THE BRAUER CHARACTERS OF THE SPORADIC SIMPLE HARADA-NORTON GROUP AND ITS AUTOMORPHISM GROUP IN CHARACTERISTICS 2 AND 3

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Dedicated to the memory of Herbert Pahlings

Abstract. We determine the 2-modular and 3-modular character tables of the sporadic simple Harada-Norton group and its automorphism group.

1. Introduction

The present paper is a contribution to the modular ATLAS project [31]. The aim of the project is to classify the modular representations of the simple groups and their bicyclic extensions given in the ATLAS [2], by computing their Brauer characters, or equivalently, their decomposition numbers.

In particular for the sporadic simple groups this is a challenging task. In lieu of a unified theory for these groups, the solution of this problem has always relied on computational methods, and especially on the MeatAxe (see [25]) developed by Parker and Thackray in the late 1970's. However, even though the MeatAxe has evolved into a host of programmes to analyse modules of matrix algebras, the open problems in the modular ATLAS project defy a direct application: owing to the magnitude of the computational problems, memory and time constraints render a naive approach utterly infeasible.

To regain computational tractability, Parker and Thackray established the condensation method (see [28]). The idea of condensation is that given a module V for an algebra A, one considers the *condensed* module Ve of the algebra eAe for some suitably chosen idempotent $e \in A$. The condensation functor from $\mathsf{mod}\text{-}A$ to $\mathsf{mod}\text{-}eAe$ has a number of interesting properties, among the most important of which is that it maps simple modules to simple modules or zero, and composition series to series whose layers are simple or zero. For a detailed introduction to condensation see [27], for example. Unfortunately, the use of condensation introduces the *generation problem* (see [20], for example): for computational purposes we need to work with a feasibly small generating set for the *condensed algebra* eAe, yet such a set is not known in general, and we therefore devote time and energy to verify our condensation results.

Results. In this paper we determine the decomposition numbers of the sporadic simple Harada-Norton group HN and its automorphism group $\mathsf{HN}.2$ in characteristics 2 and 3. More precisely, the decomposition matrices of the 2-blocks of positive defect of HN and $\mathsf{HN}.2$ are determined in Section 2, where the results for HN are given in Tables 3 and 6, and those for $\mathsf{HN}.2$ are given in Tables 7 and 8. For the reader's convenience we give the degrees of Brauer characters lying in the principal

1	132	132	760	2 650
2 650	3 344	15 904	31 086	31 086
34 352	34 352	43 416	43 416	177 286
217 130	1 556 136			

Table 1. Brauer character degrees in the principal 2-block of HN

1	133	133	760	3 344
8 778	8 778	9 139	12 264	12 264
31 768	31 768	137 236	147 061	255 037
339 702	339 702	496 924	496 924	783 696

Table 2. Brauer character degrees in the principal 3-block of HN

2-block of HN in Table 1. The characteristic 3 case is dealt with in Section 3, where the results for HN are given in Tables 9, 16 and 17, and those for the principal block of HN.2, being the only block not Morita equivalent to a block of HN, are given in Table 19. The degrees of the irreducible Brauer characters in the principal 3-block of HN are restated in Table 2.

The results obtained here are also accessible in [31] and will be through the CTblLib database [1] in GAP [4]. Since the Brauer trees of HN and HN.2 have already been determined in [7] and the characteristic 5 case has been tackled in [14], this completes the modular ATLAS project for HN. Moreover, since the decomposition matrix of the non-principal 3-block of HN of defect 2 was known prior to the completion of the present paper, it was the starting point of further investigations into this block carried out in [9].

The **proofs** for the two characteristics are quite different:

In **characteristic 2**, the non-principal block may be treated entirely theoretically. It possesses a semi-dihedral defect group of order 16, and we may therefore use Erdmann's classification [3] of possible decomposition matrices for such blocks to determine its decomposition matrix. The true challenge lies in the principal block. Here we use condensation, but since there are no condensation subgroups readily available which allow the application of [20, Theorem 2.7], we are forced to verify the condensation results. As the dimensions of the simple modules with the exception of the last are small enough, we construct bases for these simple modules in order to show that the degrees of the Brauer atoms produced (see [6, Definition 3.2.1]), which are lower bounds for the degrees of the Brauer characters, are maximal, i.e. the atoms are in fact characters. We can then show that the last atom is a character, too.

For the automorphism group the result is immediate from the ordinary character table.

In **characteristic 3**, we start by investigating the block of defect 2 by choosing a basic set of projective characters and improving it pursuing a strategy first employed in [18]: We consider the endomorphism ring E of a suitably chosen projective lattice, and by Fitting correspondence compare the decomposition numbers of E with those of the block in question (see [10, Ch.II.12]). ¿From the practical side, we consider a direct summand of a projective permutation module, so that the regular representation of E is just the associated condensed module. Actually, here we are better off with the generation problem: since the regular action of E is faithful we are able to determine the dimension of the condensed algebra in question. Alone, this strategy turns out to be computationally tractable only for projective modules leading to non-faithful trace idempotents, so that after this analysis we are still left with several possible cases for the decomposition matrix.

These cases are then greatly reduced in parallel to our treatment of the principal block, which relies entirely on condensation. There are several condensation subgroups available which are normal in maximal subgroups, making them ideal candidates for [20, Theorem 2.7]. However, we still do not find any single suitable one whose trace idempotent is faithful. Therefore, we choose two condensation subgroups such that no simple module is annihilated by both condensation idempotents. This introduces a new problem: given the simple modules of one condensed algebra, we have to match them to the simple modules of the other condensed algebra, in order to gain the complete picture of the composition factors of the original module. We use [22] to overcome this problem (see also [21]).

For the automorphism group HN.2, a simple application of Clifford theory already gives the bulk of its Brauer characters. The remaining ones are once more determined through condensation, reusing the condensation infrastructure we have established for the simple group.

Background. In the sequel, we assume the reader to be familiar with the basics of ordinary and modular group representation theory, and in particular its computational aspects, as are for example exposed in [6, 15]. We employ the particular choice of Brauer lifts using Conway polynomials as in [8]. Character theoretic computations are carried out in GAP, whose facilities are also used to explicitly compute with permutation and matrix groups; in particular we use the orb package [19]. Module theoretic computations are done using the MeatAxe and its extensions [26], and as a computational workhorse we apply condensation of tensor products, respectively homomorphism spaces, as described in [12] and [16]. For a general introduction to condensation we refer the reader to [27], and to [13] for a treatise on peakword condensation. As a source of explicit data we use the database CTblLib in GAP, as well as the web-based database [29], also accessible via the GAP package AtlasRep [30]. Next to matrix and permutation representations for many of the groups in [2], given in *standard generators* in the sense of [32] where available, the latter in particular also contains *straight line programmes* to find maximal subgroups and conjugacy class representatives.

Let us fix some notation which will hold throughout the paper. Let G denote the Harada-Norton group HN. Characters, representations, and modules are given by their degrees, respectively dimensions. By a slight abuse of notation, we therefore do not distinguish between a module, the representation it affords, and the associated Brauer character. If there is more than one character of the same degree we affix subscripts to distinguish them. The ordinary characters are numbered as they are numbered in the character tables of CTblLib, which generally corresponds to the numbering given in [2]. In the decomposition matrices we also indicate this numbering next to the character degrees if necessary. If χ is an ordinary character we let χ' denote its restriction to the p-regular classes, where p is the characteristic of the field currently considered. Moreover, we stick to group theoretic notation as is used in [2].

¿From the ordinary character table of G and $G.2$, see [2], we see that all character fields are 2-elementary abelian, thus by [8, Section I.5] the quadratic extension $\mathrm{GF}(p^2)$ of any finite prime field $\mathrm{GF}(p)$ is a splitting field for G and $G.2$. We let F denote $\mathrm{GF}(p^2)$ in the sequel, and let (Q, R, F) denote a p-modular splitting system for G and $G.2$.

2. Characteristic 2

The 2-modular characters of G are distributed over three blocks, two of which have positive defect: the principal block B_0 of defect $d = 14$, having $k = 45$ ordinary and $l = 17$ modular characters, and the block B_1 of defect $d = 4$, having $k = 8$ ordinary and $l = 3$ modular characters. The block of defect 0 consists of the ordinary character χ_{46}.

2.1. **Proof for the block** B_1**.** By [11, Section 9] the defect group of Block B_1 is a semidihedral group of order 16. In [3, Section 11] Erdmann lists all possible decomposition matrices for such blocks, such that we are left with to decide which of the four cases in Lemmas 11.4, 11.6, 11.9, and 11.11 holds:

To distinguish Erdmann's notation from ATLAS notation, we write here ϑ where she uses χ. In this notation [3, Theorem 11.1], work of Olsson [24] implies that B_1 has four characters $\vartheta_1, \ldots, \vartheta_4$ of height zero, three characters $\vartheta_1^*, \ldots, \vartheta_3^*$ of height one for which $\vartheta^{*\prime} := \vartheta_1^{*\prime} = \vartheta_2^{*\prime} = \vartheta_3^{*\prime}$, and a single character $\hat{\vartheta}$ of height two. Moreover, $\vartheta_1, \ldots, \vartheta_4$ can be chosen such that

$$\begin{array}{rcccl} \delta_1\vartheta_1' + \delta_2\vartheta_2' & = & -\delta_3\vartheta_3' - \delta_4\vartheta_4' & = & \vartheta^{*\prime}, \\ \delta_2\vartheta_2' + \delta_4\vartheta_4' & = & -\delta_1\vartheta_1' - \delta_3\vartheta_3' & = & \hat{\vartheta}', \end{array}$$

where $\delta_1, \ldots, \delta_4 \in \{\pm 1\}$. In particular, choosing any $\vartheta_i \in \{\vartheta_1, \ldots, \vartheta_4\}$ and $\vartheta_r^* \in \{\vartheta_1^*, \ldots, \vartheta_3^*\}$, we conclude that $\{\vartheta_i', \vartheta_r^{*\prime}, \hat{\vartheta}'\}$ is a basic set of Brauer characters. Actually, as is easily seen, the above conditions determine $\vartheta_1, \ldots, \vartheta_4$ and $\delta_1, \ldots, \delta_4$ uniquely up to the permutation interchanging $\delta_1\vartheta_1 \leftrightarrow -\delta_4\vartheta_4$ and $\delta_2\vartheta_2 \leftrightarrow -\delta_3\vartheta_3$, and by the positivity of character degrees we are left with the four cases

$$[\delta_1, \ldots, \delta_4] \in \{[1, 1, -1, -1], [1, -1, -1, 1], [-1, 1, -1, 1], [1, 1, -1, 1]\},$$

corresponding to the Lemmas cited above.

By computing the actual heights of the characters in B_1, we conclude that $\{\vartheta_1, \ldots, \vartheta_4\} = \{\chi_{17}, \chi_{37}, \chi_{45}, \chi_{49}\}$, and $\{\vartheta_1^*, \ldots, \vartheta_3^*\} = \{\chi_{34}, \chi_{35}, \chi_{36}\}$, and $\hat{\vartheta} = \chi_{44}$. Decomposing the restrictions of the ordinary characters in B_1 into the basic set $\{\chi_{17}', \chi_{34}', \chi_{44}'\}$ we obtain the matrix of Table 3, where the characters belonging to the basic set are indicated by bold face. This shows that we have

$$\begin{array}{lllllllllll} \vartheta_1 & = & \chi_{37}, & \vartheta_2 & = & \chi_{17}, & \vartheta_3 & = & \chi_{49}, & \vartheta_4 & = & \chi_{45}, \\ \delta_1 & = & 1, & \delta_2 & = & -1, & \delta_3 & = & -1, & \delta_4 & = & 1, \end{array}$$

that is, by [3, Lemma 11.6] the decomposition matrix of B_1 is as given in Table 3.

