Broué's abelian defect group conjecture holds for the double cover of the Higman-Sims sporadic simple group

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Dedicated to the memory of Herbert Pahlings

Abstract

In the representation theory of finite groups, there is a well-known and important conjecture, due to Broué', saying that for any prime p, if a p-block A of a finite group G has an abelian defect group P, then A and its Brauer corresponding block B of the normaliser $N_G(P)$ of P in G are derived equivalent. We prove in this paper, that Broué's abelian defect group conjecture, and even Rickard's splendid equivalence conjecture are true for the faithful 3-block A with an elementary abelian defect group P of order 9 of the double cover $2.\mathsf{HS}$ of the Higman-Sims sporadic simple group. It then turns out that both conjectures hold for all primes p and for all p-blocks of $2.\mathsf{HS}$.

Keywords: Broué's conjecture; abelian defect group; splendid derived equivalence, double cover of the Higman-Sims sporadic simple group.

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1. Introduction and notation

In the representation theory of finite groups, one of the most important and interesting problems is to give an affirmative answer to a conjecture which was introduced by Broué around 1988 [5]. He actually conjectures the following, where the various notions of equivalences used are recalled more precisely in **1.8**:

Conjecture 1.1 (Broué's Abelian Defect Group Conjecture [5])**.** *Let $(\mathcal{K}, \mathcal{O}, k)$ be a splitting p-modular system, where p is a prime, for all subgroups of a finite group G. Assume that A is a block algebra of $\mathcal{O}G$ with a defect group P and that A_N is a block algebra of $\mathcal{O}N_G(P)$ such that A_N is the Brauer correspondent of A, where $N_G(P)$ is the normaliser of P in G. Then A and A_N should be derived equivalent provided P is abelian.*

In fact, a stronger conclusion than **1.1** is expected:

Conjecture 1.2 (Rickard's Splendid Equivalence Conjecture [48, 49])**.** *Keeping the notation, we suppose that P is abelian as in* **1.1**. *Then there should be a splendid derived equivalence between the block algebras A of $\mathcal{O}G$ and A_N of $\mathcal{O}N_G(P)$.*

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There are several cases where Conjectures **1.1** and **1.2** have been verified, albeit the general conjecture is widely open; for an overview, containing suitable references, see [7]. As for general results concerning blocks with a fixed defect group, by [29, 47, 51, 52] Conjectures **1.1** and **1.2** are proved for blocks with cyclic defect groups in arbitrary characteristic.

Moreover, in [18, (0.2)Theorem] it is shown that **1.1** and **1.2** are true for the principal block algebra of an arbitrary finite group when the defect group is elementary abelian of order 9. In view of the strategy used in [18], and of a possible future theory reducing **1.1** and **1.2** to the quasi-simple groups, it seems worthwhile to proceed with this class of groups, as far as non-principal 3-blocks with elementary abelian defect group of order 9 are concerned. Indeed, for these cases there are partial results already known, see [14, 21, 22, 23, 25, 27, 39] for instance. The present paper is another step in that programme, our main result being the following:

Theorem 1.3. *Let G be the double cover* 2.HS *of the Higman-Sims sporadic simple group, and let $(\mathcal{K}, \mathcal{O}, k)$ be a splitting* 3*-modular system for all subgroups of G. Suppose that A is the faithful block algebra of $\mathcal{O}G$ with elementary abelian defect group $P = C_3 \times C_3$ of order* 9*, and that B is the block algebra of $\mathcal{O}N_G(P)$ such that B is the Brauer correspondent of A. Then, A and B are splendidly derived equivalent, hence Conjectures* **1.1** *and* **1.2** *of Broué and Rickard hold.*

As an immediate corollary we get:

Corollary 1.4. *Broué's abelian defect group conjecture* **1.1***, and even Rickard's splendid equivalence conjecture* **1.2** *are true for all primes p and for all block algebras of $\mathcal{O}G$.*

Our strategy to prove **1.3** is similar to the ones pursued, for example, for the Janko sporadic simple group J_4 in [23, 1.6.Theorem] or the Harada-Norton sporadic simple group HN in [25, 1.3.Theorem]. Our starting point was actually to realise that the 3-decomposition matrix of A coincides (up to a suitable order of rows and columns) with the 3-decomposition matrix of the principal 3-block A' of the alternating group $\mathfrak{A}_8$ on eight letters:

A	A'					
176	1	1	.	.	.	.
176*	7	.	1	.	.	.
616	14	1	.	1	.	.
616*	20	.	1	1	.	.
56	28	.	.	.	1	.
1000	35	.	.	.	.	1
1792	56	1	1	1	.	1
1232	64	1	.	.	1	1
1232*	70	.	1	.	1	1

Here, we indicate ordinary irreducible characters just by their degrees, and complex conjugation by *. Therefore, it is quite natural to suspect that the block algebras A and A' are Morita equivalent, or even Puig equivalent. If this were true, then since Conjectures **1.1** and **1.2** have been solved for A' in [42, 43], this would immediately entail their validity for A as well. Indeed, we are able to prove:

Theorem 1.5. *We keep the notation and the assumptions as in* **1.3***, and let $G' = \mathfrak{A}_8$ be the alternating group on eight letters. Then, the block algebra A of $\mathcal{O}G$ and the principal block algebra A' of $\mathcal{O}G'$ are Puig equivalent.*

Remark 1.6. A few remarks on **1.5** are appropriate:

(a) In order to prove **1.5** in its full strength, the detailed local analysis leads to a problem similar to the one already encountered in [23, 6.14.Question]: Viewing G and G' as (unrelated) abstract groups would only allow to prove that the block algebras A and A' are Morita equivalent, but not necessarily Puig equivalent. In its consequence this would mean that we were only able to verify Broué's conjecture **1.1**, but not Rickard's conjecture **1.2** for G. To remedy this, and to circumvent [23, 6.14.Question], we use the fact that G' can be embedded as a subgroup into G, leading to an explicit configuration of groups allowing for compatible local analysis.

(b) Note that complex conjugation induces a non-trivial permutation both on the irreducible ordinary and Brauer characters of A; in terms of columns of the decomposition matrix this amounts to interchanging the first two columns. But all ordinary and Brauer characters of A' are real-valued. Hence the Puig equivalence asserted by **1.5** does not commute with the self-equivalences of the module categories of A and A' induced by taking contragredient modules.

Actually, our proof of **1.5** provides two distinct Puig equivalences, one inducing the bijection between the simple A- and A'-modules as indicated in the decomposition matrix above, the other one inducing the bijection obtained by interchanging its first two columns.

(c) As far as we have experienced, it looks that most of all non-principal 3-blocks with elementary abelian defect group P of order 9 are just Morita equivalent to certain principal 3-blocks with defect group P, see [21, 22, 23, 25], for instance. One might be tempted to say that these non-principal blocks are *pseudo-principal*. So, the non-principal block algebra A considered here is even pseudo-principal in two ways, each leading to a different 'trivial' character; and the principal block algebra A' is also pseudo-principal with a different 'trivial' character.

However, there are non-principal 3-blocks of finite groups with defect group P which are not pseudo-principal in the above sense, that is they are not Morita equivalent to any principal 3-block: For example, it has been already noted in [17, Remark 4.4] that the non-principal 3-block with defect group P of the Higman-Sims sporadic simple group HS has this property, and the faithful 3-blocks of $4.\mathsf{M}_{22}$ described in [39] have as well.

Contents 1.7. This paper is organised as follows: In §2 we recall a few of the most important ingredients of our proofs. In §3 we present the local data related to $G' = \mathfrak{A}_8$. In §4 we present the local data related to $G = 2.\mathsf{HS}$, and relate the groups G' and G; in particular we comment on how the explicit embedding is achieved in a computational setting. In §5 we proceed to give a stable equivalence for A and its Brauer correspondent. In §6 we determine the images of the simple A-modules with respect to this stable equivalence. In §7 we finally complete the proofs of **1.3**, **1.4** and **1.5**, and we also give details on the phenomena in **1.6**(a) and **1.6**(b).

A further comment on the computational contents of the present paper is in order: The general paradigm of course being to proceed towards theoretical insights, the presentation of our results is based as much as possible on general principles. Still, computations are an important and non-trivial part of the picture, either by paving a way for subsequent theoretical analysis, or by setting in as soon as theoretical principles fail.

As tools, we use the computer algebra system GAP [10], to calculate with permutation groups as well as with ordinary and Brauer characters. We also make use of the data library [4], in particular allowing for easy access to the data compiled in [8, 13, 58], and of the interface [57] to the data library [59]. Moreover, we use the computer algebra system MeatAxe [50] to handle matrix representations over finite fields, as well as its extensions to compute submodule lattices [33, 38], radical and socle series [36], and homomorphism spaces, endomorphism rings and direct sum decompositions [34, 35].

In order to facilitate explicit computations, we use 'small' finite fields, but we always make sure, silently, that these are chosen such that the computational results thus obtained remain valid without change after scalar extension to the fixed field of positive characteristic which is 'large enough' in the sense of **1.8** below.

Notation/Definition 1.8. Throughout this paper, we use the standard notation and terminology as is used in [8, 40, 54]. We recall a few for convenience:

If A and B are finite dimensional k-algebras, where k is a field, we denote by mod-A, A-mod and A-mod-B the categories of finitely generated right A-modules, left A-modules and (A, B)-bimodules, respectively. We write M_A, ${}_AM$ and ${}_AM_B$ when M is a right A-module, a left A-module and an (A, B)-bimodule. A module always refers to a finitely generated right module, unless stated otherwise. We let $M^\vee = \mathrm{Hom}_A(M_A, A_A)$ be the A-dual of the A-module M, so that $M^\vee$ becomes a left A-module via $(a\phi)(m) = a{\cdot}\phi(m)$ for $a \in A$, $\phi \in M^\vee$ and $m \in M$. We denote by $\mathrm{soc}(M)$ and $\mathrm{rad}(M)$ the socle and the radical of M, respectively. For simple A-modules $S_1, \cdots, S_n$, and positive integers $a_1, \cdots, a_n$, we write that '$M = a_1 \times S_1 + \cdots + a_n \times S_n$, *as composition factors*' when the set of all composition factors are a_1 times S_1, $\cdots$, a_n times S_n. For another A-module L, we write $M \Big| L$ when M is isomorphic to a direct summand of L as an A-module. If A is self-injective, the stable module category $\underline{\text{mod}}$-A is the quotient category of mod-A with respect to the projective A-homomorphisms, that is those factoring through a projective module.

