ is injective.

Next we consider the FP_n-epimorphism $\pi' := \bigoplus_{j=1}^k \pi_j : M \longrightarrow \bigoplus_{j=1}^k M_{n_j}$. Assume that there is some $0 \neq \tilde{v} \in \hat{V}_1 \cap \ker(\pi')$. We write $\tilde{v} = \sum_{i=1}^n a_i \delta_i$, for appropriate $a_1, \ldots, a_n \in F$. By the injectivity of $\pi_{|\hat{V}_1}$ we have $\tilde{v} \notin \ker(\pi)$, that is, $a_{n-1} \neq 0$ or $a_n \neq 0$. We set $v_0 := \sum_{j=k+1}^l \gamma_j^+ = \sum_{j=k+1}^l \delta_j^+$, then $v_0 \in \ker(\pi')$. Moreover, $v_0 = \gamma^+ + \sum_{j=1}^k \gamma_j^+$, and thus also $v_0 \in \hat{V}_1$. If $a_{n-1} = a_n \neq 0$, then we may suppose that $a_{n-1} = 1 = a_n$, and thus

$$\tilde{v} + v_0 = \sum_{j=n_1+\cdots+n_k+1}^{n_1+\cdots+n_{l-1}} (a_j+1)\delta_j \in \hat{V}_1 \cap \ker(\pi) = \{0\},$$

hence $\tilde{v} = \sum_{j=k+1}^l \gamma_j^+ = v_0$. Now assume that $a_{n-1} \neq a_n$. Let $j \in \{k+1, \ldots, l-1\}$. Then

$$0 \neq v := \tilde{v}(y_{n_j}'+1) = (\sum_{i=n_1+\cdots+n_{j-1}+2}^{n_1+\cdots+n_j} (a_i + a_{i-1})\delta_i)$$

$$+ (a_{n_1+\cdots+n_{j-1}+1} + a_{n_1+\cdots+n_j})\delta_{n_1+\cdots+n_{j-1}+1} + (a_{n-1} + a_n)(\delta_{n-1} + \delta_n) \in \hat{V}_1 \cap \ker(\pi').$$

Hence we have $v = bv_0$, for some $0 \neq b \in F$, which in turn means that $j = k+1 = l-1$. But, since $k < l-2$, this is a contradiction. Hence we actually have $a_{n-1} = a_n \neq 0$, and thus $\hat{V}_1 \cap \ker(\pi') = \langle v_0 \rangle_F$.

We now show that $\pi'_{|\hat{V}_1}$ is surjective: For $j \in \{k+1, \ldots, l-1\}$ and $i \in \{1, \ldots, k\}$ we have $\hat{V}_j^{\pi_i} \leq \mathrm{Rad}(\mathrm{Res}_{Y'_{n_i}}^{P_n}(M_{n_i}))$, since otherwise taking $w \in \hat{V}_j$ such that $w^{\pi_i} \in M_{n_i} \setminus \mathrm{Rad}(\mathrm{Res}_{Y'_{n_i}}^{P_n}(M_{n_i}))$ yields $w(y'_{n_i}+1)^{n_i-1} = a\gamma_i^+ \in \hat{V}_j$, for some $0 \neq a \in F$, a contradiction. Hence we have

$$(\hat{V}_{k+1} + \cdots + \hat{V}_{l-1})^{\pi'} \leq \bigoplus_{i=1}^{k} \mathrm{Rad}(\mathrm{Res}_{Y'_{n_i}}^{P_n}(M_{n_i})) = \mathrm{Rad}(\mathrm{Res}_{Y'_n}^{P_n}(\bigoplus_{i=1}^{k} M_{n_i})).$$

From $\hat{V}_1 + \cdots + \hat{V}_{l-1} = \mathrm{Res}_{Y'_n}^{\mathfrak{S}_n}(M')$ we get $(\hat{V}_1 + \hat{V}_{k+1} + \cdots + \hat{V}_{l-1})^{\pi'} = \mathrm{Res}_{Y'_n}^{P_n}(\bigoplus_{i=1}^{k} M_{n_i})$, hence $\hat{V}_1^{\pi'} = \mathrm{Res}_{Y'_n}^{P_n}(\bigoplus_{i=1}^{k} M_{n_i})$.

Thus there is some $w \in \hat{V}_1$ such that $w^{\pi'} = \gamma_1 = \delta_1$. We write $w = \delta_1 + \sum_{i=n_1+\cdots+n_k+1}^{n} a_i \delta_i$, for some $a_{n_1+\cdots+n_k+1}, \ldots, a_n \in F$. Furthermore, $\sum_{i=n_1+\cdots+n_k+1}^{n} a_i = 1$, since $w \in \hat{V}_1 \leq M'$. Assume there is $j \in \{k+1, \ldots, l-1\}$ such that $h_j(w^{\pi_j}) > 1$. Then let

$$\tilde{w} := w(y'_{n_j}+1) \in (\hat{V}_1 \cap \ker(\pi')) \setminus \{0\}.$$

Hence we have $\tilde{w}^{\pi_j} \neq 0$, and $\tilde{w}^{\pi_r} = 0$ for $j \neq r \in \{k+1, \ldots, l-1\}$, while $\tilde{w}^{\pi_l} = (a_{n-1} + a_n)(\delta_{n-1} + \delta_n)$. But $\hat{V}_1 \cap \ker(\pi') = \langle v_0 \rangle_F$ and our hypothesis $k < l-2$ yield a contradiction, thus we have $h_j(w^{\pi_j}) \leq 1$ for all $j \in \{k+1, \ldots, l-1\}$. In particular, $\sum_{i=n_1+\cdots+n_{j-1}+1}^{n_1+\cdots+n_j} a_i = 0$, for $j \in \{k+1, \ldots, l-1\}$. This forces $a_{n-1} + a_n = 1$, that is $h_l(w^{\pi_l}) = 2$. Thus, for any $j \in \{k+1, \ldots, l-1\}$, we get $w(y'_{n_j}+1) = (a_{n-1} + a_n)(\delta_{n-1} + \delta_n) \in (\hat{V}_1 \cap \ker(\pi')) \setminus \{0\}$, contradicting $\hat{V}_1 \cap \ker(\pi') = \langle v_0 \rangle_F$. $\qquad\square$

Proposition 6.2. *Let $n_l = 2$ and $l \geq 4$. Then E has vertex Q_n and source $\mathrm{Res}_{Q_n}^{\mathfrak{A}_n}(E)$.*