χ	$\chi(1)$	Φ_1	Φ_2	Φ_3
17	214 016	1	.	.
34	1 361 920	.	1	.
35	1 361 920	.	1	.
36	1 361 920	.	1	.
37	1 575 936	1	1	.
44	2 985 984	.	.	1
45	3 200 000	1	.	1
49	4 561 920	1	1	1

TABLE 3. The decomposition matrix of the 2-block B_1 of HN

2.2. **Proof for the principal block B_0.** Our proof for the principal block relies on the tried and tested formula of applying the MeatAxe in conjunction with condensation. Owing to the modest dimensions of the first few irreducible representations of G, the first five nontrivial ones are available through [29]. This way we obtain 132_1, 132_2, 760, $2\,650_1$, and $2\,650_2$; where $2\,650_1$ yields $2\,650_2$ by taking the contragredient. Also available from the same source is the smallest non-trivial irreducible representation of the sporadic Baby Monster group B in characteristic 2. The restriction of this 4370-dimensional module yields the composition factors

$$4\,370{\downarrow_G} = 3\,344 + 760 + 132_1 + 132_2,$$

thus adding 3 344 to our collection of simple modules we have available for further computations. Also owing to the small size of these modules, we may compute their Brauer characters explicitly by lifting eigenvalues, where the necessary straight line programmes to restrict from B to HN and to find conjugacy class representatives are also available at [29].

The eighth Brauer character in the principal block may also be computed with a plain application of the MeatAxe: Chopping the tensor product $132_1 \otimes 132_2$ gives the composition factors

$$132_1 \otimes 132_2 = 15\,904 + 2 \times 760$$

and lets us derive the Brauer character of 15 904 from the tensor product of the Brauer characters. As the direct analysis of even larger tensor products with the MeatAxe is considerably slowed and ultimately rendered infeasible due to the increased module dimensions, we determine the remaining nine simple modules and their Brauer characters with the help of condensation.

As our condensation subgroup K we choose a Sylow 3-subgroup of order 243 of the alternating group on 12 letters which in turn is the largest maximal subgroup of G, and condense with the corresponding trace idempotent e; here, a straight line programme to find the largest maximal subgroup of G is available in [29], and Sylow subgroups of permutation groups can be computed in GAP. Unfortunately, for our choice of condensation subgroup, there are at present no computationally efficient methods available to deal with the generation problem (see [20]). Here we arbitrarily choose a few elements $g_i \in G$, and have to allow for the possibility that the corresponding *condensed elements* eg_ie generate a proper subalgebra $\mathcal{C}$ of

	1	132_1	132_2	760	$2\,650_1$	$2\,650_2$	$3\,344$	$132_1 \otimes 132_2$	$132_1 \otimes 760$	$132_2 \otimes 760$	$760 \otimes 760$	$132_1 \otimes 2\,650_1$	$132_1 \otimes 3\,344$	$132_2 \otimes 3\,344$	$760 \otimes 2\,650_1$	$760 \otimes 3\,344$	$2\,650_1 \otimes 2\,650_2$
$k1$	1	.	.	.	.	.	.	.	2	2	20	6	6	6	40	12	46
$k2_1$	.	1	.	.	.	.	.	.	2	2	6	3	4	4	22	6	23
$k2_2$	.	.	1	.	.	.	.	.	2	2	6	4	4	4	22	6	23
$k6_1$	.	.	.	.	1	.	.	.	.	.	8	.	.	.	5	2	8
$k6_2$	.	.	.	.	.	1	.	.	.	.	8	1	.	.	5	2	8
$k12$	.	.	.	1	.	.	.	2	.	.	3	1	2	2	10	5	11
$k28$	.	.	.	.	.	.	1	.	.	.	10	.	.	.	2	4	8
$k62$	.	.	.	.	.	.	.	1	.	.	4	1	2	2	12	4	10
$k118_1$	.	.	.	.	.	.	.	.	.	1	.	1	.	.	.	.	2
$k118_2$	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	2
$k160_1$	.	.	.	.	.	.	.	.	.	.	.	1	1	1	7	1	5
$k160_2$	.	.	.	.	.	.	.	.	.	.	.	1	1	1	6	1	5
$k164_1$	.	.	.	.	.	.	.	.	2	.	.	.	1	2	1	2	4
$k164_2$	.	.	.	.	.	.	.	.	.	2	.	1	2	1	1	2	4
$k706$	.	.	.	.	.	.	.	.	.	.	.	1	.	.	4	.	3
$k922$	.	.	.	.	.	.	.	.	.	.	2	.	1	1	2	2	2
$k6\,344$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1
$k908$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	2	.
$k5\,652$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.
$k14\,072$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1

TABLE 4. Condensation results for $p = 2$ for HN

the condensed group algebra $eFGe$. We apply the condensation of tensor products technique described in [12, 16].

The results are presented in Table 4, in which we display the multiplicities of the composition factors of the condensed tensor products given, where we denote the simple $\mathcal{C}$-modules by their dimension, preceded by the letter 'k'. We also condense the known simple FG-modules of the principal block individually in order to match them to simple $\mathcal{C}$-modules. Note that we condense the tensor product $132_1 \otimes 132_2$, even though we were able to compute its decomposition directly: it is more efficient to condense the tensor product than to deal with the module $15\,904$ alone.

Our next task is to determine which simple $\mathcal{C}$-modules are composition factors of simple $eFGe$-modules coming from simple FG-modules of its principal block. To this end, it turns out that the restrictions of the ordinary characters indicated by bold face in Table 6 are a basic set of Brauer characters for the principal block. A basic set of Brauer characters for block B_1 was already given in Table 3, and adding χ'_{46} we obtain a complete basic set of Brauer characters.

Then, by decomposing the Brauer characters of the tensor products considered into this basic set, we see that only the last two contain composition factors which lie outside of the principal block. As a manner of speaking, we say that a simple $\mathcal{C}$-module belongs to a block of FG if it is a composition factor of the restriction to $\mathcal{C}$ of a condensed simple FG-module lying in that block.

By computing the multiplicity of the trivial character of K in the restrictions to K of the basic set characters and the known Brauer characters, we determine the dimensions of the corresponding condensed modules. Of course, these are upper bounds for the dimensions of their simple $\mathcal{C}$-constituents.

Also, decomposing the characters of the last two tensor products in the basic set of B_0 and the irreducible Brauer characters of the other blocks, gives that 214 016 occurs twice, and 1 361 920 occurs once as a constituent of $760 \otimes 3\,344$. The single defect zero module 3 424 256 appears once in $2\,650_1 \otimes 2\,650_2$.

We compute the dimensions of the condensed modules whose Brauer characters constitute the basic set. From this it is immediate that $k14\,072$ is a composition factor of the condensed defect zero module 3 424 256 restricted to $\mathcal{C}$: it is simply too large to fit into any of the other modules. As the dimension of the $\mathcal{C}$-module is equal to the dimension computed of 3 424 256 condensed, we conclude that $k14\,072$ is the irreducible restriction of the latter module, and therefore belongs to the defect zero block.

The simple 1 361 920-dimensional modules of B_1 condense to modules of dimension 5 652. By summing up the dimensions of the composition factors by their multiplicities as given in the second to last column of Table 4, it is plain to see that $k5\,652$ appears in one of the condensed simple modules of the same dimension, as the sum obtained is less. Thus $k5\,652$ too is an irreducible restriction of a condensed simple module lying in B_1 and therefore belongs to B_1.

For $k908$ we verify computationally that it also originates from a module outside of the principal block: employing the MeatAxe and peakword techniques, we find a uniserial submodule of $760 \otimes 3\,344$ condensed with head and socle isomorphic to $k908$ and containing $k5\,652$ as the only additional composition factor. Therefore there is a non-trivial extension of $k5\,652$ by $k908$, and as $k5\,652$ belongs to B_1 so must $k908$.

Thus all appearing $\mathcal{C}$-modules, except $k908$, $k5\,652$, and $k14\,072$, belong to the principal block. Hence the matrix A comprising the first 17 rows of Table 4 describes the condensed modules of the B_0-components of the modules given at its top. It is immediate that A has full rank. Therefore, as B_0 has $l = 17$ Brauer characters, by [5, Proposition 1] condensation with respect to our chosen subgroup K defines a faithful functor from mod-B_0 to mod-eB_0e. Furthermore, by [14, Theorem 4.1] taking as b the B_0-components of the Brauer characters afforded by the modules indexing the columns of Table 4, the product bA^{-1} gives a basic set of Brauer atoms for B_0.

As the possibility that $\mathcal{C}$ is a proper subalgebra of $eFGe$ remains, the degrees of the Brauer atoms are lower bounds for the degrees of the irreducible Brauer characters. Therefore in order to verify that the Brauer atoms are in fact the Brauer characters, we need to show that the simple FG-modules possess these dimensions. This is achieved by constructing bases for almost all simple modules through uncondensation: With the help of the MeatAxe we can determine the socle series of the condensed tensor products considered, and may therefore seek a suitably small

$k118_2$	$k922$	$k706$	$k2_1 \oplus k12 \oplus k160_2$
$k2_2$	$k1 \oplus k62$	$k1$	$k164_2$
$k1$	$k2_1 \oplus k2_2 \oplus k28$	$k2_2$	
$k2_1 \oplus k164_1$	$k1 \oplus k1$		
	$k6_1 \oplus k6_2$		
	$k6_1 \oplus k6_2$		
	$k28$		

Table 5. Socle series of select submodules of some tensor products

submodule U of the condensed module Ve which contains a condensed version of the simple module S we would like to construct. The subspace embedding of Ve into V therefore embeds a basis of U into V, and with standard MeatAxe functionality we may construct a basis of the smallest G-invariant subspace of V containing the embedded basis of U. Ideally, we have $U = Se$, and we are done after having constructed a basis of S. However, if Se does not lie in the socle of V, we need to iterate this procedure by working along an ascending composition series of U.

Recall that we have already chopped up the tensor product $132_1 \otimes 132_2$ completely, hence we are done with that. Within the condensed tensor products $132_1 \otimes 760$, $760 \otimes 760$, $132_1 \otimes 2\,650_1$, and $132_1 \otimes 3\,344$, respectively, we find suitable submodules whose socle series we give in Table 5. Note that for every pair of modules conjugate under the outer automorphism, which of course is seen on the level of characters, it is sufficient to construct a basis of only one member. Therefore the four modules of Table 5 allow us to verify the dimensions of almost all Brauer characters except the one corresponding to $k6\,344$. For the practical realisation of these computations, it is important to note that we do not need to construct a representation of the action of G on any of the huge tensor product spaces: using the natural isomorphism $M \otimes N \cong \mathrm{Hom}_F(M^*, N)$ (see [17, Lemma 2.6]) we only need to work with the available representations of the tensor factors. The computational demand of the necessary calculations involved is bounded by the construction of the $250\,589$-dimensional module containing the uncondensed $k922$ which itself is $217\,130$-dimensional. As we are able to work over the field with two elements in this case, the memory needed to construct and store this space is less than 8 GB of RAM.

With only the last Brauer atom of degree $1\,556\,136$ left to prove to be a Brauer character, we can avoid the unwieldy construction of a basis for this module: We additionally condense the tensor product $V := 2\,650_1 \otimes 3\,344$ instead. Let S denote the simple FG-module such that its Brauer character φ contains the Brauer atom α of degree $1\,556\,136$ as a summand. Since φ is a constituent of the basic set character χ'_{39}, and the latter has degree $2\,031\,480$, we conclude that φ contains α as a summand with multiplicity precisely one.

A calculation with the MeatAxe gives that head and socle of $Ve{\downarrow_{\mathcal{C}}}$ are isomorphic, and have structure

$$k160_1 \oplus k6\,344 \oplus k12\,288.$$

By the above we have $43\,416_1 e = k160_1$ and $2\,985\,984 e = k12\,288$, and $Se{\downarrow_{\mathcal{C}}}$ has $k6\,344$ as a composition factor. Recall that e is a faithful condensation idempotent, and that the Brauer atom α alone already accounts for the condensed dimension $6\,344$, that is $\varphi = \alpha$ is equivalent to $Se = k6\,344$, but due to the generation problem this does not immediately follow from the above statement, and neither it is clear how socle and head of Ve and thus of V look like. But, with the possible exception of φ, we have proved that all simple $eFGe$-modules restrict irreducibly to $\mathcal{C}$, and that two such $eFGe$-modules are isomorphic if their restrictions to $\mathcal{C}$ are. Hence we conclude that both the head and the socle of V have at most the constituents $43\,416_1$, $2\,985\,984$ and S; in particular, the constituents $132_{1/2}$ and 760 do not occur.