By G we always denote a finite group, and we fix a prime number p. Assume that $(\mathcal{K}, \mathcal{O}, k)$ is a splitting p-modular system for all subgroups of G, that is to say, $\mathcal{O}$ is a complete discrete valuation ring of rank one such that its quotient field $\mathcal{K}$ is of characteristic zero, and its residue field $k = \mathcal{O}/\mathrm{rad}(\mathcal{O})$ is of characteristic p, and that $\mathcal{K}$ and k are splitting fields for all subgroups of G. We denote by k_G the trivial kG-module. If X is a kG-module, then we write $X^* = \mathrm{Hom}_k(X, k)$ for the *contragredient* of X, namely, X^* is again a kG-module via $(\varphi g)(x) = \varphi(xg^{-1})$ for $x \in X$, $\varphi \in X^*$ and $g \in G$; if no confusion may arise we also call this the *dual* of X. For kG-modules X and Y we set $[X, Y]^G = \dim_k[\mathrm{Hom}_{kG}(X, Y)]$. Let H be a subgroup of G, and let M and N be a kG-module and a kH-module, respectively. Then let $M{\downarrow}_H^G = M{\downarrow}_H$ be the restriction of M to H, and let $N{\uparrow}_H^G = N{\uparrow}^G = (N \otimes_{kH} kG)_{kG}$ be the induction (induced module) of N to G. We say that a kG-module X is a *trivial source* module if X is indecomposable and X has a trivial source, see [28, II Definition 12.1]. Note that the definition here is slightly different from [54, §27 p.218] where an indecomposability is not assumed.

We denote by $\mathrm{Irr}(G)$ and $\mathrm{IBr}(G)$ the sets of all irreducible ordinary and Brauer characters of G, respectively; we write 1_G for the trivial character of G. Since the character field $\mathbb{Q}(\chi) := \mathbb{Q}(\chi(g)\ ;\ g \in G) \subseteq \mathcal{K}$ of any character $\chi \in \mathrm{Irr}(G)$ is contained in a cyclotomic field, we may identify $\mathbb{Q}(\chi)$ with a subfield of the complex number field $\mathbb{C}$, hence we may think of characters having values in $\mathbb{C}$. In particular, we write χ^* for the complex conjugate of χ, where of course χ^* is the character of the $\mathcal{K}G$-module contragredient to a $\mathcal{K}G$-module affording χ, see [40, Chap.3 §1.3 p.74]. If A is a block algebra of $\mathcal{O}G$, then we write $\mathrm{Irr}(A)$ and $\mathrm{IBr}(A)$ for the sets of all characters in $\mathrm{Irr}(G)$ and $\mathrm{IBr}(G)$ which belong to A, respectively.

Let G' be another finite group, and let V be an $(\mathcal{O}G, \mathcal{O}G')$-bimodule. Then we can regard V as a right $\mathcal{O}[G \times G']$-module via $v{\cdot}(g, g') = g^{-1}vg'$ for $v \in V$ and $g, g' \in G$. Let A and A' be block algebras of $\mathcal{O}G$ and $\mathcal{O}G'$, respectively, such that A and A' have a defect group P in common, that is, if P is considered as an abstract group, we have fixed embeddings $P \to G$ and $P \to G'$. Identifying P with the images of these embeddings, we set $\Delta P = \{(g, g) \in G \times G' \,|\, g \in P\}$. Then we say that A and A' are *Puig equivalent* if there is a Morita equivalence between A and A' which is induced by an indecomposable (A, A')-bimodule $\mathfrak{M}$ such that, as a right $\mathcal{O}[G \times G']$-module, $\mathfrak{M}$ is a trivial source module and ΔP-projective. This is equivalent to the condition that A and A' have source algebras which are isomorphic as interior P-algebras, see [46, Remark 7.5] and [32, Theorem 4.1]. We say that A and A' are *stably equivalent of Morita type* if there exists an indecomposable (A, A')-bimodule $\mathfrak{M}$ such that both ${}_A\mathfrak{M}$ and $\mathfrak{M}_{A'}$ are projective and that ${}_A(\mathfrak{M} \otimes_{A'} \mathfrak{M}^\vee)_A \cong {}_AA_A \oplus (\text{proj } (A, A)\text{-bimod})$ and ${}_{A'}(\mathfrak{M}^\vee \otimes_A \mathfrak{M})_{A'} \cong {}_{A'}A'_{A'} \oplus (\text{proj } (A', A')\text{-bimod})$. We say that A and A' are *splendidly stably equivalent of Morita type* if the stable equivalence of Morita type is induced by an indecomposable (A, A')-bimodule $\mathfrak{M}$ which is a trivial source $\mathcal{O}[G \times G']$-module and is ΔP-projective, see [32, Theorem 3.1].

We say that A and A' are *derived equivalent* if $\mathrm{D}^b(\text{mod-}A)$ and $\mathrm{D}^b(\text{mod-}A')$ are equivalent as triangulated categories, where $\mathrm{D}^b(\text{mod-}A)$ is the bounded derived category of mod-A. In that case, there even is a *Rickard complex* $M^\bullet \in \mathrm{C}^b(A\text{-mod-}A')$, where the latter is the category of bounded complexes of finitely generated (A, A')-bimodules, all of whose terms are projective both as left A-modules and as right A'-modules, such that $M^\bullet \otimes_{A'} (M^\bullet)^\vee \cong A$ in $K^b(A\text{-mod-}A)$ and $(M^\bullet)^\vee \otimes_A M^\bullet \cong A'$ in $K^b(A'\text{-mod-}A')$, where $K^b(A\text{-mod-}A)$ is the homotopy category associated with $\mathrm{C}^b(A\text{-mod-}A)$; in other words, in that case we even have $K^b(\text{mod-}A) \cong K^b(\text{mod-}A')$. We say that A and A' are *splendidly derived equivalent* if $K^b(\text{mod-}A)$ and $K^b(\text{mod-}A')$ are equivalent via a Rickard complex $M^\bullet \in \mathrm{C}^b(A\text{-mod-}A')$ as above, such that additionally each of its terms is a direct sum of ΔP-projective trivial source modules as an $\mathcal{O}[G \times G']$-module; see [31, 32]. Note that a Morita equivalence entails a derived equivalence, and that a Puig equivalence entails a splendid derived equivalence.

2. Preliminaries

Lemma 2.1 (Scott). *The following holds:*

(i) *If M is a trivial source kG-module, then M lifts uniquely (up to isomorphism) to a trivial source $\mathcal{O}G$-lattice $\widehat{M}$.*

(ii) *If M and N are both trivial source kG-modules, then $[M,N]^G = (\chi_{\widehat{M}}, \chi_{\widehat{N}})^G$.*

Proof. See [28, II Theorem 12.4 and I Proposition 14.8] and [3, Corollary 3.11.4]. □

Lemma 2.2 (Linckelmann). *Let A and B be finite-dimensional k-algebras (where deviating from our general assumption k is an arbitrary field), such that A and B are both self-injective and indecomposable as algebras, but not simple. Suppose that there is an (A,B)-bimodule M such that M induces a stable equivalence between the algebras A and B.*

(i) *If M is indecomposable then for any simple A-module S, the B-module $(S \otimes_A M)_B$ is non-projective and indecomposable.*

(ii) *If for any simple A-module S the B-module $S \otimes_A M$ is simple, then M induces a Morita equivalence between A and B.*

Proof. (i) and (ii) respectively are given in [30, Theorem 2.1(ii) and (iii)]. □

Lemma 2.3 (Fong-Reynolds). *Let H be a normal subgroup of G, and let A and B be block algebras of $\mathcal{O}G$ and $\mathcal{O}H$, respectively, such that A covers B. Let $T = T_G(B)$ be the inertial subgroup (stabiliser) of B in G. Then, there is a block algebra $\tilde{A}$ of $\mathcal{O}T$ such that $\tilde{A}$ covers B, $1_A 1_{\tilde{A}} = 1_{\tilde{A}} 1_A = 1_{\tilde{A}}$, $A = \tilde{A}^G$ (block induction), and the block algebras A and $\tilde{A}$ are Morita equivalent via a pair $(1_A{\cdot}\mathcal{O}G{\cdot}1_{\tilde{A}},\ 1_{\tilde{A}}{\cdot}\mathcal{O}G{\cdot}1_A)$, that is, the Morita equivalence is a Puig equivalence and induces a bijection*

$$\mathrm{Irr}(\tilde{A}) \to \mathrm{Irr}(A),\quad \tilde{\chi} \mapsto \tilde{\chi}{\uparrow}^G; \qquad \mathrm{Irr}(A) \to \mathrm{Irr}(\tilde{A}),\quad \chi \mapsto \chi{\downarrow}_T{\cdot}1_{\tilde{A}}$$

between $\mathrm{Irr}(\tilde{A})$ *and* $\mathrm{Irr}(A)$, *and a bijection*

$$\mathrm{IBr}(\tilde{A}) \to \mathrm{IBr}(A),\quad \tilde{\phi} \mapsto \tilde{\phi}{\uparrow}^G; \qquad \mathrm{IBr}(A) \to \mathrm{IBr}(\tilde{A}),\quad \phi \mapsto \phi{\downarrow}_T{\cdot}1_{\tilde{A}}$$

between $\mathrm{IBr}(\tilde{A})$ *and* $\mathrm{IBr}(A)$*,*

Proof. See [22, 1.5.Theorem] and [40, Chap.5, Theorem 5.10]. □

Remark 2.4. In **2.3** $\tilde{A}$ is called a *Fong-Reynolds correspondent* of A and vice versa. Note that there can be more than one Fong-Reynolds correspondent in general.