Proof. We follow the strategy given in Remark 1.3. By Lemma 6.1, we already know that $\mathrm{Res}_{Q_n}^{\mathfrak{A}_n}(E)$ is indecomposable. Assume that $R < Q_n$ is a maximal subgroup such that E is relatively R-projective, hence $X_n = \prod_{j=1}^{l} X_{n_j} \leq \Phi(Q_n) \leq R$ and $Q_{n-4} \leq R$. We fix an indecomposable direct sum decomposition of $\mathrm{Res}_R^{\mathfrak{A}_n}(E)$. By Remark 4.4, part (a), there is either exactly one summand U or there are exactly two summands U' and U'' in this decomposition which are not annihilated by $x_{n_1}^+$. In the first case $(FX_{n_1} \oplus FX_{n_1}) | \mathrm{Res}_{X_{n_1}}^R(U)$ implies $\dim(U) \geq n_1 > \dim(E)/2$, a contradiction. In the second case, Remark 4.4, part (b), implies that $R \leq \ker(p_1 q_1)$. But since $l \geq 3$ and $n_l = 2$ we have $Q_{n-4} = (P_{n_1} \times \cdots \times P_{n_{l-2}} \times P_{2^{i_{l-1}}-1} \times P_{2^{i_{l-1}}-2} \times \cdots \times P_4 \times P_2) \cap Q_n$, thus $w_{n_1,1} \in Q_{n_1} \leq Q_{n-4} \leq R$, a contradiction. $\quad\square$

7 The case $n_l = 2$ and $l = 3$

Lemma 7.1. *Let $n_l = 2$ and $l = 3$. Then $\mathrm{Res}_{Y'_n}^{\mathfrak{A}_n}(E) = U_1 \oplus U_2$ where U_1 and U_2 are indecomposable of dimension n_1 and n_2, respectively, and both have vertex Y'_n.*

Proof. We construct the following subspaces M_1 and M_2 of M': M_1 has F-basis $\mathfrak{B}_1 := \{b_1, \ldots, b_{n_1}\}$ with

$$b_1 := \delta_1 + \delta_{n-1} + \sum_{i=0}^{n_2/2-1} \delta_{n_1+1+2i} \quad \text{and} \quad b_j := \delta_j + \delta_{j-1} + \delta_{n-1} + \delta_n,$$

for $j = 2, \ldots, n_1$. Obviously, $\mathfrak{B}_1$ is indeed linearly independent. Moreover, $M_1 + \langle \delta^+ \rangle_F$ is an FY'_n-module, since

$$b_j y'_{n_1} = \begin{cases} b_1 + b_2, & \text{for } j = 1, \\ b_{j+1}, & \text{for } j = 2, \ldots, n_1 - 1, \\ \sum_{i=2}^{n_1} b_i, & \text{for } j = n_1, \end{cases} \quad b_j y'_{n_2} = \begin{cases} b_1 + \sum_{i=1}^{n_1/2} b_{2i} + \delta^+, & \text{for } j = 1, \\ b_j, & \text{for } j = 2, \ldots, n_1. \end{cases}$$

Similarly, we define the subspace M_2 of M' with basis $\mathfrak{B}_2 := \{\tilde{b}_1, \ldots, \tilde{b}_{n_2}\}$ where

$$\tilde{b}_1 := \delta_{n_1+1} + \delta_{n-1} + \sum_{i=0}^{n_1/2-1} \delta_{1+2i} \quad \text{and} \quad \tilde{b}_j := \delta_{n_1+j} + \delta_{n_1+j-1} + \delta_{n-1} + \delta_n,$$

for $j = 2, \ldots, n_2$. Also $M_2 + \langle \delta^+ \rangle_F$ is an FY'_n-module and, by construction, $M_1 + M_2 + \langle \delta^+ \rangle_F$ is an FY'_n-submodule of $\mathrm{Res}_{Y'_n}^{\mathfrak{S}_n}(M')$. We show that $M_1 + M_2 + \langle \delta^+ \rangle_F$ contains an F-basis of M', implying that we actually have $M_1 + M_2 + \langle \delta^+ \rangle_F = \mathrm{Res}_{Y'_n}^{\mathfrak{S}_n}(M')$: First of all, we have $\delta_1^+ = \sum_{i=1}^{n_1/2} b_{2i} \in M_1$ and $\delta_2^+ = \sum_{i=1}^{n_2/2} \tilde{b}_{2i} \in M_2$. Thus $\delta_{n-1} + \delta_n = \delta_1^+ + \delta_2^+ + \delta^+ \in M_1 + M_2 + \langle \delta^+ \rangle_F$, and

$$\delta_i + \delta_{i+1} = \begin{cases} b_{i+1} + \delta_{n-1} + \delta_n \in M_1 + M_2 + \langle \delta^+ \rangle_F, & \text{for } i = 1, \ldots, n_1 - 1, \\ \tilde{b}_{i+1} + \delta_{n-1} + \delta_n \in M_1 + M_2 + \langle \delta^+ \rangle_F, & \text{for } i = n_1 + 1, \ldots, n_1 + n_2 - 1. \end{cases}$$

Hence we have $s_1 := \sum_{i=0}^{n_2/2-1} \delta_{n_1+1+2i} \in M_1 + M_2 + \langle \delta^+ \rangle_F$ and $s_2 := \sum_{i=0}^{n_1/2-1} \delta_{1+2i} \in M_1 + M_2 + \langle \delta^+ \rangle_F$, thus we get $\delta_1 + \delta_{n-1} = b_1 + s_1 \in M_1 + M_2 + \langle \delta^+ \rangle_F$ and $\delta_{n_1+1} + \delta_{n-1} = \tilde{b}_1 + s_2 \in M_1 + M_2 + \langle \delta^+ \rangle_F$, therefore,

$$\mathrm{Res}_{Y'_n}^{\mathfrak{A}_n}(E) = \mathrm{Res}_{Y'_n}^{\mathfrak{S}_n}(M'/M'') = (M_1 + M''/M'') + (M_2 + M''/M'').$$

Letting $U_i := M_i + M''/M''$, we from $\dim(E) = n - 2 = n_1 + n_2 = \dim(M_1) + \dim(M_2)$ get $U_i \cong M_i$, for $i \in \{1, 2\}$, and hence $\mathrm{Res}_{Y'_n}^{\mathfrak{A}_n}(E) = U_1 \oplus U_2$. For $j = 1, \ldots, n_1$ we set $\bar{b}_j := b_j + M''$, and for $j = 1, \ldots, n_2$ we set $\bar{\tilde{b}}_j := \tilde{b}_j + M''$. Then $\overline{\mathfrak{B}}_1 := \{\bar{b}_1, \ldots, \bar{b}_{n_1}\}$ and $\overline{\mathfrak{B}}_2 := \{\bar{\tilde{b}}_1, \ldots, \bar{\tilde{b}}_{n_2}\}$ are bases for U_1 and U_2, respectively. Furthermore, both U_1 and U_2 are indecomposable. Namely, $b_1 y'^+_{n_1} = \delta_1^+ = \sum_{i=1}^{n_1/2} b_{2i} \neq 0$ so that M_1 is not annihilated by $y'^+_{n_1}$. In particular, $FY'_{n_1} \mid \mathrm{Res}_{Y'_{n_1}}^{Y'_n}(M_1)$. Comparing dimensions, we deduce $FY'_{n_1} \cong \mathrm{Res}_{Y'_{n_1}}^{Y'_n}(M_1)$. In particular, $\mathrm{Res}_{Y'_{n_1}}^{Y'_n}(M_1)$ and thus also $M_1 \cong U_1$ is uniserial, hence indecomposable. The indecomposability of U_2 is proved analogously.