We now additionally choose another condensation subgroup, namely the normal subgroup 5_+^{1+4} of the sixth maximal subgroup $M_6 := 5_+^{1+4} \colon 2_-^{1+4}.5.4$. To distinguish this condensation from the previous one, we denote the trace idempotent of 5_+^{1+4} by e' and prefix the simple $e'FGe'$-modules by the letter 'c'. By [20, Theorem 2.7] we obtain a generating system of $e'FGe'$ of 128 elements, if we condense the 126 non-identity double coset representatives of $M_6 \backslash G / M_6$ and the two generators for M_6, obtained through a straight line programme available at [29].

As we know all irreducible Brauer characters except φ, we can determine the dimensions of their condensed modules with respect to e'. This shows that e' is not faithful; indeed, precisely the simple modules $132_{1/2}$ and 760 condense to the zero module. We identify the possible composition factors of the $2\,772$-dimensional condensed module Ve' by applying the MeatAxe. We again find that head and socle of Ve' are isomorphic, and have structure

$$c8 \oplus c1\,152 \oplus c960,$$

where $43\,416_1 e' = c8$, $2\,985\,984 e' = c960$ and $Se' = c1\,152$. Again, a consideration of condensed dimensions shows that α alone accounts for the dimension $1\,152$, which implies that $\varphi - \alpha$ is a sum of multiples of $132_{1/2}$ and 760.

Moreover, as we look at Ve' as an $e'FGe'$-module, uncondensing (in theory, not in practice!) the submodule of Ve' isomorphic to $c1\,152$ yields a submodule U of V having a simple head isomorphic to S, whose radical has only constituents condensing to the zero module, that is amongst $132_{1/2}$ and 760. But since none of the latter occur in the socle of V we conclude that U is isomorphic to S, in other words S occurs in the socle of V, and similarly in the head of V as well.

We now return to our established faithful condensation setup and reconsider Ve: Assume that $\varphi \neq \alpha$. Then, as α is a summand of φ with multiplicity one, hence $k6\,344$ occurs only once as a constituent of $Se{\downarrow_{\mathcal{C}}}$, we conclude that the head or the socle of $Ve{\downarrow_{\mathcal{C}}}$ (or both) contain a condensed module of at least one of $132_{1/2}$ and 760, which is not the case, and thus is a contradiction. Hence we conclude that the last Brauer atom α coincides with φ, thus is an irreducible Brauer character.

The decomposition matrix of the principal block is given in Table 6.

2.3. **The Automorphism Group in Characteristic 2.** The 2-modular characters of $G.2$ are distributed over three blocks, each covering precisely one of the 2-blocks of G: the principal block B_0 of defect $d = 15$, having $k = 63$ ordinary and $l = 12$ modular characters, the block B_1 of defect $d = 5$, having $k = 13$ ordinary and $l = 3$ modular characters, and the block B_2 of defect $d = 1$, having $k = 2$ ordinary and $l = 1$ modular characters.

The task of determining the decomposition numbers of $G.2$ in characteristic 2 is easy: every Brauer character of G invariant under the outer automorphism gives

χ	$\chi(1)$	φ_1	φ_2	φ_3	φ_4	φ_5	φ_6	φ_7	φ_8	φ_9	φ_{10}	φ_{11}	φ_{12}	φ_{13}	φ_{14}	φ_{15}	φ_{16}	φ_{17}
1	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
2	133	1	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.
3	133	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
4	760	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.
5	3 344	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.
6	8 778	2	.	1	1	1	.	1	.	.	.	.	.	.	.	.	.	.
7	8 778	2	1	.	1	1	.	1	.	.	.	.	.	.	.	.	.	.
8	8 910	2	1	1	1	1	.	1	.	.	.	.	.	.	.	.	.	.
9	9 405	1	.	.	1	1	1	1	.	.	.	.	.	.	.	.	.	.
10	16 929	1	1	1	.	.	1	.	1	.	.	.	.	.	.	.	.	.
11	35 112	.	.	.	.	.	1	.	.	.	.	.	.	.	1	.	.	.
12	35 112	.	.	.	.	.	1	.	.	.	.	.	.	1	.	.	.	.
13	65 835	1	2	1	.	.	.	.	.	1	.	.	.	.	1	.	.	.
14	65 835	1	1	2	.	.	.	.	.	.	1	.	.	1	.	.	.	.
15	69 255	3	2	2	1	1	1	1	1	.	.	1	.	.	.	.	.	.
16	69 255	3	2	2	1	1	1	1	1	.	.	.	1	.	.	.	.	.
18	267 520	6	2	2	2	2	1	2	2	.	.	.	.	.	.	.	1	.
19	270 864	6	2	2	2	2	1	3	2	.	.	.	.	.	.	.	1	.
20	365 750	8	3	3	2	2	3	1	1	.	.	1	1	1	1	1	.	.
21	374 528	6	2	2	1	1	2	.	.	.	.	1	1	2	1	1	.	.
22	374 528	6	2	2	1	1	2	.	.	.	.	1	1	1	2	1	.	.
23	406 296	6	4	4	.	.	1	.	2	.	.	1	1	1	1	.	1	.
24	653 125	13	7	7	2	2	3	2	4	.	.	2	2	.	.	1	1	.
25	656 250	12	6	6	1	2	3	1	3	.	.	1	2	1	1	1	1	.
26	656 250	12	6	6	2	1	3	1	3	.	.	2	1	1	1	1	1	.
27	718 200	14	8	9	2	2	2	2	4	.	1	2	2	1	.	1	1	.
28	718 200	14	9	8	2	2	2	2	4	1	.	2	2	.	1	1	1	.
29	1 053 360	26	12	12	6	6	7	5	6	.	.	3	3	1	1	2	1	.
30	1 066 527	19	11	11	2	2	5	1	5	1	.	3	3	1	2	2	1	.
31	1 066 527	19	11	11	2	2	5	1	5	.	1	3	3	2	1	2	1	.
32	1 185 030	20	12	12	2	2	5	1	4	1	1	3	3	3	3	2	1	.
33	1 354 320	24	14	14	2	2	6	1	8	.	.	4	4	1	1	2	2	.
38	1 625 184	30	16	16	4	4	7	4	10	.	.	4	4	1	1	2	3	.
39	2 031 480	14	8	8	2	2	2	2	4	.	.	2	2	.	.	.	1	1
40	2 375 000	20	8	8	4	4	6	2	2	.	.	3	3	2	2	2	.	1
41	2 407 680	50	26	26	8	8	13	6	14	.	.	7	7	2	2	4	3	.
42	2 661 120	48	28	28	5	5	8	4	14	1	1	7	7	2	2	4	4	.
43	2 784 375	25	14	14	3	3	5	2	5	1	1	4	4	2	2	2	1	1
47	3 878 280	48	27	27	7	7	10	6	12	1	1	7	7	3	3	3	3	1
48	4 156 250	52	29	29	7	7	11	5	13	1	1	8	8	3	3	4	3	1
50	4 809 375	65	36	36	9	9	14	7	17	1	1	10	10	3	3	5	4	1
51	5 103 000	68	38	38	8	8	14	6	18	1	1	10	10	4	4	5	5	1
52	5 103 000	68	38	38	8	8	14	6	18	1	1	10	10	4	4	5	5	1
53	5 332 635	71	41	41	8	8	16	5	17	2	2	11	11	6	6	6	4	1
54	5 878 125	81	47	47	9	9	16	7	22	2	2	12	12	5	5	6	6	1

Table 6. The decomposition matrix of the principal 2-block B_0 of HN

just one Brauer character of $G.2$, while the pairs of Brauer characters of G whose members are interchanged by the outer automorphism induce to one Brauer character of $G.2$. Therefore, using the results of the previous section we can immediately give the three decomposition matrices in Tables 7 and 8.

3. Characteristic 3

In characteristic 3, the simple group G possesses three blocks of positive defect: the principal block B_0 of defect $d = 6$, having $k = 33$ ordinary and $l = 20$ modular characters, the block B_1 of defect $d = 2$, having $k = 9$ ordinary and $l = 7$ modular characters, and the block B_2 of defect $d = 1$, having $k = 3$ ordinary and $l = 2$ modular characters. Moreover, there are nine blocks of defect 0, consisting of the ordinary characters $\{\chi_{15}, \chi_{16}, \chi_{30}, \chi_{31}, \chi_{44}, \chi_{47}, \chi_{51}, \chi_{52}, \chi_{53}\}$.

χ	$\chi(1)$	Φ_1	Φ_2	Φ_3	Φ_4	Φ_5	Φ_6	Φ_7	Φ_8	Φ_9	Φ_{10}	Φ_{11}	Φ_{12}
1	1	1	.	.	.	.	.	.	.	.	.	.	.
2	1	1	.	.	.	.	.	.	.	.	.	.	.
3	266	2	1	.	.	.	.	.	.	.	.	.	.
4	760	.	.	.	1	.	.	.	.	.	.	.	.
5	760	.	.	.	1	.	.	.	.	.	.	.	.
6	3 344	.	.	.	.	1	.	.	.	.	.	.	.
7	3 344	.	.	.	.	1	.	.	.	.	.	.	.
8	17 556	4	1	2	.	2	.	.	.	.	.	.	.
9	8 910	2	1	1	.	1	.	.	.	.	.	.	.
10	8 910	2	1	1	.	1	.	.	.	.	.	.	.
11	9 405	1	.	1	1	1	.	.	.	.	.	.	.
12	9 405	1	.	1	1	1	.	.	.	.	.	.	.
13	16 929	1	1	.	1	.	1	.	.	.	.	.	.
14	16 929	1	1	.	1	.	1	.	.	.	.	.	.
15	70 224	.	.	.	2	.	.	.	.	1	.	.	.
16	131 670	2	3	.	.	.	.	1	.	1	.	.	.
17	138 510	6	4	2	2	2	2	.	1	.	.	.	.
20	267 520	6	2	2	1	2	2	.	.	.	.	1	.
21	267 520	6	2	2	1	2	2	.	.	.	.	1	.
22	270 864	6	2	2	1	3	2	.	.	.	.	1	.
23	270 864	6	2	2	1	3	2	.	.	.	.	1	.
24	365 750	8	3	2	3	1	1	.	1	1	1	.	.
25	365 750	8	3	2	3	1	1	.	1	1	1	.	.
26	749 056	12	4	2	4	.	.	.	2	3	2	.	.
27	406 296	6	4	.	1	.	2	.	1	1	.	1	.
28	406 296	6	4	.	1	.	2	.	1	1	.	1	.
29	653 125	13	7	2	3	2	4	.	2	.	1	1	.
30	653 125	13	7	2	3	2	4	.	2	.	1	1	.
31	1 312 500	24	12	3	6	2	6	.	3	2	2	2	.
32	1 436 400	28	17	4	4	4	8	1	4	1	2	2	.
33	1 053 360	26	12	6	7	5	6	.	3	1	2	1	.
34	1 053 360	26	12	6	7	5	6	.	3	1	2	1	.
35	2 133 054	38	22	4	10	2	10	1	6	3	4	2	.
36	1 185 030	20	12	2	5	1	4	1	3	3	2	1	.
37	1 185 030	20	12	2	5	1	4	1	3	3	2	1	.
38	1 354 320	24	14	2	6	1	8	.	4	1	2	2	.
39	1 354 320	24	14	2	6	1	8	.	4	1	2	2	.
45	1 625 184	30	16	4	7	4	10	.	4	1	2	3	.
46	1 625 184	30	16	4	7	4	10	.	4	1	2	3	.
47	2 031 480	14	8	2	2	2	4	.	2	.	.	1	1
48	2 031 480	14	8	2	2	2	4	.	2	.	.	1	1
49	2 375 000	20	8	4	6	2	2	.	3	2	2	.	1
50	2 375 000	20	8	4	6	2	2	.	3	2	2	.	1
51	2 407 680	50	26	8	13	6	14	.	7	2	4	3	.
52	2 407 680	50	26	8	13	6	14	.	7	2	4	3	.
53	2 661 120	48	28	5	8	4	14	1	7	2	4	4	.
54	2 661 120	48	28	5	8	4	14	1	7	2	4	4	.
55	2 784 375	25	14	3	5	2	5	1	4	2	2	1	1
56	2 784 375	25	14	3	5	2	5	1	4	2	2	1	1
63	3 878 280	48	27	7	10	6	12	1	7	3	3	3	1
64	3 878 280	48	27	7	10	6	12	1	7	3	3	3	1
65	4 156 250	52	29	7	11	5	13	1	8	3	4	3	1
66	4 156 250	52	29	7	11	5	13	1	8	3	4	3	1
69	4 809 375	65	36	9	14	7	17	1	10	3	5	4	1
70	4 809 375	65	36	9	14	7	17	1	10	3	5	4	1
71	5 103 000	68	38	8	14	6	18	1	10	4	5	5	1
72	5 103 000	68	38	8	14	6	18	1	10	4	5	5	1
73	5 103 000	68	38	8	14	6	18	1	10	4	5	5	1
74	5 103 000	68	38	8	14	6	18	1	10	4	5	5	1
75	5 332 635	71	41	8	16	5	17	2	11	6	6	4	1
76	5 332 635	71	41	8	16	5	17	2	11	6	6	4	1
77	5 878 125	81	47	9	16	7	22	2	12	5	6	6	1
78	5 878 125	81	47	9	16	7	22	2	12	5	6	6	1