Lemma 2.5. *Let A and B be finite dimensional k-algebras (where deviating from our general assumption k is an arbitrary field), such that A and B are both self-injective. Let F be a covariant functor such that*

(1) *F is exact.*

(2) *If X is a projective A-module, then $F(X)$ is a projective B-module,*

(3) *F induces a stable equivalence from* mod-A *to* mod-B.

Then the following holds:

(i) (Stripping-off method, case of socle) *Let X be a projective-free A-module, and write $F(X) = Y \oplus (\mathrm{proj})$ for a projective-free B-module Y. Let S be a simple A-submodule of X, and set $T = F(S)$. Now, if T is a simple B-module, then we may assume that Y contains T and that*

$$F(X/S) = Y/T \oplus (\mathrm{proj}).$$

(ii) (Stripping-off method, case of radical) *Similarly, let X be a projective-free A-module, and write $F(X) = Y \oplus (\mathrm{proj})$ for a projective-free B-module Y. Let X' be an A-submodule of X such that X/X' is simple, and set $T = F(X/X')$. Now, if T is a simple B-module, then we may assume that T is an epimorphic image of Y and that*

$$\mathrm{Ker}(F(X) \twoheadrightarrow T) = \mathrm{Ker}(Y \twoheadrightarrow T) \oplus (\mathrm{proj}).$$

Proof. See [22, 1.11.Lemma] or [26, A.1.Lemma]. □

Lemma 2.6. *Let H be a proper subgroup of G, and let A and B be block algebras of kG and kH, respectively. Now, let M and M' be finitely generated (A, B)- and (B, A)-bimodules, respectively, which satisfy the following:*

(1) ${}_AM_B \mid 1_A{\cdot}kG{\cdot}1_B$ *and* ${}_BM'{}_A \mid 1_B{\cdot}kG{\cdot}1_A$.

(2) *The pair (M, M') induces a stable equivalence between* mod-A *and* mod-B.

Then we get the following:

(i) *Assume that X is a non-projective indecomposable kG-module in A with vertex Q. Then there exists a non-projective indecomposable kH-module Y in B, unique up to isomorphism, such that $(X \otimes_A M)_B = Y \oplus (\mathrm{proj})$, and Q^g is a vertex of Y for some element $g \in G$ (and hence $Q^g \subseteq H$). Since Q^g is also a vertex of X, this means that X and Y have at least one vertex in common.*

(ii) *Assume that Y is a non-projective indecomposable kH-module in B with vertex Q. Then there exists a non-projective indecomposable kG-module X in A, unique up to isomorphism, such that $(Y \otimes_B M')_A = X \oplus (\mathrm{proj})$, and Q is a vertex of X.*

(iii) *Let X, Y and Q be the as in* (i). *We may assume $Q \leqslant H$ because we can replace Q^g in* (i) *by Q. Then there is an indecomposable kQ-module L such that L is a source of both X and Y. This means that X and Y have at least one source in common.*

(iv) *Let X, Y and $Q \leqslant H$ be the same as in* (ii). *Then there is an indecomposable kQ-module L such that L is a source of both X and Y. This means that X and Y have at least one source in common.*

(v) *Let X, Y, Q and L be the same as in* (iii). *In addition, suppose that A and B have a common defect group P (and hence $P \subseteq H$) and that $H \geqslant N_G(P)$. Let f be the Green correspondence with respect to (G, P, H). If $Q \in \mathfrak{A}(G, P, H)$, see* [40, Chap.4, §4] *then we have $(X \otimes_A M)_B = f(X) \oplus (\mathrm{proj})$.*

(vi) *Let X, Y, Q and L be the same as in* (ii). *Furthermore, as in* (v), *assume that P is a common defect group of A and B, and that $H \geqslant N_G(P)$, and let f and $\mathfrak{A}$ be the same as in* (v). *Now, if $Q \in \mathfrak{A}(G, P, H)$, then we have $(Y \otimes_B M')_A = f^{-1}(Y) \oplus (\mathrm{proj})$.*

Proof. See [26, A.3.Lemma]. □

Lemma 2.7. *Set $G = \mathfrak{A}_5 \rtimes C_4 = (\mathfrak{A}_5 \times 2).2$, where the action on C_4 of $\mathfrak{A}_5$ is that C_4/C_2 acts faithfully on $\mathfrak{A}_5$. Let P be a Sylow 3-subgroup of $\mathfrak{A}_5$ (and hence $P \cong C_3$).*

(i) *There is a faithful non-principal block algebra A of kG (that is, not having the central subgroup of order 2 in its kernel) with defect group P.*

(ii) *We can write $\mathrm{Irr}(A) = \{\chi_1, \chi_2, \chi_3\}$ such that $\chi_1(1) = 1$, $\chi_2(1) = 4$, $\chi_3(1) = 5$ and $\chi_1(u) = \chi_2(u) = 1$ for any element $u \in P - \{1\}$. Moreover, we can write $\mathrm{IBr}(A) = \{\varphi_1, \varphi_2\}$ such that the 3-decomposition matrix of A is*

	φ_1	φ_2
χ_1	1	.
χ_2	.	1
χ_3	1	1

(iii) *Set $H = N_G(P)$. Then $H = \mathfrak{S}_3 \times C_4$. Let further B be the block algebra of kH that is the Brauer correspondent of A.*

(iv) *Set $M = \mathfrak{f}(A)$, where $\mathfrak{f}$ is the Green correspondence with respect to $(G \times G, \Delta P, G \times H)$. Then, M induces a Morita equivalence between A and B (and hence M induces a Puig equivalence between A and B). Further, simple kG-modules in A affording φ_1 and φ_2 are trivial source kG-modules.*

Proof. (i)-(iii) are easy. (iv) follows from (i)-(iii) and [20, Theorem 1.2]. □

3. Green correspondences for $\mathfrak{A}_8$

Notation 3.1. We introduce some further notation which we use throughout the rest of the paper. Moreover, in this section let $p = 3$.

Let G' be the alternating group on eight letters, namely, $G' = \mathfrak{A}_8$, and let A' be the principal block algebra of kG'. Note that we are abusing notation here, inasmuch as in the introductory

Section **1** we have used the same letter to denote the principal block algebra of $\mathcal{O}G'$, but no confusion will arise from that.

Since Sylow 3-subgroups of G' are isomorphic to $C_3 \times C_3$, we can assume that P is a Sylow 3-subgroup of $\mathfrak{A}_8$ as well, which is originally a Sylow 3-subgroup of $G = 2.\mathsf{HS}$, see **4.1** and **4.3**. There are exactly two conjugacy classes of G' which contain elements of order 3, that is, P has exactly two G'-conjugacy classes of subgroups of order 3. Let Q and R be representatives of these, see **5.3**(ii), and see also [8, p.22].

Let $H' = N_{G'}(P)$, and hence $H' = P \rtimes D_8$; note that the subgroups of order 3 of P still fall into two H'-conjugacy classes. Let B' be the principal block algebra of kH'; thus $B' = kH'$.

Lemma 3.2. (i) *The* 3*-decomposition matrix and the Cartan matrix of* A'*, respectively, are the following:*

	$k_{G'}$	7	13	28	35
χ'_1	1	.	.	.	.
χ'_7	.	1	.	.	.
χ'_{14}	1	.	1	.	.
χ'_{20}	.	1	1	.	.
χ'_{28}	.	.	.	1	.
χ'_{35}	.	.	.	.	1
χ'_{56}	1	1	1	.	1
χ'_{64}	1	.	.	1	1
χ'_{70}	.	1	.	1	1

	$P(k_{G'})$	$P(7)$	$P(13)$	$P(28)$	$P(35)$
$k_{G'}$	4	1	2	1	2
7	1	4	2	1	2
13	2	2	3	0	1
28	1	1	0	3	2
35	2	2	1	2	4

(ii) *All simple* kG'*-modules* $k_{G'}, 7, 13, 28, 35$ *in* A' *have* P *as their vertices.*

Proof. (i) follows from [13, A_8 (mod 3)] and [8, p.22], and for (ii) see [16, 3.7.Corollary]. □

Notation 3.3. We use the notation $\chi'_1, \cdots, \chi'_{70}$ and $k_{G'}, 7, 13, 28, 35$ as in **3.2**, where the numbers mean the degrees (dimensions) of characters (modules).

Lemma 3.4. *The following holds:*

(i) *The character table of* $H' = P \rtimes D_8 \cong (C_3 \times C_3) \rtimes D_8$ *is given as follows:*

centraliser	72	8	12	12	18	18	4	6	6
element	$1A$	$2A$	$2B$	$2C$	$3A$	$3B$	$4A$	$6A$	$6B$
χ_{1a}	1	1	1	1	1	1	1	1	1
χ_{1b}	1	1	-1	-1	1	1	1	-1	-1
χ_{1c}	1	1	-1	1	1	1	-1	-1	1
χ_{1d}	1	1	1	-1	1	1	-1	1	-1
χ_2	2	-2	0	0	2	2	0	0	0
χ_{4a}	4	0	0	2	-2	1	0	0	-1
χ_{4b}	4	0	0	-2	-2	1	0	0	1
χ_{4c}	4	0	2	0	1	-2	0	-1	0
χ_{4d}	4	0	-2	0	1	-2	0	1	0

Note that χ_{1b} *is distinguished amongst the non-trivial linear characters, for example by having an element of order* 4 *in its kernel.*

(ii) $H' = \mathrm{Inn}(H') \lhd \mathrm{Aut}(H')$ *such that* $|\mathrm{Aut}(H')/H'| = 2$*, and any non-inner automorphism of* H' *induces a non-inner automorphism of* $D_8 = H'/P$*, and interchanges the two conjugacy classes of subgroups of order* 3 *of* P*. In particular, there is an induced character table automorphism of* $\mathrm{Irr}(H')$ *interchanging*

$$\chi_{1c} \leftrightarrow \chi_{1d}, \quad \chi_{4a} \leftrightarrow \chi_{4c}, \quad \chi_{4b} \leftrightarrow \chi_{4d}.$$

(iii) *The* 3*-decomposition matrix and the Cartan matrix of* $B' = kH' = k[P \rtimes D_8]$*, respectively, are the following:*

	$1a$	$1b$	$1c$	$1d$	2
χ_{1a}	1	.	.	.	.
χ_{1b}	.	1	.	.	.
χ_{1c}	.	.	1	.	.
χ_{1d}	.	.	.	1	.
χ_{2}	.	.	.	.	1
χ_{4a}	1	.	1	.	1
χ_{4b}	.	1	.	1	1
χ_{4c}	1	.	.	1	1
χ_{4d}	.	1	1	.	1

	$P(1a)$	$P(1b)$	$P(1c)$	$P(1d)$	$P(2)$
$1a$	3	0	1	1	2
$1b$	0	3	1	1	2
$1c$	1	1	3	0	2
$1d$	1	1	0	3	2
2	2	2	2	2	5

(iv) *All simple kH'-modules $1a, 1b, 1c, 1d, 2$ in B' have P as their vertices.*

Proof. (i) and (ii) follow from an explicit computation with GAP [10], the rest is easy. $\square$

Notation 3.5. We use the notation $\chi_{1a}, \cdots, \chi_{4d}$ and $1a = k_{H'}, 1b, 1c, 1d, 2$ as in **3.4**, where the numbers mean the degrees (dimensions) of characters (modules).