Next we show that Y'_n is a vertex of both U_1 and U_2. For this, notice that Y'_n possesses precisely three maximal subgroups, these are $Z_1 := \langle y'_{n_1}, (y'_{n_2})^2 \rangle$, $Z_2 := \langle y'_{n_1} y'_{n_2}, (y'_{n_2})^2 \rangle$ and $Z_3 := \langle (y'_{n_1})^2, y'_{n_2} \rangle$. We show that neither U_1 nor U_2 can be relatively Z_i-projective, for

$i = 1, 2, 3$. By Green's Indecomposability Theorem, it suffices to verify that U_1 and U_2 restrict indecomposably to each of these groups. We investigate U_1 first. By definition, $(y'_{n_2})^2$ acts trivially on U_1. That is, we may view U_1 as a module for the factor group $\overline{Y}'_n := Y'_n / \langle (y'_{n_2})^2 \rangle$ and show that it restricts indecomposably to each of the maximal subgroups of $\overline{Y}'_n$. The latter are in natural bijection with the maximal subgroups of Y'_n, thus are $\overline{Z}_1 = \langle \overline{y}'_{n_1} \rangle$, $\overline{Z}_2 = \langle \overline{y}'_{n_1} \overline{y}'_{n_2} \rangle$ and $\overline{Z}_3 = \langle (\overline{y}'_{n_1})^2, \overline{y}'_{n_2} \rangle$, where $^- : Y'_n \longrightarrow \overline{Y}'_n$ denotes the natural epimorphism. As we have already mentioned, $\mathrm{Res}^{Y'_n}_{Y'_{n_1}}(U_1) \cong FY'_{n_1}$. Hence also $\mathrm{Res}^{\overline{Y}'_n}_{\overline{Z}_1}(U_1) \cong F\overline{Z}_1$ which is indecomposable.

We consider $\mathrm{Res}^{\overline{Y}'_n}_{\overline{Z}_2}(U_1)$ and $\mathrm{Res}^{\overline{Y}'_n}_{\overline{Z}_3}(U_1)$. For convenience, we replace the F-basis $\overline{\mathfrak{B}}_1 = \{\overline{b}_1, \ldots, \overline{b}_{n_1}\}$ of U_1 by $\overline{\mathfrak{B}}'_1 := \{\overline{b}'_1, \ldots, \overline{b}'_{n_1}\}$ where $\overline{b}'_1 := \overline{b}_1$, and $\overline{b}'_j := \overline{b}'_{j-1} + \overline{b}_j$, for $j = 2, \ldots, n_1$. In other words,

$$\overline{b}'_j = \overline{\delta}_j + \sum_{i=0}^{n_2/2-1} \overline{\delta}_{n_1+1+2i} + \begin{cases} \overline{\delta}_{n-1}, & \text{if } j \text{ is odd,} \\ \overline{\delta}_n, & \text{if } j \text{ is even.} \end{cases}$$

With this notation, we get

$$\overline{b}'_j \overline{y}'_{n_1} = \overline{b}'_j y'_{n_1} = \overline{b}'_{j+1}, \text{ for } j = 1, \ldots, n_1 - 1,$$
$$\overline{b}'_{n_1} \overline{y}'_{n_1} = \overline{b}'_{n_1} y'_{n_1} = \overline{b}'_1,$$
$$\overline{b}'_j \overline{y}'_{n_2} = \overline{b}'_j y'_{n_2} = \sum_{i \neq j} \overline{b}'_i, \text{ for } j = 1, \ldots, n_1.$$

We set $\overline{Y}''_{n_1} := \langle \overline{y}'^2_{n_1} \rangle$, and notice that $\overline{Y}''_{n_1} \leq \overline{Z}_2$ and $\overline{Y}''_{n_1} \leq \overline{Z}_3$. Since $\mathrm{Res}^{\overline{Y}'_n}_{\overline{Y}'_{n_1}}(U_1) \cong F\overline{Y}'_{n_1}$, we also have $\mathrm{Res}^{\overline{Y}'_n}_{\overline{Y}''_{n_1}}(U_1) \cong F\overline{Y}''_{n_1} \oplus F\overline{Y}''_{n_1}$. More precisely, $\mathrm{Res}^{\overline{Y}'_n}_{\overline{Y}''_{n_1}}(U_1) = V_1 \oplus V_2$ where $V_1 := \langle \overline{b}'_i \mid i \text{ odd} \rangle_F$ and $V_2 := \langle \overline{b}'_i \mid i \text{ even} \rangle_F$, and the maps $V_1 \longrightarrow F\overline{Y}''_{n_1}$, $\overline{b}'_{2i-1} \longmapsto \overline{y}'^i_{n_1}$ and $V_2 \longrightarrow F\overline{Y}''_{n_1}$, $\overline{b}'_{2i} \longmapsto \overline{y}'^i_{n_1}$, for $i = 1, \ldots, n_1/2$, are isomorphisms of $F\overline{Y}''_{n_1}$-modules. Consequently,

$$\mathcal{E} := \mathrm{End}_{F\overline{Y}''_{n_1}}(\mathrm{Res}^{\overline{Y}'_n}_{\overline{Y}''_{n_1}}(U_1)) \cong \mathrm{Mat}(2, F\overline{Y}''_{n_1}),$$

and we will from now on simply identify $\mathcal{E}$ and $\mathrm{Mat}(2, F\overline{Y}''_{n_1})$. Moreover, $\mathcal{E}$ is a $\overline{Y}'_n$-algebra with respect to the conjugation action induced by the natural embedding $F\overline{Y}'_n \longrightarrow \mathcal{E}$. Via this embedding, the elements $\overline{y}'^2_{n_1}$ and $\overline{y}'_{n_1}$ correspond to the endomorphisms