TABLE 7. The decomposition matrix of the principal 2-block B_0 of $\mathsf{HN}.2$

χ	$\chi(1)$	Φ_1	Φ_2	Φ_3	Φ_1
18	214 016	1	.	.	.
19	214 016	1	.	.	.
40	1 361 920	.	1	.	.
41	1 361 920	.	1	.	.
42	2 723 840	.	2	.	.
43	1 575 936	1	1	.	.
44	1 575 936	1	1	.	.
57	2 985 984	.	.	1	.
58	2 985 984	.	.	1	.
59	3 200 000	1	.	1	.
60	3 200 000	1	.	1	.
67	4 561 920	1	1	1	.
68	4 561 920	1	1	1	.
61	3 424 256	.	.	.	1
62	3 424 256	.	.	.	1

TABLE 8. The decomposition matrices of the 2-blocks B_1 and B_2 of HN.2

χ	$\chi(1)$	Φ_1	Φ_2
23	406 296	1	.
38	1 625 184	.	1
39	2 031 480	1	1

TABLE 9. The decomposition matrix of the 3-block B_2 of HN

The decomposition matrix of the 3-block B_2, reproduced in Table 9, follows immediately from the Brauer-Dade theory of blocks with cyclic defect groups, and has previously been published in [7].

3.1. **Proof for the block B_1, part 1.** We begin with a basic set of projective characters displayed in Table 10, where we also give a basic set of Brauer characters, comprising of the restrictions of the ordinary characters indicated by bold face. The origin of the projective characters is given there as well, where 1_{B_1} denotes the associated block idempotent, $M_6 = 5_+^{1+4}: 2_-^{1+4}.5.4$ is the sixth maximal subgroup of G (see [2]), which is a $3'$-subgroup, the character φ_2 is the linear character of M_6 of order 2, the character φ_3 is one of the two linear characters of order 4, and φ_7 is one of the two rational valued characters of degree 5. Note that the character of the B_1-component of $\mathrm{Ind}_{M_6}^G(\varphi_7)$ decomposes into the ordinary characters of B_1 with even coefficients, hence dividing by 2 still yields a projective character. The remaining projective characters are products of irreducible characters of G, one factor of which is a character of defect zero.

χ	$\chi(1)$	Ψ_1	Ψ_2	Ψ_3	Ψ_4	Ψ_5	Ψ_6	Ψ_7	Ψ
8	8 910	1	.	.	.	.	.	.	3
10	16 929	.	1	.	.	.	.	.	1
19	270 864	.	.	1	.	.	.	.	.
32	1 185 030	1	1	.	3	1	1	.	5
33	1 354 320	2	2	.	2	2	1	.	4
37	1 575 936	1	2	1	2	1	1	.	1
43	2 784 375	2	2	1	5	3	2	.	5
49	4 561 920	1	2	2	9	6	4	1	3
50	4 809 375	1	3	3	9	5	4	1	4

$\Psi_1 = \mathrm{Ind}_{M_6}^G(\varphi_2)\cdot 1_{B_1}, \quad \Psi_3 = \mathrm{Ind}_{M_6}^G(\varphi_3)\cdot 1_{B_1}, \quad \Psi_5 = (\mathrm{Ind}_{M_6}^G(\varphi_7)\cdot 1_{B_1})/2$

$\Psi_2 = (\chi_2\cdot\chi_{31})\cdot 1_{B_1}, \quad \Psi_4 = (\chi_2\cdot\chi_{47})\cdot 1_{B_1}, \quad \Psi_6 = (\chi_5\cdot\chi_{15})\cdot 1_{B_1}, \quad \Psi_7 = (\chi_4\cdot\chi_{15})\cdot 1_{B_1}$

TABLE 10. A first basic set of projective characters for B_1

We now pursue a strategy first employed in [18]. Let M_6' denote the unique subgroup of M_6 of index 2. Hence the RG-permutation module on the cosets of M_6' is a projective RG-lattice. Put

$$E := \mathrm{End}_{RG}(\mathrm{Ind}_{M_6'}^G(R)\cdot 1_{B_1});$$

note that (Q, R, F) is a splitting system for E. The B_1-component of the associated permutation character $\Psi := \mathrm{Ind}_{M_6}^G(\varphi_1 + \varphi_2)$, where φ_1 is the trivial character, decomposes into the ordinary characters of B_1 as given in Table 10. In particular, $Q \otimes_R E$ has eight irreducible characters.

We consider $\bar{E} := F \otimes_R E$, and proceed to determine its regular representation. In practice, a straight line programme to find M_6 is available in [29]. Using the facilities provided by GAP it is then easy to find M_6', and the permutation representation of G on the 273 030 912 cosets in $M_6'\backslash G$, realised as a G-orbit of vectors in a suitable matrix representation of G. Then the regular representation of $\bar{E}$ is obtained by a direct condense technique, using the orb package [19]; for details see [18].

It turns out that $\bar{E}$ has five simple modules $3a, 1a, 1b, 2c, 2a$ of the respective dimensions, and the Cartan matrix C of $\bar{E}$ is found to be as given in Table 11. Let $D \in \mathbb{N}_0^{8\times 5}$ be the decomposition matrix of E. It is easy to check that the matrix equation $D^tD = C$ has, up to a permutation of rows, the matrix given in Table 11 as its only solution. (We have already chosen the ordering with some foresight such that a later reordering of rows or columns is unnecessary.)

Hence it remains to show how the rows of the matrix in Table 11 correspond to rows of Table 10. The regular character of E shows that

$$\{1_1, 1_2\} \leftrightarrow \{\chi_{10}, \chi_{37}\}, \quad \{3_1, 3_2\} \leftrightarrow \{\chi_8, \chi_{49}\},$$
$$\{4_1, 4_2\} \leftrightarrow \{\chi_{33}, \chi_{50}\}, \quad \{5_1, 5_2\} \leftrightarrow \{\chi_{32}, \chi_{43}\},$$

	Φ_{3a}	Φ_{1a}	Φ_{1b}	Φ_{1c}	Φ_{2a}
	$3a$	$1a$	$1b$	$1c$	$2a$
3_1	1	.	.	.	.
1_1	.	1	.	.	.
5_1	1	1	1	.	.
4_1	1	.	.	1	.
1_2	.	.	.	1	.
5_2	1	.	1	1	.
3_2	.	.	1	.	1
4_2	.	1	1	.	1

$$C := \begin{pmatrix} 4 & 1 & 2 & 2 & . \\ 1 & 3 & 2 & . & 1 \\ 2 & 2 & 4 & 1 & 2 \\ 2 & . & 1 & 3 & . \\ . & 1 & 2 & . & 2 \end{pmatrix}$$

TABLE 11. Decomposition matrix and Cartan matrix of E

χ	$\chi(1)$	Φ_1	Φ_2	Ψ_3	Φ_4	Φ_5	Ψ_6	Φ_7	Ψ'
8	8 910	1	.	.	.	.	.	.	.
10	16 929	.	1	.	.	.	.	.	.
19	270 864	.	.	1	.	.	.	.	1
32	1 185 030	1	1	.	1	.	1	.	1
33	1 354 320	1	.	.	.	1	2	.	.
37	1 575 936	.	.	1	.	1	1	.	1
43	2 784 375	1	.	1	1	1	3	.	2
49	4 561 920	.	.	2	1	.	6	1	2
50	4 809 375	.	1	3	1	.	5	1	3

TABLE 12. A second basic set of projective characters for B_1

while χ_{19} is not met. The basic set of Table 10 shows in particular that χ_8' is an irreducible Brauer character, and the associated projective indecomposable character Φ_1 is contained three times as a summand in the projective character Ψ. Thus Φ_{3a} is the Fitting correspondent of Φ_1, and part of the matching is

$$3_1 \leftrightarrow \chi_8 \quad \text{and} \quad 3_2 \leftrightarrow \chi_{49}.$$

This yields 8 remaining cases. To be a valid matching, the projective indecomposable characters of E have to decompose into the basic set of projective characters in Table 10 with integral coefficients. It is easily checked using GAP that this necessary condition is only fulfilled for

$$1_1 \leftrightarrow \chi_{10}, \quad 1_2 \leftrightarrow \chi_{37}, \quad 4_1 \leftrightarrow \chi_{33}, \quad 4_2 \leftrightarrow \chi_{50}, \quad 5_1 \leftrightarrow \chi_{32}, \quad 5_2 \leftrightarrow \chi_{43}.$$

This gives five of the seven projective indecomposable characters for block B_1, and together with Ψ_3 and Ψ_6 we obtain the basic set of projective characters displayed in Table 12.

χ	$\chi(1)$	Φ_1	Φ_2	Ψ'_3	Φ_4	Φ_5	Ψ_6	Φ_7
8	8 910	1	.	.	.	.	.	.
10	16 929	.	1	.	.	.	.	.
19	270 864	.	.	1	.	.	.	.
32	1 185 030	1	1	.	1	.	1	.
33	1 354 320	1	.	.	.	1	2	.
37	1 575 936	.	.	1	.	1	1	.
43	2 784 375	1	.	1	1	1	3	.
49	4 561 920	.	.	1	1	.	6	1
50	4 809 375	.	1	2	1	.	5	1

TABLE 13. A third basic set of projective characters for B_1

The projective character Ψ' given in the last column of Table 12 is obtained as $(\chi_2 \cdot \chi_{30}) \cdot 1_{B_1}$. We have

$$\Psi' = \Psi_3 + \Phi_4 - \Phi_7.$$

Since Φ_7 and Φ_4 are projective indecomposable, this relation implies that

$$\Psi'_3 := \Psi_3 - \Phi_7$$

is a projective character. We obtain the new basic set of projective characters displayed in Table 13. ¿From this table it is evident that in order to obtain the projective indecomposable character Φ_3 contained in Ψ'_3, we may subtract Φ_7 at most once from Ψ'_3. Likewise, we consider the decomposition

$$\Phi_6 = \Psi_6 - a \cdot \Phi_4 - b \cdot \Phi_5 - c \cdot \Phi_7, \quad \text{where } 0 \leq a, b \leq 1 \text{ and } 0 \leq c \leq 5 - a.$$

This leaves 44 possibilities for the decomposition matrix of B_1. We postpone the proof which of these possibilities holds until after our treatment of the principal block, as with little effort we can eliminate three quarters of these possibilities in parallel to determining the Brauer characters of B_0.