Lemma 3.6. *The block algebra $B' = kH' = k[P \rtimes D_8]$ has exactly 18 non-isomorphic trivial source modules over k. In fact, they are given in the following list, where the diagrams are Loewy and socle series:*

(i) *Five PIM's $P(1a)$, $P(1b)$, $P(1c)$, $P(1d)$, $P(2)$:*

$$\boxed{\begin{matrix} 1a \\ 2 \\ 1a\ 1c\ 1d \\ 2 \\ 1a \end{matrix}}, \quad \boxed{\begin{matrix} 1b \\ 2 \\ 1b\ 1c\ 1d \\ 2 \\ 1b \end{matrix}}, \quad \boxed{\begin{matrix} 1c \\ 2 \\ 1a\ 1b\ 1c \\ 2 \\ 1c \end{matrix}}, \quad \boxed{\begin{matrix} 1d \\ 2 \\ 1a\ 1b\ 1d \\ 2 \\ 1d \end{matrix}}, \quad \boxed{\begin{matrix} 2 \\ 1a\ 1b\ 1c\ 1d \\ 2\ 2\ 2 \\ 1a\ 1b\ 1c\ 1d \\ 2 \end{matrix}}.$$

(ii) *Five trivial source modules with vertex P: The simple modules $1a$, $1b$, $1c$, $1d$, 2.*

(iii) *Eight trivial source modules with cyclic vertex of order 3, where we also give the associated trivial source characters, see* **2.1***:*

$$\begin{array}{cccc} \boxed{\begin{matrix} 1a\ \ 1d \\ 2 \\ 1a\ \ 1d \end{matrix}}, & \boxed{\begin{matrix} 1b\ \ 1c \\ 2 \\ 1b\ \ 1c \end{matrix}}, & \boxed{\begin{matrix} 2 \\ 1a\ \ 1d \\ 2 \end{matrix}}, & \boxed{\begin{matrix} 2 \\ 1b\ \ 1c \\ 2 \end{matrix}}, \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ \chi_{1a}+\chi_{1d}+\chi_{4c} & \chi_{1b}+\chi_{1c}+\chi_{4d} & \chi_{2}+\chi_{4c} & \chi_{2}+\chi_{4d} \end{array}$$

$$\begin{array}{cccc} \boxed{\begin{matrix} 1a\ \ 1c \\ 2 \\ 1a\ \ 1c \end{matrix}}, & \boxed{\begin{matrix} 1b\ \ 1d \\ 2 \\ 1b\ \ 1d \end{matrix}}, & \boxed{\begin{matrix} 2 \\ 1a\ \ 1c \\ 2 \end{matrix}}, & \boxed{\begin{matrix} 2 \\ 1b\ \ 1d \\ 2 \end{matrix}}. \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ \chi_{1a}+\chi_{1c}+\chi_{4a} & \chi_{1b}+\chi_{1d}+\chi_{4b} & \chi_{2}+\chi_{4a} & \chi_{2}+\chi_{4b} \end{array}$$

Proof. (i) The structure of the PIM's is immediate as soon as we know that $\mathrm{Ext}^1_{kH'}(1x, 1y) = 0$ for all $x, y \in \{a, b, c, d\}$. This in turn follows from the Ext-quiver of B', which is given as a quiver with relations in [42, Section 4, Case 2].

To find the non-projective trivial source modules, we employ [40, Chap.4, Exc.10]. From that (ii) is immediate. Moreover, this also yields the trivial source characters given in (iii), from which it is easy to see, using the vanishing of $\mathrm{Ext}^1_{kH'}(1x, 1y)$ again, that the associated modules are indecomposable. $\square$

Lemma 3.7. *An (A', B')-bimodule $\mathcal{M}'$ defined by $\mathcal{M}' = f_{(G'\times G', \Delta P, G'\times H')}(A')$ induces a splendid stable equivalence of Morita type between A' and B', namely by*

$$\mathcal{F}' : \text{mod-}A' \to \text{mod-}B' : X_{A'} \mapsto (X \otimes_{A'} \mathcal{M}')_{B'}.$$

In particular, $\mathcal{F}'$ fulfills the assumptions of **2.6***, and hence its assertions as well.*

Proof. This follows from [42, Example 4.3] and [43, Corollary 2]. $\square$

Lemma 3.8. *Let f' and g' be the mutually inverse Green correspondences with respect to (G', P, H'). Then the Green correspondents of simple modules are the following:*

$$\begin{array}{rcccc|rcc}
g'(1a) & = & \boxed{k_{G'}} & \leftrightarrow & \chi'_1 & f'(k_{G'}) & = & 1a \\
g'(1b) & = & \boxed{7} & \leftrightarrow & \chi'_7 & f'(7) & = & 1b \\
g'(1c) & = & \boxed{\begin{matrix} 13 \\ k_{G'} \;\; 7 \\ 13 \end{matrix}} & \leftrightarrow & \chi'_{14} + \chi'_{20} & f'(13) & = & \boxed{\begin{matrix} 1c \\ 2 \\ 1c \end{matrix}} \\
g'(1d) & = & \boxed{28} & \leftrightarrow & \chi'_{28} & f'(28) & = & 1d \\
g'(2) & = & \boxed{35} & \leftrightarrow & \chi'_{35} & f'(35) & = & 2
\end{array}$$

On the left hand we also give the associated trivial source characters, see **2.1***. Recall that by considering H' just as an abstract group $\{1c, 1d\}$ are indistinguishable, see* **3.4***, but now note that by fixing $H' \leqslant G'$ and specifying f', this defines $1c$ and $1d$ uniquely.*

Proof. This follows from [55, Theorem] and [42, Example 4.3]. $\square$

4. 3-Local structure for 2.HS

Notation 4.1. From now on, we assume that G is the covering group $2.\mathsf{HS}$ of the sporadic simple Higman-Sims group HS, and hence $|G| = 2^{10}{\cdot}3^2{\cdot}5^3{\cdot}7{\cdot}11$, see [8, p.80].

Lemma 4.2. *We obtain the following:*

(i) *In order to prove Broué's abelian defect group conjecture* **1.1** *and Rickard's conjecture* **1.2** *for $G = 2.\mathsf{HS}$, it suffices to prove them for the case $p = 3$.*

(ii) *There exists a unique faithful 3-block with non-cyclic abelian defect group P, and P is elementary abelian of order 9, and in order to prove Broué's abelian defect group conjecture for $G = 2.\mathsf{HS}$, it suffices to prove it for this 3-block.*

Proof. (i) Since the conjecture is proved when the defect group is cyclic, we know from **4.1** that it is enough to check it for the primes $p \in \{2, 3, 5\}$. For $p = 2$ it follows from [13, HS, (mod 2)] that there are a couple of blocks of G and both have non-abelian defect groups. For $p = 5$ it follows from [13, HS (mod 5)] that there are a couple of blocks of G which have noncyclic defect groups and the defect group is non-abelian.

(ii) Assume that $p = 3$. Then, again by [13, HS (mod 3)], there are three 3-blocks of G which have noncyclic defect groups. Those 3-blocks have defect groups which are elementary abelian of order 9. Two of them are non-faithful and therefore these two blocks show up in HS. For the principal 3-block of HS, the conjectures have been checked by Okuyama [42, Example 4.8]. For the non-principal 3-block of HS, they have been verified in our previous paper [21, 0.2 Theorem(ii)]. Thus, the remaining untreated case is a unique faithful 3-block of G with noncyclic defect group. $\square$

Notation 4.3. From now on, we assume $p = 3$ and let A be the block algebra of kG with defect group $P \cong C_3 \times C_3$ mentioned in **4.2**; note that we are again abusing notation here, since in the introductory Section **1** we have used the same letter for the associated block of $\mathcal{O}G$, but this will not lead to any confusion.

Set $N = N_G(P)$, and let A_N be the block algebra of kN which is the Brauer correspondent of A. Let (P, e) be a maximal A-Brauer pair in G, namely, e is a block idempotent of $kC_G(P)$ such that $\mathrm{Br}_P(1_A){\cdot}e = e$, see [1], [6] and [54, §40]. Set $H = N_G(P, e)$, namely, $H = \{g \in N_G(P) | g^{-1}eg = e\}$. Let B be a block algebra of kH which is a Fong-Reynolds correspondent of A_N, see **2.3**; note that there are exactly two distinct Fong-Reynolds correspondents of A_N, see **4.10**(iii).