$$\begin{pmatrix} \overline{y}'^2_{n_1} & 0 \\ 0 & \overline{y}'^2_{n_1} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ \overline{y}'^2_{n_1} & 0 \end{pmatrix},$$

respectively. Now $\overline{y}'_{n_2}$ acts on U_1 as $1 + (\overline{y}'_{n_1})^+ = 1 + (\overline{y}'^2_{n_1})^+ + \overline{y}'_{n_1} \cdot (\overline{y}'^2_{n_1})^+$. Letting $s := (\overline{y}'^2_{n_1})^+ \in \mathrm{Soc}(F\overline{Y}''_{n_1})$, the elements $\overline{y}'_{n_2}$ and $\overline{y}'_{n_1} \overline{y}'_{n_2}$ correspond to the endomorphisms

$$\psi := \begin{pmatrix} 1+s & s \\ s & 1+s \end{pmatrix} \quad \text{and} \quad \psi' := \begin{pmatrix} s & 1+s \\ \overline{y}'^2_{n_1} + s & s \end{pmatrix},$$

respectively.

We now determine the elements in $\mathcal{E}$ which are fixed by $\overline{y}'_{n_2}$. These are precisely the elements in $\operatorname{End}_{F\overline{Z}_3}(U_1)$. Let $\varphi \in \mathcal{E}$ such that $\varphi = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, for some $a, b, c, d \in F\overline{Y}''_{n_1}$. Then $\varphi \in \operatorname{End}_{F\overline{Z}_3}(U_1)$ if and only if $\varphi\psi = \psi\varphi$, or equivalently, if $(a + d)s = 0 = (b + c)s$. That is,

$$\mathcal{F} := \operatorname{End}_{F\overline{Z}_3}(U_1) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \operatorname{Mat}(2, F\overline{Y}''_{n_1}) \mid (a + d)s = 0 = (b + c)s \right\}.$$

Obviously, $I := \mathcal{F} \cap \operatorname{Rad}(\operatorname{Mat}(2, F\overline{Y}''_{n_1}))$ is a nilpotent ideal in $\mathcal{F}$ such that

$$\begin{aligned}
\mathcal{F}/I &= \mathcal{F}/\mathcal{F} \cap \operatorname{Rad}(\operatorname{Mat}(2, F\overline{Y}''_{n_1})) \cong \mathcal{F} + \operatorname{Rad}(\operatorname{Mat}(2, F\overline{Y}''_{n_1}))/\operatorname{Rad}(\operatorname{Mat}(2, F\overline{Y}''_{n_1})) \\
&\subseteq \operatorname{Mat}(2, F\overline{Y}''_{n_1})/\operatorname{Rad}(\operatorname{Mat}(2, F\overline{Y}''_{n_1})).
\end{aligned}$$

Since $F\overline{Y}''_{n_1}$ is a local F-algebra, there is an isomorphism

$$\operatorname{Mat}(2, F\overline{Y}''_{n_1})/\operatorname{Rad}(\operatorname{Mat}(2, F\overline{Y}''_{n_1})) \longrightarrow \operatorname{Mat}(2, F).$$

Moreover, since the annihilator of s in $F\overline{Y}''_{n_1}$ coincides with $\operatorname{Rad}(F\overline{Y}''_{n_1})$, the above isomorphism maps the algebra $\mathcal{F}/I$ onto

$$\left\{ \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix} \in \operatorname{Mat}(2, F) \mid \bar{a} = \bar{d}, \bar{b} = \bar{c} \right\} \cong FC_2.$$

In consequence, $\mathcal{F}/I$ and thus also $\mathcal{F} = \operatorname{End}_{F\overline{Z}_3}(U_1)$ is a local F-algebra, and $\operatorname{Res}_{\overline{Z}_3}^{\overline{Y}'_n}(U_1)$ is indecomposable.

Similarly, $\varphi \in \operatorname{End}_{F\overline{Z}_2}(U_1)$ if and only if $\varphi\psi' = \psi'\varphi$, or equivalently, if $(a + d)(s + 1) = (a + d)(s + \overline{y}'^2_{n_1}) = 0$ and $c(s + 1) = b(s + \overline{y}'^2_{n_1})$. Since both $s + 1, s + \overline{y}'^2_{n_1} \in F\overline{Y}''_{n_1}$ are units, we get

$$\mathcal{F}' := \operatorname{End}_{F\overline{Z}_2}(U_1) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \operatorname{Mat}(2, F\overline{Y}''_{n_1}) \mid a = d, c = b\overline{y}'^2_{n_1} \right\}.$$

Let $0 \neq e = a \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b \cdot \begin{pmatrix} 0 & 1 \\ \overline{y}'^2_{n_1} & 0 \end{pmatrix} \in \mathcal{F}'$, for some $a, b \in F\overline{Y}''_{n_1}$, be an idempotent. Then

$$a \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b \cdot \begin{pmatrix} 0 & 1 \\ \overline{y}'^2_{n_1} & 0 \end{pmatrix} = e = e^2 = a^2 \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b^2 \cdot \begin{pmatrix} \overline{y}'^2_{n_1} & 0 \\ 0 & \overline{y}'^2_{n_1} \end{pmatrix}$$

shows that $b = 0$ and $a^2 = a \neq 0$ is an idempotent in $F\overline{Y}''_{n_1}$. Since $F\overline{Y}''_{n_1}$ is a local F-algebra we conclude that $a = 1$, thus we have $e = 1$, implying that $\mathcal{F}' = \operatorname{End}_{F\overline{Z}_2}(U_1)$ is a local F-algebra as well, and $\operatorname{Res}_{\overline{Z}_2}^{\overline{Y}'_n}(U_1)$ is indecomposable.

Therefore, we have now shown that U_1 is an indecomposable $F\overline{Y}'_n$-module with vertex $\overline{Y}'_n$. This in turn implies that U_1, as FY'_n-module, is also indecomposable with vertex Y'_n (cf. [12]). Replacing y'_{n_1} by y'_{n_2}, we deduce that also U_2 has vertex Y'_n. $\qquad\square$

Proposition 7.2. *Let $n_l = 2$ and $l = 3$. Then E has vertex Q_n and source $\mathrm{Res}^{\mathfrak{A}_n}_{Q_n}(E)$.*

Proof. We first show that $\mathrm{Res}^{\mathfrak{A}_n}_{Q_n}(E)$ is indecomposable. For this, fix an indecomposable direct sum decomposition of $\mathrm{Res}^{\mathfrak{A}_n}_{Q_n}(E)$. For $j \in \{1, 2\}$, by Proposition 4.3, there is precisely one indecomposable direct summand V_j in this decomposition which is not annihilated by $y'^+_{n_j}$. We denote the preimage of V_j under the natural epimorphism $M' \longrightarrow M'/M''$ by $\hat{V}_j$. From $\overline{\gamma}^+_j \in V_j$ we, by Proposition 4.2, conclude that $\mathrm{Soc}(\mathrm{Res}^{\mathfrak{A}_n}_{Y'_n}(E)) \leq V_1 + V_2$, so that $\mathrm{Res}^{\mathfrak{A}_n}_{Y'_n}(E) = V_1 + V_2$. Hence it suffices to show that $V_1 = V_2$.