3.2. **Proof for the principal block B_0.** As in our treatment of G in characteristic 2, we begin by constructing several small dimensional representations. In [29] we readily find 133_1 and 133_2, which are conjugate under the outer automorphism of G, and 760. Furthermore, restricting the smallest non-trivial irreducible representation of the sporadic Baby Monster group B also available from [29] gives the composition factors

$$4\,371{\downarrow_G} = 1 + 133_1 + 133_2 + 760 + 3\,344,$$

all of which are liftable. As the tensor products $133_1 \otimes 133_1$ and $133_1 \otimes 133_2$ allow a direct treatment with the MeatAxe, we also chop these modules obtaining

$$\begin{aligned} 133_1 \otimes 133_1 &= 1 + 8\,778_1 + 8\,910 \\ 133_1 \otimes 133_2 &= 760 + 16\,929 \end{aligned}$$

in the process. Choosing basic sets of Brauer characters for B_0 and B_1 as indicated in bold face in Tables 13 and 16, respectively, we see that 8 910 and 16 929 afford two of the irreducible Brauer characters of B_1 already known. Hence, we readily obtain the Brauer character of $8\,778_1$, and applying the outer automorphism to $8\,778_1$ gives $8\,778_2$.

To determine the remaining Brauer characters, we employ condensation again. As condensation subgroups we choose $K_1 \cong 2^3.2^2.2^6$ of order 2 048, which is the largest normal 2-subgroup of the ninth maximal subgroup $M_9 \cong 2^3.2^2.2^6.(3 \times L_3(2))$ of G, and $K_2 \cong 5^2.5.5^2$ of order 3 125, which is the largest normal 5-subgroup of the tenth maximal subgroup $M_{10} \cong 5^2.5.5^2.4A_5$ of G (see [2]). A straight line programme to find M_{10} is available in [29]. Following the construction given in [23, Section 3, p. 365] we can similarly procure a straight line programme for the ninth maximal subgroup by computing its intersection with the maximal subgroup A_{12}. The normal subgroups in question are then easily found using the facilities to deal with permutation groups provided by GAP.

The reason for choosing two condensation subgroups becomes evident, if we calculate the dimensions of the condensed simple modules known via their Brauer characters: we see that the corresponding trace idempotent e_1 annihilates the pairs $(133_1, 133_2)$ and $(8\,778_1, 8\,778_2)$, whereas the trace idempotent e_2 of the second condensation subgroup K_2 annihilates 760 and 3 344. Condensing every module with both idempotents allows us to counteract the blind spots produced by condensing with just either of the two.

By [20, Theorem 2.7] condensing the non-identity representatives of the double cosets in $M_9\backslash G/M_9$ and $M_{10}\backslash G/M_{10}$ together with generators for the maximal subgroups gives generating sets for e_1FGe_1 and e_2FGe_2, consisting of 387 and 643 elements, respectively. Words for the double coset representatives can be computed using orb, realising the permutation representations of G on the 264 515? 625 cosets in $M_9\backslash G$, and the 364 041 216 cosets in $M_{10}\backslash G$, as G-orbits of vectors in suitable matrix representations of G; again we spare the details of the actual computations.

As working with 1030 generators simultaneously is cumbersome to impossible, we adopt the strategy to consider the algebras $\mathcal{C}_1 \leq e_1FGe_1$ and $\mathcal{C}_2 \leq e_2FGe_2$, each generated by the condensed elements corresponding to the following six elements

$$a,\ b,\ ab,\ ba,\ a^2,\ aba, \tag{1}$$

where a and b are standard generators of G. We apply the condensation of tensor products technique described in [12, 16] to the modules listed in Table 14, where the simple module $12\,264_1$ which we also use will be constructed in the process, and chop the condensed modules using the six group elements of (1).

For any FG-module M the idea now is to consider a composition series of $Me_i{\downarrow_{\mathcal{C}_i}}$ for $i = 1, 2$, which can be obtained with the MeatAxe. Let $\mathcal{B}_i$ be a basis of Me_i adapted to the composition series computed, i.e. with respect to $\mathcal{B}_i$ the action of elements of $\mathcal{C}_i$ on $Me_i{\downarrow_{\mathcal{C}_i}}$ is given by lower block-triangular matrices, for which the block-diagonal gives the representations on the composition factors. We can prove that every composition factor of Me_i restricts irreducibly to $\mathcal{C}_i$ by checking if the lower block-diagonal structure is preserved by all generators of e_iFGe_i with respect to the basis $\mathcal{B}_i$. Moreover, we check that two composition factors are isomorphic as e_iFGe_i-modules if they are as $\mathcal{C}_i$-modules. This is applied to the tensor products $760 \otimes 8\,778_1$ and $3\,344 \otimes 3\,344$. Thus all $\mathcal{C}_1$- and $\mathcal{C}_2$-modules appearing as composition factors in the modules can be uniquely extended to e_1FGe_1- and e_2FGe_2-modules, respectively.

For the identification which simple e_1FGe_1-module corresponds to which simple e_2FGe_2-module, we now adopt the improved matching method detailed in [22].

Using our basic sets of B_0 and B_1, together with the Brauer characters of B_2 and the characters of defect zero, it is plain to see which composition factors outside of B_0 and B_1 appear in the tensor products $760 \otimes 8\,778_1$ and $3\,344 \otimes 3\,344$. The dimensions of their condensed modules are also readily determined. These dimensions are almost sufficient to identify these modules, and thus to omit them immediately from the condensation results, except that there is a constituent of dimension 96 for the condensation subgroup K_2 being the condensed module of the irreducible Brauer character 406 296 in the block B_2 of defect 1. Running the matching algorithm shows that this constituent matches with a constituent of dimension 243 for the condensation subgroup K_1, hence can be uniquely identified. Moreover, the decompositions of both tensor products into our basic sets of B_0 and B_1 (see Table 15) show that the character χ'_{49} of the basic set of Table 13 does not occur in either decomposition. Hence by the results obtained in 3.1, the irreducible Brauer character corresponding to the projective indecomposable Φ_7 is not a constituent of either of these tensor products. Thus their restrictions to $B_0 \oplus B_1$ contain at most 26 distinct constituents.

The result of the matching algorithm run on the components in the condensed blocks of B_0 and B_1 yields the pairs of 'matched composition factors' given by the second and third columns of Table 14. Note that the algorithm does not identify the simple FG-module to which a condensed module corresponds; this will be proved in the following. More precisely, we indeed find the matching between the constituents as shown, where only $k1_1$ and $k9$ for the condensation subgroup K_1, and $c3_{1/2}$, $c8_{1/2}$ and $c16_{1/2}$ for the condensation subgroup K_2 remain 'unmatched'. This means that either we have missed existing matchings, or there is none because of non-faithful condensation. Since we have already constructed the simple modules $133_{1/2}$, 760, 3 344, $8\,778_{1/2}$, and will do so for the simple modules $12\,264_{1/2}$ later on, we can identify all of the 'unmatched' constituents as condensed simple modules, and thus show by using their Brauer characters that there indeed is no matching for them.

Thus we count 26 non-isomorphic simple modules occurring as 'matched composition factors' in the condensation results. Hence, all simple modules of the principal block B_0, and all but one simple module of B_1 arise as composition factors in the two tensor products. Furthermore, we have a bijection between these simple FG-modules and the above 'matched composition factors'.

The strategy now is as follows: We proceed through the columns of Tables 14 and 15, and compare the multiplicities of the 'matched composition factors' with the decomposition of the Brauer characters of the tensor products in question into the basic sets of B_0 and B_1. This way we collect successively the information how the basic set characters decompose into irreducible Brauer characters. Using this information, it is straightforward to determine the Brauer characters which correspond to the 'matched composition factors'. We therefore omit the elementary calculations and only state the individual results:

► Condensing the simple modules, which are explicitly known as matrix representations, first reveals

$$1e_1 \mapsto k1_2,\ 760e_1 \mapsto k1_1,\ 3\,344e_1 \mapsto k9,\ 133_1e_2 \mapsto c3_2,\ 133_2e_2 \mapsto c3_1.$$

	K_1	K_2	$133_1 \otimes 133_1$	$133_1 \otimes 133_2$	$133_1 \otimes 3\,344$	$133_1 \otimes 760$	$760 \otimes 760$	$1_{A_{12}}^G$	$760 \otimes 3\,344$	$1_{2.\mathsf{HS}.2}^G$	$133_1 \otimes 8\,778_1$	$3\,344 \otimes 3\,344$	$133_1 \otimes 12\,264_1$	$760 \otimes 8\,778_1$
760	$k1_1$	–	.	1	.	1	.	2	2	3	5	16	1	3
1	$k1_2$	$c1$	1	.	.	1	1	2	1	4	1	6	.	3
133_2	–	$c3_1$	.	.	.	1	1	2	.	2	.	4	.	1
133_1	–	$c3_2$	.	.	.	.	1	2	.	2	2	4	.	1
9 139	$k8$	$c7$	.	.	.	.	2	3	1	7	2	13	1	3
3 344	$k9$	–	.	.	2	1	2	5	4	6	2	15	.	3
$12\,264_1$	–	$c8_1$	.	.	.	1	.	1	.	2	3	10	1	2
$12\,264_2$	–	$c8_2$	.	.	.	1	.	1	.	2	3	10	1	2
$31\,768_1$	$k26_1$	$c8_3$	.	.	1	.	.	2	2	3	2	9	1	3
$31\,768_2$	$k26_2$	$c8_4$	.	.	.	2	.	2	2	3	1	9	.	4
$8\,778_1$	–	$c16_1$	1	.	.	.	1	.	.	.	4	4	.	.
$8\,778_2$	–	$c16_2$	.	.	.	1	1	.	.	.	.	4	.	.
137 236	$k50$	$c40$	.	.	.	.	.	.	1	2	.	4	1	2
147 061	$k89$	$c33$	.	.	.	.	.	.	.	1	.	2	1	2
$339\,702_2$	$k138_1$	$c96_2$	.	.	.	.	.	.	.	.	2	2	.	.
$339\,702_1$	$k138_2$	$c96_1$	.	.	.	.	.	.	.	.	.	2	.	.
255 037	$k173$	$c89$	.	.	.	.	1	2	.	3	1	6	.	.
$496\,924_1$	$k260_1$	$c168_1$	.	.	.	.	.	.	.	.	.	.	1	1
$496\,924_2$	$k260_2$	$c168_2$	.	.	.	.	.	.	.	.	.	.	.	1
783 696	$k322$	$c272$	.	.	.	.	.	.	.	.	.	1	1	1
40 338	$k18$	$c6$	.	.	.	.	.	.	.	.	.	.	.	1
8 910	$k27_1$	$c18$	1	.	.	.	1	1	.	1	.	2	.	1
16 929	$k27_2$	$c9$	.	1	.	.	.	1	.	1	.	2	.	1
270 864	$k105$	$c96$	.	.	.	.	1	.	2	.	.	2	.	1
1 159 191	$k618$	$c387$	.	.	.	.	.	.	.	.	.	1	.	.
1 305 072	$k702$	$c384$	.	.	.	.	.	.	1	.	.	1	.	2

TABLE 14. Condensation results for $p = 3$ for HN

► $133_1 \otimes 133_1$ and $133_1 \otimes 133_2$: These are condensed to match the composition factors, which we have already determined but not computed as explicit matrix representations:

$$8\,778_1 e_2 \mapsto c16_1,\ 8\,778_2 e_2 \mapsto c16_2,\ 8\,910 e_1 \mapsto k27_1,\ 16\,929 e_1 \mapsto k27_2.$$