Lemma 4.4. (i) *The 3-decomposition matrix of A is given as follows:*

degree	[8, p.81]	S_1	$S_2 = {S_1}^*$	S_3	S_4	S_5
176	χ_{26}	1	.	.	.	.
176^*	χ_{27}	.	1	.	.	.
616	χ_{28}	1	.	1	.	.
616^*	χ_{29}	.	1	1	.	.
56	χ_{25}	.	.	.	1	.
1000	χ_{32}	.	.	.	.	1
1792	χ_{35}	1	1	1	.	1
1232	χ_{33}	1	.	.	1	1
1232^*	χ_{34}	.	1	.	1	1

where $S_1, \cdots, S_5$ are non-isomorphic simple kG-modules in A whose k-dimensions are 176, 176, 440, 56, 1000, *respectively. The simples S_1 and S_2 are dual to each other, while the remaining are self-dual. There are three pairs (χ_{26}, χ_{27}), (χ_{28}, χ_{29}) and (χ_{33}, χ_{34}) of complex conjugate characters, and all the other χ_i's are real-valued.*

(ii) *All simple kG-modules $S_1, \cdots, S_5$ in A have P as a vertex.*

Proof. (i) follows from [13, *HS* (mod 3)] and [8, p.81], and for (ii) see [16, 3.7.Corollary]. □

Notation 4.5. We use the notation $\chi_{26}, \chi_{27}, \chi_{28}, \chi_{29}, \chi_{25}, \chi_{32}, \chi_{35}, \chi_{33}, \chi_{34}$, and $S_1, \cdots, S_5$ as in **4.4**.

Lemma 4.6. *The following holds:*

(i) $N = N_G(P) = 2.\Big(2 \times (P \rtimes SD_{16})\Big) = P \rtimes L$ *for a subgroup L of N with $|L| = 2^6$ such that $L \rhd Z \cong C_4$ and $L/Z \cong SD_{16}$. Moreover, L/Z acts non-trivially on Z, with kernel isomorphic to D_8. (Recall that SD_{16} has a unique subgroup isomorphic to D_8.)*

(ii) $C_G(P) = Z \times P$ *and L/Z acts faithfully on P. (Note that SD_{16} is a Sylow 2-subgroup of* $\mathrm{GL}_2(3)$*.)*

(iii) $Z = O_{3'}(C_G(P)) = O_{3'}(N)$, *and we can write* $\mathrm{Irr}(Z) = \{\psi_0, \psi_1, \psi_2, \psi_3\}$ *such that $\psi_i(z) = \sqrt{-1}^{\,i}$ for $i = 0, 1, 2, 3$, where z is a generator of $Z \cong C_4$, and $\sqrt{-1} \in \mathcal{O}$ is a fixed 4-th root of unity. Moreover, we have $T_N(\psi_i) = G$ for $i = 0, 2$, while for $j = 1, 3$ we have*

$$T_N(\psi_1) = T_N(\psi_3) \gneqq G \text{ such that } T_N(\psi_j)/C_G(P) \cong D_8.$$

(iv) $kN = A_0 \oplus A_2 \oplus A_N$ *as block algebras, having inertial quotients SD_{16}, SD_{16} and D_8, respectively. Here, A_0 is the principal block algebra of kN, covering ψ_0, while A_2 covers ψ_2; hence A_N is the faithful block algebra being the Brauer correspondent of A. Moreover, $A_0 \cong A_2 \cong k[P \rtimes SD_{16}]$ as k-algebras, and*

$$A_N \cong \mathrm{Mat}_2(k[P \rtimes D_8])$$

as k-algebras, where A_N has $k[P \rtimes D_8]$ as its source algebra.

(v) *The 3-decomposition matrix of A_N is given as follows:*

	2α	2β	2γ	2δ	4
$\chi_{2\alpha}$	1	.	.	.	.
$\chi_{2\beta} = \chi_{2\alpha}^*$	.	1	.	.	.
$\chi_{2\gamma}$	.	.	1	.	.
$\chi_{2\delta}$	.	.	.	1	.
χ_4	.	.	.	.	1
$\chi_{8\alpha}$	1	.	1	.	1
$\chi_{8\beta}$	.	1	.	1	1
$\chi_{8\gamma} = \chi_{8\beta}^*$	1	.	.	1	1
$\chi_{8\delta} = \chi_{8\alpha}^*$	.	1	1	.	1

where the numbers mean the degrees (dimensions) of characters (modules). Note that $2\beta = 2\alpha^*$ and that 2γ, 2δ and 4 are all self-dual, but apart from this the characters of degree 2 are indistinguishable: Apart from the character table automorphism of $\mathrm{Irr}(A_N)$ induced by complex conjugation there is another one interchanging

$$\chi_{2\gamma} \leftrightarrow \chi_{2\delta}, \quad \chi_{8\alpha} \leftrightarrow \chi_{8\gamma}, \quad \chi_{8\beta} \leftrightarrow \chi_{8\delta}.$$

Proof. (i)–(ii) follow from explicit computation with GAP [10], and (iii) is an immediate consequence.

(iv)–(v) It follows from (iii) and [37, Theorem 2] that $kG = A_0 \oplus A_2 \oplus A_N$ where

$$A_0 \cong \mathrm{Mat}_{|G:T_G(\psi_0)|\psi_0(1)}\Big(k^{\alpha}[T_G(\psi_0)/Z]\Big) \cong k^{\alpha}[P \rtimes SD_{16}],$$
$$A_1 \cong \mathrm{Mat}_{|G:T_G(\psi_2)|\psi_2(1)}\Big(k^{\beta}[T_G(\psi_2)/Z]\Big) \cong k^{\beta}[P \rtimes SD_{16}],$$
$$A_N \cong \mathrm{Mat}_{|G:T_G(\psi_1)|\psi_1(1)}\Big(k^{\gamma}[T_G(\psi_1)/Z]\Big) \cong \mathrm{Mat}_2\Big(k^{\gamma}[P \rtimes D_8]\Big),$$

as k-algebras, for some $\alpha, \beta \in \mathrm{Z}^2(SD_{16}, k^\times)$, and $\gamma \in \mathrm{Z}^2(D_8, k^\times)$. Since $|\mathrm{H}^2(SD_{16}, k^\times)| = 1$ by [15, Proof of Corollary (2J)], we have $\alpha \equiv \beta \equiv 1 \pmod{\mathrm{B}^2(SD_{16}, k^\times)}$. On the other hand, $|\mathrm{H}^2(D_8, k^\times)| = 2$ by [12, V Satz 25.6]. But now the assertion in (v) follows by explicit computation with GAP [10], in particular we get that there are 9 irreducible ordinary characters belonging to A_N, and 5 irreducible Brauer characters. Hence, by [15, Page 34 Table 1] we infer $\gamma \equiv 1 \pmod{\mathrm{B}^2(D_8, k^\times)}$. Finally, the statement about source algebras follows from [45, Proposition 14.6], see [54, (45.12)Theorem] and [2, Theorem 13]. □

Remark 4.7. Note that the decomposition matrix of A_N given above coincides with that of B' in **3.4**. This will of course turn out to be no accident, but by the current state of knowledge we cannot avoid the explicit computation to proceed as above.

Notation 4.8. We use the notation L and Z, as in **4.6**. Moreover, let z be a generator of $Z \cong C_4$. We also use the notation $\chi_{2\alpha}, \chi_{2\beta}, \chi_{2\gamma}, \chi_{2\delta}, \chi_4, \chi_{8\alpha}, \chi_{8\beta}, \chi_{8\gamma}, \chi_{8\delta}$ and $2\alpha, 2\beta, 2\gamma, 2\delta, 4$ as in **4.6**.

Lemma 4.9. *The block algebra A_N has exactly 18 non-isomorphic trivial source modules over k. In fact, they are given in the following list, in which the diagrams are Loewy and socle series and we use the same notation as in* **4.8**.

(i) *Five PIM's: $P(2\alpha)$, $P(2\beta)$, $P(2\gamma)$, $P(2\delta)$, $P(4)$.*

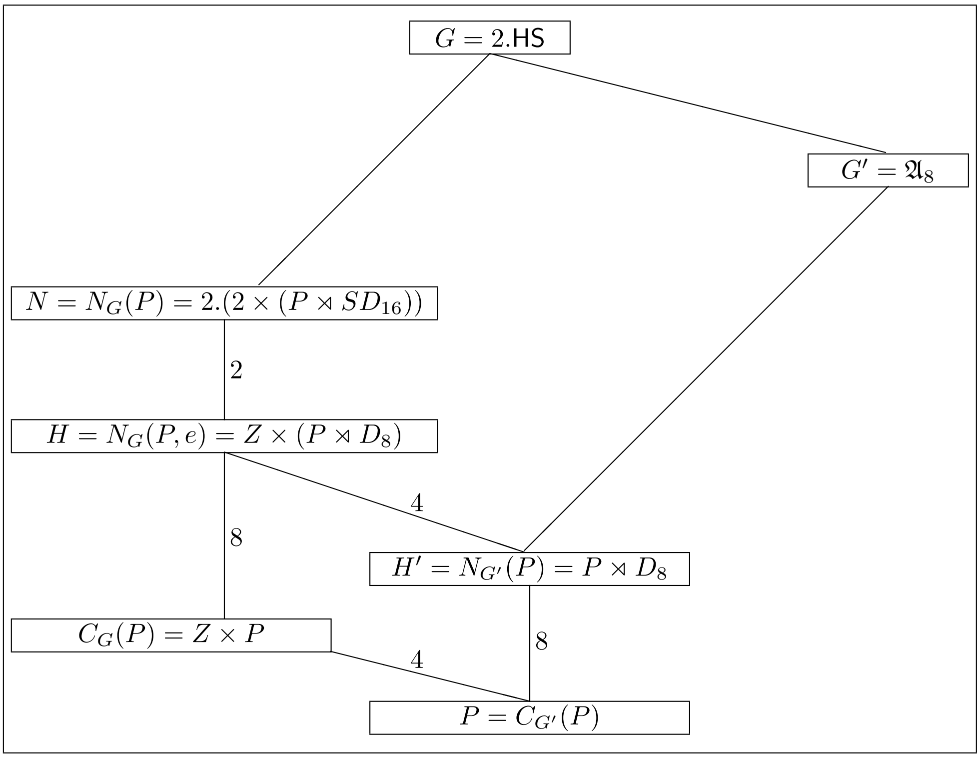
$$\boxed{\begin{matrix} 2\alpha \\ 4 \\ 2\alpha\ 2\gamma\ 2\delta \\ 4 \\ 2\alpha \end{matrix}}, \quad \boxed{\begin{matrix} 2\beta \\ 4 \\ 2\beta\ 2\gamma\ 2\delta \\ 4 \\ 2\beta \end{matrix}}, \quad \boxed{\begin{matrix} 2\gamma \\ 4 \\ 2\alpha\ 2\beta\ 2\gamma \\ 4 \\ 2\gamma \end{matrix}}, \quad \boxed{\begin{matrix} 2\delta \\ 4 \\ 2\alpha\ 2\beta\ 2\delta \\ 4 \\ 2\delta \end{matrix}}, \quad \boxed{\begin{matrix} 4 \\ 2\alpha\ 2\beta\ 2\gamma\ 2\delta \\ 4\ 4\ 4 \\ 2\alpha\ 2\beta\ 2\gamma\ 2\delta \\ 4 \end{matrix}}.$$