Assume that $V_1 \neq V_2$. We first conclude from Lemma 7.1 that $\mathrm{Res}^{\mathfrak{A}_n}_{Q_n}(E) = V_1 \oplus V_2$, where $\dim(V_j) = n_j$. As in the proof of Lemma 6.1, for $j = 1, 2, 3$, we use the FP_n-epimorphism $\pi_j : M \longrightarrow M_{n_j}$. We show that $(\pi_1)_{|\hat{V}_1}$ is surjective: We have $\hat{V}^{\pi_1}_2 \leq \mathrm{Rad}(\mathrm{Res}^{P_n}_{Y'_{n_1}}(M_{n_1}))$, since otherwise taking $w \in \hat{V}_2$ such that $w^{\pi_1} \in M_{n_1} \setminus \mathrm{Rad}(\mathrm{Res}^{P_n}_{Y'_{n_1}}(M_{n_1}))$ yields $w(y'_{n_1} + 1)^{n_1 - 1} = a\gamma^+_1 \in \hat{V}_2$, for some $0 \neq a \in F$, a contradiction. From $\hat{V}_1 + \hat{V}_2 = M'$ we get $(\hat{V}_1 + \hat{V}_2)^{\pi_1} = M_{n_1}$, thus $\hat{V}^{\pi_1}_1 = M_{n_1}$. Hence from $\dim(\hat{V}_1) = n_1 + 1 = \dim(M_{n_1}) + 1$ we get $\dim(\hat{V}_1 \cap \ker(\pi_1)) = 1$. We set $v_0 := \gamma^+_2 + \gamma^+_3$. Then clearly $v_0 \in \ker(\pi_1)$ and, since $v_0 = \gamma^+ + \gamma^+_1$, we also have $v_0 \in \hat{V}_1$, thus $\hat{V}_1 \cap \ker(\pi_1) = \langle v_0 \rangle_F$.

Now there is some $w \in \hat{V}_1$ such that $w^{\pi_1} = \gamma_1 = \delta_1$. We write $w = \delta_1 + \sum^n_{i=n_1+1} a_i \delta_i$ with appropriate $a_{n_1+1}, \ldots, a_n \in F$. Then we have

$$\tilde{w} := w(y'_{n_2} + 1) = \sum^{n_1+n_2}_{i=n_1+2} (a_i + a_{i-1})\delta_i + (a_{n_1+n_2} + a_{n_1+1})\delta_{n_1+1}$$
$$+ (a_{n-1} + a_n)(\delta_{n-1} + \delta_n) \in \hat{V}_1 \cap \ker(\pi_1).$$

Since $w \in M'$, we have $\tilde{w} \neq 0$, hence we get $\tilde{w} = av_0$, for some $0 \neq a \in F$, where we may suppose that $a = 1$. Thus we have $a_{i+1} = a_i + 1$ for all $i \in \{n_1 + 1, \ldots, n - 3\}$ and $a_n = a_{n-1} + 1$. Hence adding a suitable multiple of v_0 we may assume that $w = \delta_1 + (b + 1)\delta_{n-1} + b\delta_n + \sum^{n_2/2-1}_{i=0} \delta_{n_1+1+2i}$, for some $b \in F$. Since $\hat{V}_1$ is an FQ_n-module, we also have

$$ww_{n_2,2} = \delta_1 + (b+1)\delta_{n-1} + b\delta_n + \sum^{n_2/2}_{i=1} \delta_{n_1+2i} = w + \gamma^+_2 \in \hat{V}_1,$$

implying $\gamma^+_2 \in \hat{V}_1 \cap \ker(\pi_1)$, a contradiction. Hence we have $V_1 = V_2$.

It remains to show that Q_n is a vertex of E. We follow the strategy given in Remark 1.3, and assume that E is relatively R-projective, for some maximal subgroup $R < Q_n$. By Lemma 7.1 we have $\mathrm{Res}^{Q_n}_{Y'_n}(V_1) = U_1 \oplus U_2$ where the U_i are indecomposable with vertex Y'_n. Since $\mathrm{Res}^{\mathfrak{A}_n}_{Q_n}(E) = V_1$ is indecomposable, we infer that $Y'_n \leq R$. Thus, again by Lemma 7.1, $\mathrm{Res}^{Q_n}_R(V_1)$ is either indecomposable or the direct sum of an indecomposable module of dimension n_1 and an indecomposable module of dimension $n_2 \neq n_1$, a contradiction. $\square$

8 The case $n_l = 2$ and $l = 2$

In order to complete the proof of Theorem 1.1, we are now left with the case where $l = 2 = n_l$ which is treated in the following.

Proposition 8.1. *Let $n_l = 2$ and $l = 2$. If $n > 6$ then E has vertex Q_n and source $\operatorname{Res}^{\mathfrak{A}_n}_{Q_n}(E)$.*

Proof. We follow the strategy given in Remark 1.3, and assume that E is relatively R-projective for some maximal subgroup $R < Q_n$. Hence we have $\langle \Phi(Q_n), Q_{n-4} \rangle \leq R$ and, in particular, $X_{n_1} = X_n \leq \Phi(Q_n) \leq R$. Using $\Phi(Q_n) = \Phi(P_n) = \Phi(P_{n_1})$ and $Q_{\frac{n_1}{2}} \leq Q_{n-4}$, Proposition 2.6 yields $B'_{n_1} = Q_{\frac{n_1}{2}}\Phi(P_{n_1}) \leq Q_{n-4}\Phi(Q_n) \leq R$. Since Q_n/B'_{n_1} is elementary abelian of order 4, there are precisely three maximal subgroups of Q_n containing $B'_{n_1} = B'_n$:

$$R'_1 := B'_n\langle y'_{n_1}\rangle, \quad R'_2 := B'_n\langle w_{n_1,1}\rangle = Q_{n_1}, \quad R'_3 := B'_n\langle w_{2,1}w_{2,2}\rangle = (B_n \times P_2) \cap Q_n.$$