χ	$\chi(1)$	$133_1 \otimes 133_1$	$133_1 \otimes 133_2$	$133_1 \otimes 3\,344$	$133_1 \otimes 760$	$760 \otimes 760$	$1_{A_{12}}^G$	$760 \otimes 3\,344$	$1_{2.\mathrm{HS}.2}^G$	$133_1 \otimes 8\,778_1$	$3\,344 \otimes 3\,344$	$133_1 \otimes 12\,264_1$	$760 \otimes 8\,778_1$	$3\,344 \otimes 9\,139_1$
1	1	1	.	.	.	1	1	.	1	.	1	1	1	2
2	133	.	.	.	.	.	1	.	.	1	.	−2	−1	−1
3	133	.	.	.	1	.	1	.	.	−1	.	−2	−1	−1
4	760	.	1	.	.	.	1	1	.	.	.	−1	.	−2
5	3344	.	.	1	.	1	.	1	.	−1	−2	−1	−2	−2
6	8778	1	.	.	−1	1	−1	.	−1	1	−1	1	.	1
7	8778	.	.	.	1	1	.	.	.	.	1	−2	−1	−3
9	9405	.	.	.	.	1	1	.	2	1	4	2	2	3
11	35112	.	.	1	−1	.	1	1	.	1	3	3	2	3
12	35112	.	.	.	2	.	2	1	1	1	5	−1	2	−1
13	65835	.	.	.	1	.	1	.	1	1	2	−3	−1	−4
17	214016	.	.	.	.	.	.	1	1	.	1	.	1	.
18	267520	.	.	.	.	1	2	.	2	1	4	.	.	.
21	374528	.	.	.	.	.	.	.	.	.	2	1	.	2
22	374528	.	.	.	.	.	.	.	.	2	2	1	.	2
24	653125	.	.	.	.	.	.	.	1	.	2	.	.	3
25	656250	.	.	.	.	.	.	.	.	.	.	1	1	1
26	656250	.	.	.	.	.	.	.	.	.	.	1	1	1
29	1053360	.	.	.	.	.	.	.	.	.	1	1	1	1
35	1361920	.	.	.	.	.	.	.	.	.	.	−1	.	.
8	8910	1	.	.	.	1	1	.	1	.	1	.	.	.
10	16929	.	1	.	.	.	1	.	1	.	1	.	1	.
19	270864	.	.	.	.	1	.	1	.	.	1	.	.	1
32	1185030	.	.	.	.	.	.	.	.	.	1	.	.	1
33	1354320	.	.	.	.	.	.	.	.	.	.	.	1	.
37	1575936	.	.	.	.	.	.	1	.	.	1	.	1	1
49	4561920	.	.	.	.	.	.	.	.	.	.	.	.	1

TABLE 15. Decompositions into basic sets for B_0 and B_1

► $133_1 \otimes 3\,344$: This yields the Brauer characters $31\,768_1$ and its conjugate $31\,768_2$:

$$31\,768_1 e_1 \mapsto k26_1,\ 31\,768_2 e_1 \mapsto k26_2.$$

► $133_1 \otimes 760$: The only basic set character needed to express the Brauer character of $133_1 \otimes 760$ whose modular constituents are not yet known is χ_{13}, of degree $65\,835$.

Hence we obtain

$$\chi'_{13} = 1 + 760 + 8\,778_1 + c8_1 + c8_2 + 31\,768_1, \tag{2}$$

where we denote the unknown Brauer characters in this decomposition by their corresponding condensed modules. To determine these Brauer characters, we analyse the condensed tensor product with the MeatAxe. Its socle series is revealed to be

$$\boxed{\begin{array}{c} c8_4 \\ c8_2 \\ c16_2 \\ c8_1 \oplus c1 \\ c8_4 \oplus c3_1 \end{array}}. \tag{3}$$

Analogously to our approach in Section 2.2, we construct a basis for the uncondensed submodule of Loewy length two, with head $c8_1$ and socle $c8_4 \oplus c3_1$. With it we construct a representation of the module condensing to $c8_1$, giving $12\,264_1$. As $133_1 \otimes 760$ is self-dual, but the contragredient of $12\,264_1$ gives a new simple module $12\,264_2$, which can be checked with the representation, we can derive from the socle series in (3) that

$$12\,264_1 e_2 \mapsto c8_1,\ 12\,264_2 e_2 \mapsto c8_2.$$

¿From (2) we explicitly obtain the Brauer character of the sum $12\,264_1 + 12\,264_2$. Let σ denote the trivial extension of the nontrivial Galois automorphism of $\mathbb{Q}(\sqrt{5})$ to the algebraic number field containing all values of the ordinary characters. The character table of G (see [2]) reveals that the outer automorphism acts on the ordinary characters as the product of σ and complex conjugation. Now, since $12\,264_2$ is the contragredient of $12\,264_1$, and both are interchanged by the action of the outer automorphism (which can again be checked with the representations obtained and a straightline programme for the action of the outer automorphism available in [29]), their characters are invariant under the action of the Galois group of $\mathbb{Q}(\sqrt{5})$. It therefore follows that $12\,264_1$ and $12\,264_2$ are rational on all classes except the pairs $19A/B$ and $40A/B$ (note that irrationalities on $35A/B$ only appear as values of defect zero characters). Since we have constructed representations for $12\,264_1$, and straight line programs for conjugacy class representatives of G are available in [29], we can compute representatives for $19A$ and $40A$, and the Brauer character value of the respective representative. We find that $12\,264_1$ assumes the value $b19$ on $19A$ and $i10$ on $40A$, using the notation in [2]. The values on all classes with rational values are simply obtained by dividing all the values of the character sum $12\,264_1 + 12\,264_2$ by 2. Thus, this construction yields the Brauer characters $12\,264_1$ and $12\,264_2$.

► $760 \otimes 760$ and $1_{A_{12}}^G$: Considering both columns of Tables 14 and 15 gives the equations

$$\begin{aligned} \chi'_9 + \chi'_{18} &= 133_1 + 133_2 + 3\,344 + 2 \times k8 + k173, \\ \chi'_9 + 2\chi'_{18} &= 133_1 + 133_2 + 2 \times 3\,344 + 3 \times k8 + 2 \times k173. \end{aligned}$$

This allows us to derive the constituents of χ'_9 and χ'_{18} as

$$\begin{aligned} \chi'_9 &= 9\,139 + 133_1 + 133_2, \\ \chi'_{18} &= 255\,037 + 9\,139 + 3\,344, \end{aligned}$$

and considering the component of $760 \otimes 760$ in B_1 we recover the Brauer character 270 864, hence we see that

$$9\,139\,e_1 \mapsto k8,\ 255\,037\,e_1 \mapsto k173,\ 270\,864\,e_1 \mapsto k105.$$

▶ $760 \otimes 3\,344$ and $1^G_{2.\mathsf{HS}.2}$: From $1^G_{2.\mathsf{HS}.2}$ we obtain the relation

$$\begin{aligned} k50 + k89 = \chi'_{24} - 1 - 2 \times 9\,139 - 2 \times 3\,344 - 12\,264_1 - 12\,264_2 \\ - 31\,768_1 - 31\,768_2 - 255\,037. \end{aligned}$$

In particular we derive from this decomposition that both $k89$ and $k50$ belong to the principal block.

Now considering $760 \otimes 3\,344$, from the decomposition into the basic sets in Table 15 we infer that, since $k50$ lies in B_0, the other occurring unknown simple module $k702$ lies in B_1. With the help of the basic sets we may separate the character of the tensor product restricted to B_0 and B_1 into its block components, and obtain via

$$\begin{aligned} k50 &= \chi'_{17} - 1 - 760 - 3\,344 - 31\,768_1 - 31\,768_2 - 9\,139, \\ k702 &= \chi'_{37} - 270\,864 \end{aligned}$$

the Brauer characters 137 236 of B_0 and 1 305 072 of B_1.

Reconsidering the above relation derived from $1^G_{2.\mathsf{HS}.2}$ yields the Brauer character 147 061 of B_0, thus

$$137\,236\,e_1 \mapsto k50,\ 147\,061 \mapsto k89,\ 1\,305\,072\,e_1 \mapsto k702.$$

▶ $133_1 \otimes 8\,778_1$: Here we obtain the decomposition

$$\chi'_{22} = 2 \times 760 + 12\,264_1 + 12\,264_2 + 8\,778_1 + k138_1.$$

Hence we get an irreducible Brauer character of degree 339 702, which we call $339\,702_2$ as it appears in the second basic set character of degree 374 528. Applying the outer automorphisms gives $339\,702_1$, hence

$$339\,702_2\,e_1 \mapsto k138_1,\ 339\,702_1\,e_1 \mapsto k138_2.$$

▶ $3\,344 \otimes 3\,344$: Again separating the character into its block components with our basic sets, we see that this tensor product yields one yet unknown simple module of B_0 and likewise one yet unknown simple module of B_1. Thus we obtain a Brauer character 783 696 of B_0, and another Brauer character 1 159 191 of B_1. Computing the dimensions of the corresponding simple condensed modules with these characters, we see that

$$783\,696\,e_1 \mapsto k322,\ 1\,159\,191\,e_1 \mapsto k618.$$

▶ $133_1 \otimes 12\,264_1$: This tensor product gives the relation

$$\chi'_{25} + \chi'_{26} - \chi'_{35} = k260_1 + 1 + 12\,264_1 + 147\,061.$$

Therefore we have a new Brauer character $496\,924_1$. ¿From the latter we obtain its contragredient $496\,924_2$, hence

$$496\,924_1\,e_1 \mapsto k260_1,\ 496\,924_2\,e_1 \mapsto k260_2.$$

Thus we have determined all irreducible Brauer characters of B_0 at this stage, and we give the 3-modular decomposition matrix of this Block in Table 16.

► $760 \otimes 8\,778_1$: Finally, considering the last column of Table 14 we obtain another Brauer character of B_1: we derive the decomposition

$$\chi'_{33} = k18 + 1\,305\,072 + 8\,910,$$

which immediately yields the Brauer character 40 338, hence

$$40\,338\, e_1 \mapsto k18.$$

χ	$\chi(1)$	Φ_1	Φ_2	Φ_3	Φ_4	Φ_5	Φ_6	Φ_7	Φ_8	Φ_9	Φ_{10}	Φ_{11}	Φ_{12}	Φ_{13}	Φ_{14}	Φ_{15}	Φ_{16}	Φ_{17}	Φ_{18}	Φ_{19}	Φ_{20}
1	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
2	133	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
3	133	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
4	760	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
5	3 344	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
6	8 778	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.
7	8 778	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.
9	9 405	.	1	1	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.
11	35 112	.	.	.	.	1	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.
12	35 112	.	.	.	.	1	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.
13	65 835	1	.	.	1	.	1	.	.	1	1	1	.	.	.	.	.	.	.	.	.
14	65 835	1	.	.	1	.	.	1	.	1	1	.	1	.	.	.	.	.	.	.	.
17	214 016	1	.	.	1	1	.	.	1	.	.	1	1	1	.	.	.	.	.	.	.
18	267 520	.	.	.	.	1	.	.	1	.	.	.	.	.	.	1	.	.	.	.	.
20	365 750	1	1	1	1	1	.	.	2	1	1	1	1	.	.	1	.	.	.	.	.
21	374 528	.	.	.	2	.	.	1	.	1	1	.	.	.	.	.	1	.	.	.	.
22	374 528	.	.	.	2	.	1	.	.	1	1	.	.	.	.	.	.	1	.	.	.
24	653 125	1	.	.	1	2	.	.	2	1	1	1	1	1	1	1	.	.	.	.	.
25	656 250	1	.	.	.	.	.	.	.	1	.	.	.	.	1	.	.	.	1	.	.
26	656 250	1	.	.	.	.	.	.	.	.	1	.	.	.	1	.	.	.	.	1	.
27	718 200	.	.	1	2	2	.	1	2	1	1	1	1	.	.	1	1	.	.	.	.
28	718 200	.	1	.	2	2	1	.	2	1	1	1	1	.	.	1	.	1	.	.	.
29	1 053 360	.	.	.	3	.	1	1	.	2	2	1	1	1	.	.	.	.	.	.	1
34	1 361 920	.	.	.	3	3	.	.	2	2	2	1	1	1	1	1	1	1	.	.	.
35	1 361 920	.	.	.	2	1	.	.	1	1	1	.	.	.	1	.	1	1	.	1	.
36	1 361 920	.	.	.	2	1	.	.	1	1	1	.	.	.	1	.	1	1	1	.	.
40	2 375 000	2	1	1	3	3	.	.	3	2	2	1	1	.	2	1	1	1	1	1	.
41	2 407 680	2	1	1	1	1	.	.	2	2	2	2	2	1	2	.	.	.	1	1	1
42	2 661 120	2	1	1	4	2	.	.	4	3	3	1	1	.	2	2	1	1	1	1	.
45	3 200 000	1	1	1	3	2	.	.	2	3	3	1	1	1	3	.	1	1	1	1	1
46	3 424 256	2	1	1	5	2	1	1	3	4	4	2	2	1	2	1	1	1	1	1	1
48	4 156 250	1	.	.	7	4	1	1	2	5	5	2	2	2	3	.	2	2	1	1	1
54	5 878 125	2	1	1	9	5	1	1	3	6	6	2	2	1	4	.	3	3	2	2	1

TABLE 16. The decomposition matrix of the principal 3-block B_0 of HN.