(ii) *Five trivial source modules with a vertex P: 2α, 2β, 2γ, 2δ, 4.*

(iii) *Eight trivial source modules with cyclic vertex of order 3, where we also give the associated trivial source characters, see* **2.1***:*

$$\begin{matrix} \boxed{\begin{matrix} 2\alpha\ \ 2\delta \\ 4 \\ 2\alpha\ \ 2\delta \end{matrix}} \\ \updownarrow \\ \chi_{2\alpha} + \chi_{2\delta} + \chi_{8\gamma} \end{matrix}, \quad \begin{matrix} \boxed{\begin{matrix} 2\beta\ \ 2\gamma \\ 4 \\ 2\beta\ \ 2\gamma \end{matrix}} \\ \updownarrow \\ \chi_{2\beta} + \chi_{2\gamma} + \chi_{8\delta} \end{matrix}, \quad \begin{matrix} \boxed{\begin{matrix} 4 \\ 2\alpha\ \ 2\delta \\ 4 \end{matrix}} \\ \updownarrow \\ \chi_4 + \chi_{8\gamma} \end{matrix}, \quad \begin{matrix} \boxed{\begin{matrix} 4 \\ 2\beta\ \ 2\gamma \\ 4 \end{matrix}} \\ \updownarrow \\ \chi_4 + \chi_{8\delta} \end{matrix},$$

TABLE 1. $G' = \mathfrak{A}_8$ as a subgroup of $G = 2.\mathsf{HS}$.



$$\begin{array}{cccc}
\boxed{\begin{array}{c} 2\alpha \;\; 2\gamma \\ 4 \\ 2\alpha \;\; 2\gamma \end{array}}\;, & \boxed{\begin{array}{c} 2\beta \;\; 2\delta \\ 4 \\ 2\beta \;\; 2\delta \end{array}}\;, & \boxed{\begin{array}{c} 4 \\ 2\alpha \;\; 2\gamma \\ 4 \end{array}}\;, & \boxed{\begin{array}{c} 4 \\ 2\beta \;\; 2\delta \\ 4 \end{array}}\;. \\
\updownarrow & \updownarrow & \updownarrow & \updownarrow \\
\chi_{2\alpha} + \chi_{2\gamma} + \chi_{8\alpha} & \chi_{2\beta} + \chi_{2\delta} + \chi_{8\beta} & \chi_4 + \chi_{8\alpha} & \chi_4 + \chi_{8\beta}
\end{array}$$

Proof. This follows from **3.6** and **4.6**(iv). $\square$

Lemma 4.10. *The following holds:*

(i) $H = Z \times (P \rtimes D_8) = C_G(P) \rtimes D_8$, *hence* $H \lhd N$ *such that* $|N/H| = 2$.

(ii) *We have the block decomposition*

$$kH \;=\; B_0 \oplus B_1 \oplus B_2 \oplus B_3,$$

where the block B_i *covers the block containing* $\psi_i \in \mathrm{Irr}(Z)$ *for* $i = 0, 1, 2, 3$.

(iii) *Hence both* B_1 *and* B_3 *are Fong-Reynolds correspondents of* A_N, *see* **4.3**.

(iv) *We can write* $\mathrm{IBr}(B_i) = \{1\alpha_i, 1\beta_i, 1\gamma_i, 1\delta_i, 2_{B_i}\}$, *for* $i = 0, 1, 2, 3$, *so that we have*

$$2x{\downarrow}_H^N = 1x_1 \oplus 1x_3, \;\; \textit{for each } x \in \{\alpha, \beta, \gamma, \delta\}, \;\; \textit{and } 4{\downarrow}_H^N = 2_{B_1} \oplus 2_{B_3}.$$

Proof. This follows from **4.6** and **2.3**. $\square$

Notation 4.11. We use the notation $1\alpha_i, 1\beta_i, 1\gamma_i, 1\delta_i, 2_{B_i}$ as in **4.10**.

Lemma 4.12. *The following holds:*

(i) *The group* $G = 2.\mathsf{HS}$ *has a unique conjugacy class of subgroups isomorphic to* $G' = \mathfrak{A}_8$.

(ii) *Fixing an embedding of* G' *into* G, *and a Sylow* 3*-subgroup* P *of* G', *we have the configuration of groups as depicted in Table* **1**, *where the numbers between two boxes are indices between the two corresponding groups.*

Proof. This follows from [8, pp.80–81], **3.1**, **4.6** and **4.10**. $\square$

Remark 4.13. In view of the group theoretic configuration given in **4.12**, a few more detailed comments on how computations in GAP [10] are actually done are in order:

The starting point is the smallest faithful permutation representation of G on 704 points, available in terms of standard generators, see [56], in [59]; we choose this realisation of G once and for all. Moreover, there is a maximal subgroup of G isoclinic to $\mathfrak{S}_8 \times 2$, a generating set of which in terms of the standard generators of G is available in [59] as well. Hence going over to the derived subgroup of the latter, we find a subgroup $G' \leqslant G$ isomorphic to $\mathfrak{A}_8$; we keep G' fixed all the time. Finally, we compute a Sylow 3-subgroup P of G', and keep this fixed as well. Since the other groups appearing in the diagram in **4.12** are uniquely determined from this by group theoretic properties, we have thus achieved a concrete realisation of the above configuration of groups.

Using this setting we have all kinds of computational tools at our disposal: In particular, we are able to compute the ordinary and Brauer character tables of the groups N, H, and H' explicitly, as well as the restriction or induction of characters between these groups. Moreover, we may fetch representations of G and G' from [59], or compute them using the MeatAxe [50], and restrict them explicitly to N, H, and H', respectively, in order to analyse the restrictions with the MeatAxe [50] and its extensions.

Note that the explicit results in **3.4** and **4.6** have been obtained in that setting already. Moreover, by restricting the representation 28, see **3.3**, from G' to H' we are able to compute its Green correspondent $f'(28)$, see **3.8**, and thus to identify the representation $1d$, see **3.5**. In the same spirit we obtain the following, where we recall that so far, we are not able to tell 2γ and 2δ apart, see **4.6**(v):

Lemma 4.14. *The following holds:*

(i) *For $i = 1, 3$ the restriction functor $\mathrm{Res}{\downarrow}^H_{H'}$ induced by the (B_i, B')-bimodule ${}_{B_i}(B_i)_{B'}$ induces a Puig equivalence* mod-$B_i \to$ mod-B'.

(ii) $\{1\alpha_1{\downarrow}^H_{H'}, 1\alpha_3{\downarrow}^H_{H'}\} = \{1a, 1b\} = \{1\beta_1{\downarrow}^H_{H'}, 1\beta_3{\downarrow}^H_{H'}\}$, *where* $1\alpha_i{\downarrow}^H_{H'} \neq 1\beta_i{\downarrow}^H_{H'}$; *hence*
$$2\alpha{\downarrow}^N_{H'} = (2\alpha^*){\downarrow}^N_{H'} = 2\beta{\downarrow}^N_{H'} = 1a \oplus 1b.$$

(iii) $1\gamma := 1\gamma_1{\downarrow}^H_{H'} = 1\gamma_3{\downarrow}^H_{H'}$ *and* $1\delta := 1\delta_1{\downarrow}^H_{H'} = 1\delta_3{\downarrow}^H_{H'}$, *where* $\{1\gamma, 1\delta\} = \{1c, 1d\}$; *hence*
$$2\gamma{\downarrow}^N_{H'} = 1\gamma \oplus 1\gamma \text{ and } 2\delta{\downarrow}^N_{H'} = 1\delta \oplus 1\delta.$$

(iv) $2_{B_1}{\downarrow}^H_{H'} = 2_{B_3}{\downarrow}^H_{H'} = 2$; *hence* $4{\downarrow}^N_{H'} = 2 \oplus 2$.

Proof. Most of the assertions follow from **4.10**, while the identification of the explicit restrictions $1x_i{\downarrow}^H_{H'}$, for $x \in \{\alpha, \beta, \gamma, \delta\}$, follows from explicit computations in GAP [10]. □

5. Stable equivalences for 2.HS

Strategy 5.1. Our practical aim in §§5–7 is to construct a functor $\widetilde{\mathcal{F}}$: mod-$A \to$ mod-A', where A is the non-principal block algebra of $kG = k[2.\mathsf{HS}]$ with defect group $P \cong C_3 \times C_3$, and A' is the principal block algebra of $k\mathfrak{A}_8$, such that $\widetilde{\mathcal{F}}$ induces a stable equivalence of Morita type between A and A', and that $\widetilde{\mathcal{F}}$ transfers each simple kG-module in A to a *simple* $k\mathfrak{A}_8$-module in A'. If it has been done, then Linckelmann's theorem **2.2**(ii) yields that $\widetilde{\mathcal{F}}$ realizes a *Morita* equivalence between A and A'.

Notation 5.2. Recall the notation G, A, P, N, H, B, e as in **4.1**, **4.3** and **4.8**. Let i and j respectively be source idempotents of A and B with respect to P. As remarked in [32, pp.821–822], we can take i and j such that $\mathrm{Br}_P(i){\cdot}e = \mathrm{Br}_P(i) \neq 0$ and that $\mathrm{Br}_P(j){\cdot}e = \mathrm{Br}_P(j) \neq 0$. Set $G_P = C_G(P) = C_H(P) = H_P$.