We next show that E restricts indecomposably to R'_1 and R'_2: By Remark 4.4, part (a), and using $\dim(E) = n_1 = 2\dim(FX_{n_1})$ we get $\operatorname{Res}^{\mathfrak{A}_n}_{X_{n_1}}(E) \cong FX_{n_1} \oplus FX_{n_1}$. Now let $i \in \{1,2\}$, fix an indecomposable direct sum decomposition of $\operatorname{Res}^{\mathfrak{A}_n}_{R'_i}(E)$, and assume that $\operatorname{Res}^{\mathfrak{A}_n}_{R'_i}(E)$ is decomposable. Hence $\operatorname{Res}^{\mathfrak{A}_n}_{R'_i}(E)$ actually has two indecomposable summands, both of which are not annihilated by $x^+_{n_1}$. But, since $R'^{p_1q_1}_i = \langle w_{n_1,1}\rangle$, Remark 4.4, part (b), implies that there is precisely one such summand, a contradiction. Hence from this we also conclude that $\operatorname{Res}^{\mathfrak{A}_n}_{Q_n}(E)$ is indecomposable.

It remains to show that E is not relatively R'_3-projective. We claim that it suffices to show that E restricts indecomposably to $H' := (\mathfrak{S}_{\frac{n_1}{2}} \times \mathfrak{S}_{\frac{n_1}{2}} \times \mathfrak{S}_2) \cap \mathfrak{A}_n$. Namely, then E also restricts indecomposably to $\hat{H}' := ((\mathfrak{S}_{\frac{n_1}{2}} \wr \mathfrak{S}_2) \times \mathfrak{S}_2) \cap \mathfrak{A}_n$. Of course E is relatively $\hat{H}'$-projective, hence E and $\operatorname{Res}^{\mathfrak{A}_n}_{\hat{H}'}(E)$ have a common vertex. If E were relatively R'_3-projective then $\operatorname{Res}^{\mathfrak{A}_n}_{\hat{H}'}(E)$ would be relatively H'-projective, a contradiction since $|\hat{H}' : H'| = 2$.

Therefore, we now show that $\operatorname{End}_{FH'}(E)$ is a local F-algebra so that $\operatorname{Res}^{\mathfrak{A}_n}_{H'}(E)$ is indecomposable. Using the F-basis $\{\overline{\gamma}_1 + \overline{\gamma}_n, \ldots, \overline{\gamma}_{n_1} + \overline{\gamma}_n\}$ of D, we get

$$\operatorname{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_{\frac{n_1}{2}} \times \mathfrak{S}_{\frac{n_1}{2}}}(D) = M_1 \oplus M_2 \cong (M^{(\frac{n_1}{2}-1,1)} \boxtimes F) \oplus (F \boxtimes M^{(\frac{n_1}{2}-1,1)}).$$

Here $M^{(\frac{n_1}{2}-1,1)} \boxtimes F$ denotes the outer tensor product of the $F\mathfrak{S}_{\frac{n_1}{2}}$-modules $M^{(\frac{n_1}{2}-1,1)}$ and F. Furthermore, M_1 has F-basis $\{\overline{\gamma}_1 + \overline{\gamma}_n, \ldots, \overline{\gamma}_{\frac{n_1}{2}} + \overline{\gamma}_n\}$, and M_2 has F-basis $\{\overline{\gamma}_{\frac{n_1}{2}+1} + \overline{\gamma}_n, \ldots, \overline{\gamma}_{n_1} + \overline{\gamma}_n\}$. Both modules are indecomposable and uniserial with descending composition factors

$$(F, D^{(\frac{n_1}{2}-1,1)} \boxtimes F, F) \quad \text{and} \quad (F, F \boxtimes D^{(\frac{n_1}{2}-1,1)}, F),$$

respectively. Note that this also holds for $n = 6$, that is $n_1 = 4$, if we just let $D^{(1^2)} := \{0\}$. Now, since $n > 6$, we have $n_1 \geq 8$. Consequently $\operatorname{Res}^{\mathfrak{S}_n}_{\mathfrak{A}_{\frac{n_1}{2}} \times \mathfrak{A}_{\frac{n_1}{2}}}(D) = \tilde{M}_1 \oplus \tilde{M}_2$, where $\tilde{M}_1$ and $\tilde{M}_2$, for $n_1 > 8$, are uniserial with descending composition factors

$$(F, E_0^{(\frac{n_1}{2}-1,1)} \boxtimes F, F) \quad \text{and} \quad (F, F \boxtimes E_0^{(\frac{n_1}{2}-1,1)}, F),$$

respectively. For $n_1 = 8$ we have $\operatorname{Soc}(\tilde{M}_1) \cong \tilde{M}_1/\operatorname{Rad}(\tilde{M}_1) \cong F$ and $\operatorname{Rad}(\tilde{M}_1)/\operatorname{Soc}(\tilde{M}_1) \cong (E_+^{(3,1)} \oplus E_-^{(3,1)}) \boxtimes F$ as well as $\operatorname{Soc}(\tilde{M}_2) \cong \tilde{M}_2/\operatorname{Rad}(\tilde{M}_2) \cong F$ and $\operatorname{Rad}(\tilde{M}_2)/\operatorname{Soc}(\tilde{M}_2) \cong F \boxtimes (E_+^{(3,1)} \oplus E_-^{(3,1)})$. Thus

$$\mathcal{E} := \operatorname{End}_{F[\mathfrak{S}_{\frac{n_1}{2}} \times \mathfrak{S}_{\frac{n_1}{2}}]}(D) = \operatorname{End}_{F[\mathfrak{A}_{\frac{n_1}{2}} \times \mathfrak{A}_{\frac{n_1}{2}}]}(D)$$

has dimension 6, and F-basis $\{\varphi_1, \ldots, \varphi_6\}$ where φ_1 and φ_2 are the projections onto $\tilde{M}_1$ and $\tilde{M}_2$, respectively, φ_3 annihilates $\tilde{M}_2$ and maps $\tilde{M}_1$ onto $\mathrm{Soc}(\tilde{M}_1)$, φ_4 annihilates $\tilde{M}_1$ and maps $\tilde{M}_2$ onto $\mathrm{Soc}(\tilde{M}_2)$, φ_5 annihilates $\tilde{M}_2$ and maps $\tilde{M}_1$ onto $\mathrm{Soc}(\tilde{M}_2)$, φ_6 annihilates $\tilde{M}_1$ and maps $\tilde{M}_2$ onto $\mathrm{Soc}(\tilde{M}_1)$. That is, for $a_1, \ldots, a_{n_1} \in F$ and $v := \sum_{i=1}^{n_1} a_i(\overline{\gamma}_i + \overline{\gamma}_n) \in D$, we may suppose that