χ	$\chi(1)$	Φ_1	Φ_2	Φ_3	Φ_4	Φ_5	Φ_6	Φ_7
8	8 910	1	.	.	.	.	.	.
10	16 929	.	1	.	.	.	.	.
19	270 864	.	.	1	.	.	.	.
32	1 185 030	1	1	.	1	.	.	.
33	1 354 320	1	.	.	.	1	1	.
37	1 575 936	.	.	1	.	1	.	.
43	2 784 375	1	.	1	1	1	1	.
49	4 561 920	.	.	.	1	.	1	1
50	4 809 375	.	1	1	1	.	.	1

TABLE 17. The decomposition matrix of the 3-block B_1 of HN

3.3. **The proof for the block B_1, part 2.** At the end of Section 3.1 we were left with 44 possible decomposition matrices. Having determined all but one irreducible Brauer character of B_1, it is now elementary that $a = 1 = b$, and thus only ten cases remain.

To obtain more information on the last unknown Brauer character of B_1, we condense the tensor product $3\,344 \otimes 9\,139$, which may be done over the field with three elements, making the computation more efficient. As we have complete knowledge of all other blocks, it is trivial to only consider the part of the tensor product lying in B_1. It is furthermore sufficient to only condense with respect to the condensation subgroup K_2. We obtain the composition factors

$$8\,910,\ 16\,929,\ 40\,338,\ 2 \times 270\,864,\ 2 \times 1\,159\,191,\ 1\,305\,072,\ c1\,047,$$

where $c1\,047$ may possibly be just a composition factor of the condensed last simple module restricted to $\mathcal{C}_2$. A comparison with the basic set of B_1 (see the last column of Table 15) yields the relation

$$\chi'_{49} = 40\,338 + 1\,159\,191 + c1\,047,$$

from which we obtain a Brauer atom of degree $3\,362\,391$. The latter is a lower bound for the degree of the last irreducible Brauer character. Inspecting its possible degrees given by the ten possible decomposition matrices, we see that the obtained lower bound is in fact maximal, that is we have $c = 4$ and $\Phi_3 = \Psi'_3 - \Phi_7$. Hence the atom is the character sought, and we give the decomposition matrix of B_1 in Table 17.

3.4. **The Automorphism Group in Characteristic 3.** The characters of the automorphism group $G.2$ of G fall into seventeen 3-blocks, five of which have positive defect: the principal block B_0 of defect $d = 6$, having $k = 42$ ordinary and $l = 22$ modular characters, two blocks of defect $d = 2$, each having $k = 9$ ordinary and $l = 7$ modular characters, covering the block B_1 of defect 2 of G, and two blocks of defect $d = 1$, each having $k = 3$ ordinary and $l = 2$ modular characters, covering the block B_2 of defect 1 of G.

As G is a normal subgroup of $G.2$, by Clifford's theorem the irreducible Brauer characters of $G.2$ come in two flavours: if an irreducible Brauer character of G is not invariant under the action of the outer automorphism, then induction to $G.2$ yields an irreducible Brauer character; if the character is invariant, then it possesses two extensions to $G.2$, where one extension may be derived from the other by changing the signs of its values on the outer classes. As it turns out, we may distinguish both extensions by the sign of their value on the outer class $2C$. Hence, denoting a Brauer character by its degree, we additionally affix a superscript $+$ or $-$ to identify the extension.

As each of the blocks B_1 and B_2 of G is covered by two blocks of the automorphism group $G.2$ of the same defect, restriction defines a Morita equivalence between such a block of $G.2$ and B_1 respectively B_2 of G. Therefore the decomposition matrix for either block of defect 2 is given in Table 17, and the decomposition matrix of either block of defect 1 is given in Table 9. We may therefore focus on the principal block B_0 of $G.2$.

Owing to the abundance of pairs of characters conjugate under the outer automorphism, most Brauer characters of B_0 are immediately obtained through induction. We must therefore only determine the extensions of the invariant Brauer characters.

¿From Section 3 it is immediate that the invariant Brauer characters 1, 760, and 3 344 of G are liftable. Furthermore, we may deduce that the restriction of the ordinary character $9\,405^{\pm}$ has the constituents 266 and $9\,139^{\pm}$. Therefore we only have to establish the extensions of the four Brauer characters 137 236, 147 061, 255 037, and 783 696 of G. Of course, one extension determines the other, and therefore it suffices to just determine one member of every pair.

This is achieved by using condensation again. For the automorphism group, we choose the same setup as for the simple group, which allows us to use the computational infrastructure already in place. In other words, owing to our previous work in Section 3.2, we can again use the six elements of (1) in G plus one element α of the class $2C$ of $G.2$, which is a member of a set of standard generators of $G.2$ (see [29]). Let $\mathcal{D}_i \leq e_iFGe_i$ be the algebra generated by these elements, hence we have $\mathcal{C}_i \leq \mathcal{D}_i$. As our condensation subgroups are contained in G we have $Ve_i{\downarrow_{e_iFGe_i}} = (V{\downarrow_G})e_i{\downarrow_{e_iFGe_i}}$ for any $F[G.2]$-module V. Hence, the dimensions of condensed modules are clear from the onset.

Moreover, if a simple FG-module S is not invariant, that is there is a simple FG-module $T \ncong S$ being the image of S under the outer automorphism, and $S{\uparrow^{G.2}}$ is simple, then we have $((S{\uparrow^{G.2}})e_i{\downarrow_{\mathcal{D}_i}}){\downarrow_{\mathcal{C}_i}} = Se_i{\downarrow_{\mathcal{C}_i}} \oplus Te_i{\downarrow_{\mathcal{C}_i}}$, where $Se_i{\downarrow_{\mathcal{C}_i}} \ncong Te_i{\downarrow_{\mathcal{C}_i}}$. Thus $(S{\uparrow^{G.2}})e_i{\downarrow_{\mathcal{D}_i}}$ either is simple, of dimension twice the dimension of $Se_i{\downarrow_{\mathcal{C}_i}}$, or has two non-isomorphic constituents extending $Se_i{\downarrow_{\mathcal{C}_i}}$ and $Te_i{\downarrow_{\mathcal{C}_i}}$, respectively. If S is invariant and it extends to $G.2$ as $S^{\pm}$, then $(S^{\pm}e_i{\downarrow_{\mathcal{D}_i}}){\downarrow_{\mathcal{C}_i}} = Se_i{\downarrow_{\mathcal{C}_i}}$ implies that both $S^{\pm}e_i{\downarrow_{\mathcal{D}_i}}$ are simple. Moreover, since the actions of α on $S^{\pm}$ differ just by sign, we conclude that the actions of $e_i\alpha e_i$ on $S^{\pm}e_i$ also differ by sign only. Hence $e_i\alpha e_i$ acts by the zero map on $S^{+}e_i$ if and only if it does so on $S^{-}e_i$, and in this case we have $S^{+}e_i{\downarrow_{\mathcal{D}_i}} \cong S^{-}e_i{\downarrow_{\mathcal{D}_i}}$. Otherwise we have $S^{+}e_i{\downarrow_{\mathcal{D}_i}} \ncong S^{-}e_i{\downarrow_{\mathcal{D}_i}}$, as by Schur's Lemma any isomorphism is scalar and hence commutes with the action of $e_i\alpha e_i$. Thus to decide which case occurs we only have to consider the action of $e_i\alpha e_i$ on either of $S^{\pm}e_i$.

The modules considered are given in Table 18, all of whose restrictions to G have already been considered in Table 14, where in Table 18 we only give the composition factors lying in the principal block, since restriction to B_0 of the condensation results

	K_1	K_2	$760^+ \otimes 760^+$	$760^+ \otimes 3\,344^+$	$1_{4.\mathsf{HS}.2}^{G.2}$	$3\,344^+ \otimes 3\,344^+$
1^+	$k1'_1$	$c1'_1$	1	.	4	5
1^-	$k1'_2$	$c1'_2$	.	1	.	1
266	–	$c6'$	1	.	2	4
760^+	$k1'_3$	–	.	2	1	5
760^-	$k1'_4$	–	.	.	2	11
$3\,344^+$	$k9'_1$	–	2	1	6	11
$3\,344^-$	$k9'_2$	–	.	3	.	4
$9\,139^+$	$k8'_1$	$c7'_1$	2	.	5	8
$9\,139^-$	$k8'_2$	$c7'_2$	.	1	2	5
$17\,556$	–	$c32'$	1	.	.	4
$24\,528$	–	$c16'_1$	.	.	2	10
$63\,536$	$k52'$	$c16'_2$	.	2	3	9
$137\,236^+$	$k50'_1$	$c40'_1$	.	.	2	4
$137\,236^-$	$k50'_2$	$c40'_2$	.	1	.	.
$147\,061^+$	$k89'$	$c33'$	.	.	1	2
$147\,061^-$			.	.	.	.
$255\,037^+$	$k173'_1$	$c89'_1$	1	.	3	5
$255\,037^-$	$k173'_2$	$c89'_2$	.	.	.	1
$679\,404$	$k276'$	$c192'$	.	.	.	2
$783\,696^+$			.	.	.	.
$783\,696^-$	$k322'$	$c272'$	.	.	.	1
$993\,848$			.	.	.	.

TABLE 18. Condensation results for $p = 3$ for HN.2

is easily achieved. Thus the matching between the simple modules of the condensed algebras $\mathcal{D}_1$ and $\mathcal{D}_2$ can be derived from the matching obtained in Section 3.2. We moreover see that $(S{\uparrow}^{G.2})e_i{\downarrow}_{\mathcal{D}_i}$ is simple for all non-invariant simple FG-modules occurring, and $S^{\pm}e_i{\downarrow}_{\mathcal{D}_i}$ is simple for all invariant simple FG-modules occurring. Furthermore, we can conclude from the condensation results that $S^+e_i{\downarrow}_{\mathcal{D}_i}$ and $S^-e_i{\downarrow}_{\mathcal{D}_i}$ are non-isomorphic. The matching of a condensed module to a member of a pair of extensions follows from the characters of the considered modules and through the subsequent analysis of the columns of Table 18 as follows.

The representations needed to condense the modules of Table 18 are again obtained by restricting the 4 371-dimensional representation of the Baby Monster group B to $G.2$:

$$4\,371{\downarrow}_{G.2} = 1^- + 266 + 760^- + 3\,344^-,$$

from which we immediately obtain 760^+ and $3\,344^+$.

► To identify the condensed modules belonging to 760^+ and $3\,344^+$, we condense them individually to reveal that

$$760^+e_1 \mapsto k1'_3,\ 3\,344^+e_1 \mapsto k9'_1;$$

to distinguish the composition factors found here from earlier ones notationally we add a dash.

► From the character of $760^+ \otimes 760^+$ it is clear that the extension $255\,037^+$ is a constituent: The restriction to B_0 of $760^+ \otimes 760^+$ is the sum $\chi'_1 + \chi'_6 + \chi'_8 + \chi'_{11} + \chi'_{20}$. By the previous section and the above we know that χ'_{20} has as its constituents $3\,344^+$ and extensions of $255\,037$, and $9\,139$. As χ'_{20} takes the value 384 on the class $2C$, and $3\,344^+$ and $9\,139^{\pm}$ take the values 36 and ± 51, respectively, the consituent of χ'_{20} of degree $255\,037$ also assumes a positive value on $2C$, i.e. it is $255\,037^+$.