Moreover, letting $Q \leqslant P$ be a subgroup of order 3, we set $G_Q = C_G(Q)$ and $H_Q = C_H(Q)$. By replacing e_Q and f_Q (if necessary), we may assume that e_Q and f_Q respectively are block idempotents of kG_Q and kH_Q such that e_Q and f_Q are determined by i and j, respectively. Namely, $\mathrm{Br}_Q(i){\cdot}e_Q = \mathrm{Br}_Q(i)$ and $\mathrm{Br}_Q(j){\cdot}f_Q = \mathrm{Br}_Q(j)$. Let $A_Q = kG_Q{\cdot}e_Q$ and $B_Q = kH_Q{\cdot}f_Q$, so that $e_Q = 1_{A_Q}$ and $f_Q = 1_{B_Q}$.

Lemma 5.3. *The following holds:*

(i) *All elements in* $P-\{1\}$ *are conjugate in* N*, and hence in* G*; actually* $P-\{1\}\subseteq 3A$*, where* $3A$ *is a conjugacy class of* G *following the notation in* [8, pp.80–81]. *Thus all subgroups of* P *of order* 3 *are conjugate in* N*, and hence in* G.

(ii) *The elements in* $P-\{1\}$ *fall into two conjugacy classes of* H. *Thus* P *has exactly two* H*-conjugacy classes of subgroups of order* 3*; we fix representatives* Q *and* R. *Note that we can use the same notation* Q *and* R *as in* **3.1**.

(iii) $H_Q=C_N(Q)=Z\times Q\times\mathfrak{S}_3$ *and* $H_R=C_N(R)=Z\times R\times\mathfrak{S}_3$*, so that* $H_Q/Q\cong C_4\times\mathfrak{S}_3$ *and* $H_R/R\cong C_4\times\mathfrak{S}_3$.

(iv) $G_Q=Q\times(\mathfrak{A}_5\rtimes Z)\cong Q\times(\mathfrak{A}_5\times 2).2$*, so that* $G_Q/Q\cong(\mathfrak{A}_5\times 2).2$.

Proof. (i)–(iii) follow from **4.6** and **4.10**, (iv) follows from an explicit computation in GAP [10]. □

Lemma 5.4. *Let* $\mathcal{M}_Q$ *be the unique (up to isomorphism) indecomposable direct summand of* $A_Q{\downarrow}^{G_Q\times G_Q}_{G_Q\times H_Q}{\cdot}1_{B_Q}$ *with vertex* ΔP. *Then, the pair* $(\mathcal{M}_Q,\mathcal{M}_Q^\vee)$ *induces a Puig equivalence between* A_Q *and* B_Q.

Proof. Note first that $\mathcal{M}_Q$ exists by [23, 2.4.Lemma], and also that P is a defect group of A_Q and B_Q by [32, 7.6]. Then using **5.3**(iii) and (iv), as well as **2.7**(iv), the assertion follows as in [23, Proof of 6.2.Lemma], by going over to the central quotients G_Q/Q and H_Q/Q and their blocks dominating A_Q and B_Q, respectively, and applying [20, 1.2.Theorem] and [19, Theorem]. □

Lemma 5.5. *The* (A,B)*-bimodule* $1_A{\cdot}kG{\cdot}1_B$ *has a unique (up to isomorphism) indecomposable direct summand* $\mathcal{M}$ *with vertex* ΔP. *Moreover, the functor*

$$\mathcal{F}:\text{mod-}A\to\text{mod-}B: X_A\mapsto(X\otimes_A\mathcal{M})_B$$

induces a splendid stable equivalence of Morita type between A *and* B. *In particular,* $\mathcal{F}$ *fulfils the assumptions of* **2.6***, and hence its assertions as well.*

Proof. Note first that, again, ${}_A\mathcal{M}_B$ exists by [23, 2.4.Lemma]. Then the assertion follows as in [23, Proof of 6.3.Lemma], by applying [24, Theorem] in order to make use of **5.4**, and using gluing through [32, 3.1.Theorem]; note that the fusion condition in the latter theorem is automatically satisfied by [22, 1.15.Lemma]. □

Notation 5.6. We use the notation $\mathcal{M}$ and $\mathcal{F}$ as in **5.5**.

Lemma 5.7. *The following holds:*

(i) *The* (A,A_N)*-bimodule* $1_A{\cdot}kG{\cdot}1_{A_N}$ *has a unique (up to isomorphism) indecomposable direct summand* $\mathcal{M}_N$ *with vertex* ΔP. *Moreover, the functor*

$$\mathcal{F}_N:\text{mod-}A\to\text{mod-}A_N: X_A\mapsto(X\otimes_A\mathcal{M}_N)_{A_N}$$

induces a splendid stable equivalence of Morita type between A *and* A_N. *In particular,* $\mathcal{F}_N$ *fulfils the assumptions of* **2.6***, and hence its assertions as well.*

(ii) *Suppose that* X *is an indecomposable* kG*-module in* A *such that a vertex of* X *belongs to* $\mathfrak{A}(G,P,N)$*, and let* $\mathcal{F}_N(X)=Y\oplus(\text{proj})$ *for a non-projective indecomposable* kH*-module* Y *in* B*, and* $\mathcal{F}_N(X)=f(X)\oplus(\text{proj})$*, where* f *denotes the Green correspondence with respect to* (G,P,N). *Then,* $f(X)$ *is the correspondent of* Y *with respect to the Fong-Reynolds correspondence between* B *and* A_N*, namely* $f(X)\cong Y{\uparrow}^N$.

Proof. (i) follows by **5.5** and **2.3**, and (ii) follows from **2.6**. □

6. Images of simples via the functor $\mathcal{F}_N$

Notation 6.1. For brevity, let $F:\underline{\text{mod}}\text{-}A\to\underline{\text{mod}}\text{-}A_N$ denote the functor induced by $\mathcal{F}_N$ given in **5.7**. Recall that $F(X)=f(X)\oplus(\text{proj})$, where f is the Green correspondence with respect to (G,P,N), whenever the Green correspondent $f(X)$ is defined, see **5.7**(ii). Recall also the modules $S_1,\cdots,S_5$ defined in **4.4** and **4.5**.

Lemma 6.2. *The simples* S_1 *and* S_2 *are trivial source* kG*-modules.*

Proof. By [8, p.80] G has a subgroup U with $U \cong U_3(5)$. Then, by a computation with GAP [10], we know $1_U{\uparrow}^G{\cdot}1_A = \chi_{26} + \chi_{27}$. Therefore there is a kG-module X such that $X = (k_U{\uparrow}^G{\cdot}1_A)$ and X is liftable to an $\mathcal{O}G$-lattice affording $\chi_{26} + \chi_{27}$. Thus by **4.4**, it holds that $X = S_1 + S_2$, as composition factors. From **2.1**(ii), we have $\dim_k[\mathrm{End}_{kG}(X)] = 2$. Hence $X \cong S_1 \oplus S_2$, since $S_1 \not\cong S_2$. $\square$

Lemma 6.3. *The simple S_4 is a trivial source kG-module.*

Proof. By [8, p.80] G has a subgroup M with $M \cong \mathsf{M}_{11}$. Then, again by a computation with GAP [10], it holds that $1_M{\uparrow}^G{\cdot}1_A = \chi_{25} + \chi_{26} + \chi_{27} + \chi_{28} + \chi_{29}$. Set $X = k_M{\uparrow}^G{\cdot}1_A$. Thus it follows from [28, I Theorem 17.3] that X has a submodule S such that $S \leftrightarrow \chi_{25}$. By **4.4**, $S \cong S_4$ and $X = 2 \times S_1 + 2 \times S_2 + 2 \times S_3 + S_4$ as composition factors. Therefore the self-duality of S_4 and X implies $S \big| X$. $\square$

Lemma 6.4. *The simple S_5 is a trivial source kG-module.*

Proof. As before it follows from [8, p.80] that G has a maximal subgroup M such that $M \cong 2.\mathsf{M}_{22}$ and $|G : M| = 100$. By [8, p.39] and [13, M_{22} (mod 3)], we know that M has a 3-block $\tilde{A}$ (which is called "Block 6" in [13, M_{22} (mod 3)]) such that $\tilde{A}$ has a defect group $\tilde{P}$ with $\tilde{P} \cong C_3 \times C_3$, where we can assume $\tilde{P} = P$. Moreover, $\tilde{A}$ has an irreducible ordinary character $\tilde{\chi}_{13}$ of degree 10, and $\tilde{A}$ has a simple kM-module $\tilde{S}$ of dimension 10 corresponding to $\tilde{\chi}_{13}$. Now, it follows from [9, Proposition 3.19] that $\tilde{S}$ has a trivial source. On the other hand, a computation in GAP [10] shows $\tilde{\chi}_{13}{\uparrow}^G = \chi_{32}$. Therefore **4.4** yields that $S_5 \cong \tilde{S}{\uparrow}^G$ also has a trivial source. $\square$

Lemma 6.5. *We can assume that $f(S_1) = 2\alpha$ and $f(S_2) = f(S_1^*) = (2\alpha)^* = 2\beta$.*

Proof. It follows from **6.2**, **4.4** and [41, Lemma 2.2] that $f(S_1)$ and $f(S_2)$ are simple, so $\{f(S_1), f(S_2)\} \subseteq \{2\alpha, 2\beta, 2\gamma, 2\delta, 4\}$. Then, since $2\gamma, 2\delta, 4$ are self-dual and $2\beta = (2\alpha)^*$ by **4.6**(v), and since $S_2 = {S_1}^*$ by **4.4**, it holds that $\{f(S_1), f(S_2)\} = \{2\alpha, 2\beta\}$. Thus we get the assertion. $\square$

Lemma 6.6. *We can assume that*

$$f(S_4) = 2\delta \quad \text{and} \quad f(S_5) = 4.$$

Recall that by considering N just as an abstract group $\{2\gamma, 2\delta\}$ are indistinguishable, see **4.6**(v). *But fixing $N \leqslant G$ and specifying f serves to identify the latter uniquely.*