$$
\begin{aligned}
v^{\varphi_1} &= a_1(\overline{\gamma}_1 + \overline{\gamma}_n) + \cdots + a_{\frac{n_1}{2}}(\overline{\gamma}_{\frac{n_1}{2}} + \overline{\gamma}_n), \\
v^{\varphi_2} &= a_{\frac{n_1}{2}+1}(\overline{\gamma}_{\frac{n_1}{2}+1} + \overline{\gamma}_n) + \cdots + a_{n_1}(\overline{\gamma}_{n_1} + \overline{\gamma}_n), \\
v^{\varphi_3} &= (a_1 + \cdots + a_{\frac{n_1}{2}})((\overline{\gamma}_1 + \overline{\gamma}_n) + \cdots + (\overline{\gamma}_{\frac{n_1}{2}} + \overline{\gamma}_n)), \\
v^{\varphi_4} &= (a_{\frac{n_1}{2}+1} + \cdots + a_{n_1})((\overline{\gamma}_{\frac{n_1}{2}+1} + \overline{\gamma}_n) + \cdots + (\overline{\gamma}_{n_1} + \overline{\gamma}_n)), \\
v^{\varphi_5} &= (a_1 + \cdots + a_{\frac{n_1}{2}})((\overline{\gamma}_{\frac{n_1}{2}+1} + \overline{\gamma}_n) + \cdots + (\overline{\gamma}_{n_1} + \overline{\gamma}_n)), \\
v^{\varphi_6} &= (a_{\frac{n_1}{2}+1} + \cdots + a_{n_1})((\overline{\gamma}_1 + \overline{\gamma}_n) + \cdots + (\overline{\gamma}_{\frac{n_1}{2}} + \overline{\gamma}_n)).
\end{aligned}
$$

The multiplication in $\mathcal{E}$ is given by

$$
\begin{array}{lll}
\varphi_1^2 = \varphi_1, & \varphi_1\varphi_3 = \varphi_3 = \varphi_3\varphi_1, & \varphi_6\varphi_1 = \varphi_6 = \varphi_2\varphi_6, \\
\varphi_2^2 = \varphi_2, & \varphi_2\varphi_4 = \varphi_4 = \varphi_4\varphi_2, & \varphi_5\varphi_2 = \varphi_5 = \varphi_1\varphi_5,
\end{array}
$$

and any other product of two basis elements vanishes. Note that $H := \mathfrak{S}_{\frac{n_1}{2}} \times \mathfrak{S}_{\frac{n_1}{2}} \times \mathfrak{S}_2 = (\mathfrak{S}_{\frac{n_1}{2}} \times \mathfrak{S}_{\frac{n_1}{2}})\langle(n-1,n)\rangle$ and $H' = (\mathfrak{A}_{\frac{n_1}{2}} \times \mathfrak{A}_{\frac{n_1}{2}})\langle(1,2)(\frac{n_1}{2}+1, \frac{n_1}{2}+2), (1,2)(n-1,n)\rangle$. The algebra $\mathcal{E}$ carries an H-algebra structure with respect to the conjugation action. We thus deduce that $\tilde{\mathcal{E}} := \mathrm{End}_{FH'}(E) = \mathrm{End}_{FH}(D)$, and $\tilde{\mathcal{E}}$ consists of those $\varphi \in \mathcal{E}$ which are fixed under $(n-1,n)$. Let now $v \in D$ be as above, and let $\varphi := b_1\varphi_1 + \cdots + b_6\varphi_6 \in \mathcal{E}$ for some $b_1, \ldots, b_6 \in F$. Then

$$
v(n-1,n) = \sum_{i=1}^{n_1}(\sum_{j \neq i} a_j)(\overline{\gamma}_i + \overline{\gamma}_n) = v^{\varphi_1 + \cdots + \varphi_6}.
$$

Hence $\varphi \in \tilde{\mathcal{E}}$ if and only if $\varphi(\varphi_1 + \cdots + \varphi_6) = (\varphi_1 + \cdots + \varphi_6)\varphi$, or equivalently, if $b_1 = b_2$. We have thus shown that $\tilde{\mathcal{E}}$ has dimension 5 and F-basis $\{\varphi_1 + \varphi_2, \varphi_3, \varphi_4, \varphi_5, \varphi_6\}$. Since $\tilde{\mathcal{E}}$ is abelian and $\varphi_3, \varphi_4, \varphi_5, \varphi_6$ are nilpotent, we also deduce that $\mathrm{Rad}(\tilde{\mathcal{E}})$ has dimension 4. In particular, $\tilde{\mathcal{E}}$ is local, and the assertion follows. $\qquad\square$

Remark 8.2. It remains to consider the case $n = 6$. Let $E := E_0^{(5,1)}$. In view of the above observations, we aim to show that E has vertex $Q := (\mathfrak{S}_2 \times \mathfrak{S}_2 \times \mathfrak{S}_2) \cap \mathfrak{A}_6 = \langle(1,2)(3,4),(3,4)(5,6)\rangle \cong V_4$ and sources of dimension 2: To this end, let $Q < Q_6 := \langle(1,3)(2,4),(1,2)(3,4),(3,4)(5,6)\rangle < \mathfrak{A}_6$, where Q_6 is a dihedral group of order 8. With respect to the basis $\{\overline{\gamma}_1 + \overline{\gamma}_6, \overline{\gamma}_2 + \overline{\gamma}_6, \overline{\gamma}_3 + \overline{\gamma}_6, \overline{\gamma}_4 + \overline{\gamma}_6\}$, the action of Q_6 on E is given via the following matrices:

$$
(1,3)(2,4) \longleftrightarrow \begin{pmatrix} . & . & 1 & . \\ . & . & . & 1 \\ 1 & . & . & . \\ . & 1 & . & . \end{pmatrix},
$$

$$(1,2)(3,4) \longleftrightarrow \begin{pmatrix} . & 1 & . & . \\ 1 & . & . & . \\ . & . & . & 1 \\ . & . & 1 & . \end{pmatrix}, \quad (3,4)(5,6) \longleftrightarrow \begin{pmatrix} . & 1 & 1 & 1 \\ 1 & . & 1 & 1 \\ 1 & 1 & 1 & . \\ 1 & 1 & . & 1 \end{pmatrix}.$$