Hence for the component in B_0, Table 18 leaves the two possibilities

$$1^+ + 266 + 2 \times 3\,344^+ + 17\,556 + 2 \times 9\,139^{\pm} + 255\,037^+,$$

i.e., either $9\,139^+$ or $9\,139^-$ is a constituent. Assuming $9\,139^-$ is, it turns out that the character χ'_{20} cannot be written as a linear combination of the known Brauer characters with non-negative integer coefficients. Hence, $9\,139^+$ is a constituent, from which we obtain $255\,037^+$ explicitly. Furthermore, we identify the condensed modules

$$9\,139^+e_1 \mapsto k8'_1,\ 255\,037^+e_1 \mapsto k173'_1.$$

► Each of the three modules $760^+ \otimes 3\,344^+$, $1_{4.\mathsf{HS}.2}^{G.2}$ and $3\,344^+ \otimes 3\,344^+$ only introduces one new composition factor which is an extension of a simple FG-module. Hence we may easily deduce their corresponding Brauer characters $137\,236^-$ and $147\,061^+$, and $783\,969^-$, where

$$137\,236^-e_1 \mapsto k50'_2,\ 147\,061^+e_1 \mapsto k89',\ 783\,969^-e_1 \mapsto k322'.$$

The decomposition matrix for the principal 3-block of $G.2$ is given in Table 19.

χ	$\chi(1)$	Φ_1	Φ_2	Φ_3	Φ_4	Φ_5	Φ_6	Φ_7	Φ_8	Φ_9	Φ_{10}	Φ_{11}	Φ_{12}	Φ_{13}	Φ_{14}	Φ_{15}	Φ_{16}	Φ_{17}	Φ_{18}	Φ_{19}	Φ_{20}	Φ_{21}	Φ_{22}
1	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
2	1	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
3	266	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
4	760	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
5	760	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
6	3 344	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
7	3 344	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
8	17 556	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.
11	9 405	.	.	1	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.
12	9 405	.	.	1	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.
15	70 224	.	.	.	.	.	1	1	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.
16	131 670	1	1	.	1	1	.	.	.	.	1	2	1	.	.	.	.	.	.	.	.	.	.
18	214 016	1	.	.	.	1	1	.	1	.	.	.	1	1	.	.	.	.	.	.	.	.	.
19	214 016	.	1	.	1	.	.	1	.	1	.	.	1	.	1	.	.	.	.	.	.	.	.
20	267 520	.	.	.	.	.	1	.	1	.	.	.	.	.	.	.	.	1	.	.	.	.	.
21	267 520	.	.	.	.	.	.	1	.	1	.	.	.	.	.	.	.	.	1	.	.	.	.
24	365 750	1	.	1	1	.	1	.	1	1	.	1	1	.	.	.	.	1	.	.	.	.	.
25	365 750	.	1	1	.	1	.	1	1	1	.	1	1	.	.	.	.	.	1	.	.	.	.
26	749 056	.	.	.	2	2	.	.	.	.	1	2	.	.	.	.	.	.	.	1	.	.	.
29	653 125	1	.	.	.	1	2	.	1	1	.	1	1	1	.	1	.	1	.	.	.	.	.
30	653 125	.	1	.	1	.	.	2	1	1	.	1	1	.	1	.	1	.	1	.	.	.	.
31	1 312 500	1	1	.	.	.	.	.	.	.	.	1	.	.	.	1	1	.	.	.	.	.	1
32	1 436 400	.	.	1	2	2	2	2	2	2	1	2	2	.	.	.	.	1	1	1	.	.	.
33	1 053 360	.	.	.	3	.	.	.	.	.	1	2	1	.	1	.	.	.	.	.	1	.	.
34	1 053 360	.	.	.	.	3	.	.	.	.	1	2	1	1	.	.	.	.	.	.	.	1	.
40	1 361 920	.	.	.	1	2	2	1	1	1	.	2	1	1	.	1	.	1	.	1	.	.	.
41	1 361 920	.	.	.	2	1	1	2	1	1	.	2	1	.	1	.	1	.	1	1	.	.	.
42	2 723 840	.	.	.	2	2	1	1	1	1	.	2	.	.	.	1	1	.	.	2	.	.	1
49	2 375 000	1	1	1	2	1	2	1	2	1	.	2	1	.	.	1	1	1	.	1	.	.	1
50	2 375 000	1	1	1	1	2	1	2	1	2	.	2	1	.	.	1	1	.	1	1	.	.	1
51	2 407 680	1	1	1	.	1	1	.	.	2	.	2	2	1	.	2	.	.	.	.	1	.	1
52	2 407 680	1	1	1	1	.	.	1	2	.	.	2	2	.	1	.	2	.	.	.	.	1	1
53	2 661 120	1	1	1	2	2	1	1	2	2	.	3	1	.	.	1	1	1	1	1	.	.	1
54	2 661 120	1	1	1	2	2	1	1	2	2	.	3	1	.	.	1	1	1	1	1	.	.	1
59	3 200 000	.	1	1	2	1	1	1	.	2	.	3	1	.	1	2	1	.	.	1	1	.	1
60	3 200 000	1	.	1	1	2	1	1	2	.	.	3	1	1	.	1	2	.	.	1	.	1	1
61	3 424 256	1	1	1	2	3	2	.	1	2	1	4	2	1	.	2	.	1	.	1	1	.	1
62	3 424 256	1	1	1	3	2	.	2	2	1	1	4	2	.	1	.	2	.	1	1	.	1	1
65	4 156 250	.	1	.	4	3	2	2	1	1	1	5	2	1	1	2	1	.	.	2	1	.	1
66	4 156 250	1	.	.	3	4	2	2	1	1	1	5	2	1	1	1	2	.	.	2	.	1	1
77	5 878 125	1	1	1	4	5	3	2	1	2	1	6	2	1	.	3	1	.	.	3	1	.	2
78	5 878 125	1	1	1	5	4	2	3	2	1	1	6	2	.	1	1	3	.	.	3	.	1	2

TABLE 19. The decomposition matrix of the principal 3-block B_0 of HN.2

References

1. Thomas Breuer, *CTblLib - a GAP package, Version 1.1.3*, http://www.gap-system.org/Packages/ctbllib.html, 2004.
2. J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, and R. A. Wilson, *Atlas of finite groups*, Oxford University Press, Eynsham, 1985, Maximal subgroups and ordinary characters for simple groups, With computational assistance from J. G. Thackray.
3. Karin Erdmann, *Algebras and semidihedral defect groups. II*, Proc. London Math. Soc. (3) **60** (1990), no. 1, 123–165.
4. The GAP Group, http://www.gap-system.org, *GAP – Groups, Algorithms, and Programming, Version 4.4.12*, 2008.
5. Anne Henke, Gerhard Hiss, and Jürgen Müller, *The 7-modular decomposition matrices of the sporadic O'Nan group*, J. London Math. Soc. (2) **60** (1999), no. 1, 58–70.
6. G. Hiss, C. Jansen, K. Lux, and R. A. Parker, *Computational Modular Character theory*, http://www.math.rwth-aachen.de/MOC/CoMoChaT.
7. G. Hiss and K. Lux, *Brauer trees of sporadic groups*, Oxford Science Publications, The Clarendon Press Oxford University Press, New York, 1989.
8. C. Jansen, K. Lux, R. Parker, and R. Wilson, *An atlas of Brauer characters*, London Mathematical Society Monographs. New Series, vol. 11, The Clarendon Press Oxford University Press, New York, 1995, Appendix 2 by T. Breuer and S. Norton, Oxford Science Publications.
9. Shigeo Koshitani and Jürgen Müller, *Broué's abelian defect group conjecture holds for the Harada-Norton sporadic simple group HN*, J. Algebra **324** (2010), no. 3, 394–429. MR 2651342
10. P. Landrock, *Finite group algebras and their modules*, London Mathematical Society Lecture Note Series, vol. 84, Cambridge University Press, Cambridge, 1983. MR 737910 (85h:20002)
11. Peter Landrock, *The non-principal 2-blocks of sporadic simple groups*, Comm. Algebra **6** (1978), no. 18, 1865–1891.
12. K. Lux, M. Neunhöffer, and F. Noeske, *Condensation of homomorphism spaces*, LMS J. Comput. Math., submitted for publication.
13. Klaus Lux, Jürgen Müller, and Michael Ringe, *Peakword condensation and submodule lattices: an application of the MEAT-AXE*, J. Symbolic Comput. **17** (1994), no. 6, 529–544.
14. Klaus Lux, Felix Noeske, and Alexander J. E. Ryba, *The 5-modular characters of the sporadic simple Harada-Norton group HN and its automorphism group HN.2*, J. Algebra **319** (2008), no. 1, 320–335.
15. Klaus Lux and Herbert Pahlings, *Representations of groups*, Cambridge Studies in Advanced Mathematics, vol. 124, Cambridge University Press, Cambridge, 2010, A computational approach. MR 2680716
16. Klaus Lux and Markus Wiegelmann, *Condensing tensor product modules*, The atlas of finite groups: ten years on (Birmingham, 1995), London Math. Soc. Lecture Note Ser., vol. 249, Cambridge Univ. Press, Cambridge, 1998, pp. 174–190. MR MR1647421 (99i:20018)
17. ———, *Determination of socle series using the condensation method*, J. Symbolic Comput. **31** (2001), no. 1-2, 163–178, Computational algebra and number theory (Milwaukee, WI, 1996).
18. J. Müller, *On endomorphism rings and character tables*, Habilitationsschrift, RWTH Aachen, 2003.
19. Jürgen Müller, Max Neunhöffer, and Felix Noeske, *orb - a GAP package, Version 3.7*, http://www.gap-system.org/Packages/orb.html, 2011.
20. Felix Noeske, *Tackling the generation problem in condensation*, J. Algebra **309** (2007), no. 2, 711–722.
21. ———, *Matching simple modules of condensed algebras*, LMS J. Comput. Math. **11** (2008), 213–222.
22. ———, *Matching simple modules of condensation algebras*, (in preparation).
23. S. P. Norton and R. A. Wilson, *Maximal subgroups of the Harada-Norton group*, J. Algebra **103** (1986), no. 1, 362–376.
24. Jørn Børling Olsson, *On 2-blocks with quaternion and quasidihedral defect groups*, J. Algebra **36** (1975), no. 2, 212–241. MR MR0376841 (51 #13016)
25. R. A. Parker, *The computer calculation of modular characters (the meat-axe)*, Computational group theory (Durham, 1982), Academic Press, London, 1984, pp. 267–274.

26. Michael Ringe, *C-MeatAxe, version 2.4*, http://www.math.rwth-aachen.de/homes/MTX/, 2009.
27. A. J. E. Ryba, *Computer condensation of modular representations*, J. Symbolic Comput. **9** (1990), no. 5-6, 591–600, Computational group theory, Part 1.
28. Jon G. Thackray, *Modular representations of some finite groups*, Ph.D. thesis, University of Cambridge, 1981.
29. R. Wilson, P. Walsh, J. Tripp, I. Suleiman, R. Parker, S. Norton, S. Nickerson, S. Linton, J. Bray, and R. Abbott, *ATLAS of Finite Group Rrepresentations*, http://brauer.maths.qmul.ac.uk/Atlas/v3, 2006.
30. R. A. Wilson, R. A. Parker, S. Nickerson, J. Bray, and T. Breuer, *AtlasRep - a GAP interface to the ATLAS of Finite Group Rrepresentations, Version 1.4.0*, http://www.gap-system.org/Packages/atlasrep.html, 2008.
31. R. A. Wilson, J. G. Thackray, R. A. Parker, F. Noeske, J. Müller, K. Lux, F. Lübeck, C. Jansen, G. Hiss, and T. Breuer, *The modular Atlas project*, http://www.math.rwth-aachen.de/~MOC, 1998.
32. Robert A. Wilson, *Standard generators for sporadic simple groups*, J. Algebra **184** (1996), no. 2, 505–515.

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