Let $\omega \in F$ be a primitive third root of unity, and consider the following F-subspaces of E:

$$U := \langle (\bar{\gamma}_1 + \bar{\gamma}_6) + \omega(\bar{\gamma}_4 + \bar{\gamma}_6), (\bar{\gamma}_2 + \bar{\gamma}_6) + \omega(\bar{\gamma}_3 + \bar{\gamma}_6) \rangle_F,$$
$$V := \langle (\bar{\gamma}_3 + \bar{\gamma}_6) + \omega(\bar{\gamma}_2 + \bar{\gamma}_6), (\bar{\gamma}_4 + \bar{\gamma}_6) + \omega(\bar{\gamma}_1 + \bar{\gamma}_6) \rangle_F.$$

Then we have $E = U \oplus V$ as F-vector spaces, and the action of Q_6 on E, with respect to this new F-basis, is given via:

$$(1,3)(2,4) \longleftrightarrow \begin{pmatrix} . & . & 1 & . \\ . & . & . & 1 \\ 1 & . & . & . \\ . & 1 & . & . \end{pmatrix},$$

$$(1,2)(3,4) \longleftrightarrow \begin{pmatrix} . & 1 & . & . \\ 1 & . & . & . \\ . & . & . & 1 \\ . & . & 1 & . \end{pmatrix}, \quad (3,4)(5,6) \longleftrightarrow \begin{pmatrix} \omega & \omega^2 & . & . \\ \omega^2 & \omega & . & . \\ . & . & \omega^2 & \omega \\ . & . & \omega & \omega^2 \end{pmatrix}.$$

Thus both U and V are FQ-submodules of E, and we have $\operatorname{Res}_Q^{\mathfrak{A}_6}(E) = U \oplus V$, where the socle of U has dimension 1 so that U is indecomposable. Moreover, from $((1,2)(3,4))^{(1,3)(2,4)} = (1,2)(3,4)$ and

$$((3,4)(5,6))^{(1,3)(2,4)} = (1,2)(5,6) = (1,2)(3,4) \cdot (3,4)(5,6)$$

we infer that $\operatorname{Ind}_Q^{Q_6}(U) \cong \operatorname{Res}_{Q_6}^{\mathfrak{A}_6}(E)$. Thus, by Green's Indecomposability Theorem, $\operatorname{Res}_{Q_6}^{\mathfrak{A}_6}(E)$ is indecomposable, and E is relatively Q-projective. Moreover, each proper subgroup of Q is cyclic, and since E belongs to a block of $F\mathfrak{A}_6$ with non-cyclic defect groups, E cannot have a cyclic vertex, by [5]. So Q and U then have to be vertex and source, respectively, of E. $\qquad\square$

9   Appendix

We give a new corrected proof of [13, Thm. 1.4(a), 1.5(a)]. For the case $l = 2$ and $n_l = 2$ we reuse the observations made in the proof of Proposition 8.1, where we actually have already chosen notation appropriately.

Theorem 9.1. *Let n be even, but not a 2-power. Then D has vertex P_n and source $\operatorname{Res}_{P_n}^{\mathfrak{S}_n}(D)$.*

Proof. By [13, Prop. 3.7], $\operatorname{Res}_{P_n}^{\mathfrak{S}_n}(D)$ is indecomposable. We again follow the strategy given in Remark 1.3. Assume that $R < P_n$ is a maximal subgroup such that D is relatively R-projective, and fix an indecomposable direct sum decomposition of $\operatorname{Res}_R^{\mathfrak{S}_n}(D)$. We have $X_n \leq \Phi(P_n) \leq R$. Moreover, we have $\operatorname{Res}_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n}(D) \cong D^{(n-2,1)}$. The latter, by [13, Thm. 1.2, 1.3], has vertex P_{n-4} and trivial source. Thus we have $P_{n-4} \leq R$.

Let first $n_l > 2$, or $n_l = 2$ and $l \geq 3$; recall that $l \geq 2$ anyway. Then, in all these cases we have $P_{n_1} \leq P_{n-4} \leq R$, implying that $R^{p_1 q_1} = \langle w_{n_1,1} \rangle$. Then by Remark 4.4, part (b), there is

precisely one indecomposable direct summand U in the decomposition which is not annihilated by $x_{n_1}^+$. Thus $(FX_{n_1} \oplus FX_{n_1}) | \operatorname{Res}_{X_{n_1}}^R (U)$ implies $\dim(U) > \dim(D)/2$, a contradiction.

Let now $n_l = 2$ and $l = 2$. Then using $\Phi(P_n) = \Phi(P_{n_1})$ and $P_{\frac{n_1}{2}} \leq P_{n-4}$, Proposition 2.6 yields $B_n = B_{n_1} = P_{\frac{n_1}{2}} \Phi(P_{n_1}) \leq P_{n-4} \Phi(P_n) \leq R$. Since P_n/B_n is elementary abelian of order 4, there are precisely three maximal subgroups of P_n containing B_n, these are $R_1 := B_n \langle y'_{n_1} \rangle$ and $R_2 := B_n \langle y_{n_1} \rangle = P_{n_1}$ and $R_3 := B_n \langle w_{2,2} \rangle = B_n \times P_2$. With the notation as in the proof of Proposition 8.1, we have $R_i \cap \mathfrak{A}_n = R'_i$ for $i \in \{1, 2, 3\}$. Hence from the proof of Proposition 8.1 we infer that D restricts indecomposably to R_1 and R_2.

It remains to show that D is not relatively R_3-projective. As in the proof of Proposition 8.1, it suffices to show that D restricts indecomposably to $H := \mathfrak{S}_{\frac{n_1}{2}} \times \mathfrak{S}_{\frac{n_1}{2}} \times \mathfrak{S}_2$. Namely, then D also restricts indecomposably to $\hat{H} := (\mathfrak{S}_{\frac{n_1}{2}} \wr \mathfrak{S}_2) \times \mathfrak{S}_2$. Of course D is relatively $\hat{H}$-projective, hence D and $\operatorname{Res}_{\hat{H}}^{\mathfrak{S}_n}(D)$ have a common vertex. If D were relatively R_3-projective then $\operatorname{Res}_{\hat{H}}^{\mathfrak{S}_n}(D)$ would be relatively H-projective, a contradiction since $|\hat{H} : H| = 2$. But now the proof of Proposition 8.1 shows that $\tilde{\mathcal{E}} = \operatorname{End}_{FH}(D)$ is a local F-algebra so that $\operatorname{Res}_{H}^{\mathfrak{S}_n}(D)$ indeed is indecomposable. $\qquad\square$

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S.D.: Mathematical Institute, University of Oxford
24-29 St Giles', Oxford, OX1 3LB, UK
danz@maths.ox.ac.uk

J.M.: Lehrstuhl D für Mathematik, RWTH Aachen
Templergraben 64, D-52062 Aachen, Germany
Juergen.Mueller@math.rwth-aachen.de