Proof. It follows from **6.3**, **6.4**, **6.5**, **4.4** and [41, Lemma 2.2] that $\{f(S_4), f(S_5)\} \subseteq \{2\gamma, 2\delta, 4\}$. Now, by **4.1** and Sylow's theorem, we have $|G : N| \equiv 1 \pmod 3$. Set $T_i = f(S_i)$ for $i = 4, 5$. By the definition of Green correspondence, $T_5{\uparrow}^G = S_5 \oplus X$ for a kG-module X such that X is Q-projective. Hence, by **4.1** and [40, Chap.4, Theorem 7.5], it holds that

$$\dim(S_5) \equiv \dim(T_5{\uparrow}^G) = \dim(T_5) \cdot |G : N| \equiv \dim(T_5) \bmod 3,$$

and $\dim(S_5) = 1000 \equiv 1 \bmod 3$. Thus, $T_5 = 4$, so that $T_4 \in \{2\gamma, 2\delta\}$. $\square$

Lemma 6.7. *Using the assumption of* **6.6**, *it holds that*

$$f(S_3) = \boxed{\begin{matrix} 2\gamma \\ 4 \\ 2\gamma \end{matrix}}.$$

Proof. As noted in the proof of **6.4**, G has a maximal subgroup M such that $M \cong 2.\mathsf{M}_{22}$ and $|G : M| = 100$. By [8, p.39] and [13, M_{22} (mod 3)], we know that M has a 3-block $\tilde{B}$ (which is called "Block 7" in [13, M_{22} (mod 3)]) such that $\tilde{B}$ has a defect group $\tilde{Q}$ with $\tilde{Q} \cong C_3$, where we can assume $\tilde{Q} = Q$. Moreover, $\tilde{B}$ has an irreducible ordinary character $\tilde{\chi}_{16}$ of degree 120, and $\tilde{B}$ has a simple kM-module $\tilde{T}$ of dimension 120 corresponding to $\tilde{\chi}_{16}$. Now, it follows from [9, Proposition 3.19] that $\tilde{T}$ is a trivial source module with vertex Q. Hence the indecomposable summands of $X := \tilde{T}\!\uparrow^G \cdot 1_A$ have a trivial source as well, and are Q-projective. On the other hand, a computation in GAP [10] says that

$$X \leftrightarrow \tilde{\chi}_{16}\!\uparrow^G \cdot 1_A = \chi_{26} + \chi_{27} + \chi_{28} + \chi_{29}.$$

Therefore, **4.4** yields that $X = 2 \times S_1 + 2 \times {S_1}^* + 2 \times S_3$ as composition factors; note that this shows that X is projective-free. Recall that $\chi_{26} \leftrightarrow S_1$ and $\chi_{28} \leftrightarrow S_1 + S_3$. Hence, it holds by **2.1**, **6.2** and **4.4** that, as k-spaces,

$$\mathrm{Hom}_{kG}(S_1, X) \cong \mathrm{Hom}_{kG}({S_1}^*, X) \cong \mathrm{Hom}_{kG}(X, S_1) \cong \mathrm{Hom}_{kG}(X, {S_1}^*) \cong k,$$
$$\mathrm{Hom}_{kG}(S_4, X) = \mathrm{Hom}_{kG}(S_5, X) = \mathrm{Hom}_{kG}(X, S_4) = \mathrm{Hom}_{kG}(X, S_5) = 0.$$

Moreover, it follows from [28, II Lemma 2.7 and Corollary 2.8], **6.5** and **6.1** that, as k-spaces,

$$\mathrm{Hom}_{kN}(F(X), 2\alpha) \;\cong\; \underline{\mathrm{Hom}}_{kN}(F(X), 2\alpha) \;\cong\; \underline{\mathrm{Hom}}_{kG}(X, S_1) \;\cong\; \mathrm{Hom}_{kG}(X, S_1) \;\cong\; k.$$

Similarly, we get $\mathrm{Hom}_{kN}(F(X), (2\alpha)^*) \cong k$. Using **6.6** in the above proof, we obtain

$$\mathrm{Hom}_{kN}(F(X), 2\delta) = \mathrm{Hom}_{kN}(F(X), 4) = \mathrm{Hom}_{kN}(2\delta, F(X)) = \mathrm{Hom}_{kN}(4, F(X)) = 0.$$

Then, let Y be an indecomposable direct summand of X with $S_1 \Big| (Y/\mathrm{rad}(Y))$, hence Y is non-projective. This means that we can write $F(Y) = U \oplus (\mathrm{proj})$ for a non-projective indecomposable kN-module U in A_N, where by **2.6** we infer that U is a trivial source kN-module with vertex Q, since $\tilde{T}$ has vertex Q and Y is non-projective. Moreover, from

$$F(X) = F(Y) \oplus (\mathrm{module}) = U \oplus (\mathrm{proj}) \oplus (\mathrm{module}),$$

6.5 and **6.1**, we get $\mathrm{Hom}_{kN}(F(Y), 2\delta) = \mathrm{Hom}_{kN}(F(Y), 4) = 0$ and

$$\begin{aligned}\mathrm{Hom}_{kN}(U, 2\alpha) \;&\cong\; \underline{\mathrm{Hom}}_{kN}(U, 2\alpha) \;\cong\; \underline{\mathrm{Hom}}_{kN}(F(Y), 2\alpha)\\ &\cong\; \underline{\mathrm{Hom}}_{kG}(Y, S_1) \;\cong\; \mathrm{Hom}_{kG}(Y, S_1) \;\cong\; k.\end{aligned}$$

Thus, from **4.9**(iii) we conclude that

$$U \;=\; \boxed{\begin{array}{cc} 2\alpha & 2\gamma \\ \multicolumn{2}{c}{4} \\ 2\alpha & 2\gamma \end{array}}.$$

Since $U \Big| F(X)$ and X is self-dual, by [26, A.2.Lemma] we have $U^* \Big| F(X)^* = F(X^*) = F(X)$ as well. Since U is not self-dual, this yields that

$$\begin{aligned} F(X) =&\; U \oplus U^* \oplus (\mathrm{module}) \oplus (\mathrm{proj}) \\ =&\; \boxed{\begin{array}{cc} 2\alpha & 2\gamma \\ \multicolumn{2}{c}{4} \\ 2\alpha & 2\gamma \end{array}} \bigoplus \boxed{\begin{array}{cc} (2\alpha)^* & 2\gamma \\ \multicolumn{2}{c}{4} \\ (2\alpha)^* & 2\gamma \end{array}} \bigoplus (\mathrm{module}) \bigoplus (\mathrm{proj}), \end{aligned}$$

Note that since X is Q-projective, neither S_1 nor ${S_1}^*$ can possibly be a direct summand of X by **4.4**(ii). Hence it follows from the *stripping-off method*, see **2.5**, that there is a subquotient module Z of X such that $Z = S_3 + S_3$ as composition factors, and such that

$$F(Z) \;=\; V \,\oplus\, V^* \,\oplus\, (\mathrm{module}) \,\oplus\, (\mathrm{proj}), \qquad \text{where } V \;=\; \boxed{\begin{array}{c|c|c} 2\gamma & & \\ & 4 & \\ & & 2\gamma \end{array}}.$$

Since F induces a stable equivalence by **6.1**, we conclude that Z is decomposable, that is $Z \cong S_3 \oplus S_3$, which implies that $F(S_3) \oplus F(S_3) \;=\; V \oplus V^*$, since $F(S_3)$ is indecomposable by **2.2**(i). Hence we infer that $V \cong V^*$ is indecomposable, having Loewy and socle series

$$V \;=\; \boxed{\begin{array}{c} 2\gamma \\ 4 \\ 2\gamma \end{array}}.$$

□

Remark 6.8. (a) Note that the above analysis also shows that

$$X = \boxed{\begin{matrix} S_1 \\ S_3 \\ S_1 \end{matrix}} \bigoplus \boxed{\begin{matrix} S_1^* \\ S_3 \\ S_1^* \end{matrix}},$$

but we do not need this fact.

(b) Moreover, to prove **6.7**, as an alternative we could have proceeded as follows: Actually, the structure of the Green correspondent $f(S_3)$ and of the trivial source module X had been found by explicit computation in the first place, hence we just could have stated the outcome. But, having seen these results, subsequently we have managed to compile the above proof, which now is based as much as possible on general principles.

7. Proof of main results

Lemma 7.1. *Keeping the setting in* **4.12** *fixed, we have*

$$2\delta{\downarrow}^N_{H'} = 1\delta \oplus 1\delta = 1d \oplus 1d \quad \text{and} \quad 2\gamma{\downarrow}^N_{H'} = 1\gamma \oplus 1\gamma = 1c \oplus 1c.$$

Proof. By **4.14** we have to show that $1\delta = 1d$. In order to do so, we employ the kG-module S_4, for which we first show that $S_4{\downarrow}_{G'} = 28 \oplus 28$: By an explicit computation with GAP [10] we know that $S_4{\downarrow}_{G'} = 2 \times 28$ as composition factors, see **4.4** and **3.2**. Hence we are done as soon as we show that $\mathrm{Ext}^1_{kG'}(28, 28) = 0$. This in turn is seen as follows: Since by **3.7** the functor $\mathcal{F}'$ commutes with taking Heller translates in the stable module categories, we have

$$\mathrm{Ext}^1_{kG'}(28, 28) \;\cong\; \mathrm{Ext}^1_{kH'}(\mathcal{F}'(28), \mathcal{F}'(28)) \;\cong\; \mathrm{Ext}^1_{kH'}(1d, 1d) \;=\; 0,$$

by making use of **3.8**, and the vanishing result in the proof of **3.6**. (As an alternative, the vanishing of $\mathrm{Ext}^1_{kG'}(28, 28)$ can also be found in [53, Appendix p.3115].)

Now, on the one hand we get by **3.8** that

$$\begin{aligned} S_4{\downarrow}_{H'} &= S_4{\downarrow}_{G'}{\downarrow}_{H'} = (28 \oplus 28){\downarrow}_{H'} \\ &= \Big(f'(28) \oplus f'(28) \oplus (\mathfrak{Y}(G', P, H')\text{-proj})\Big) \\ &= 1d \oplus 1d \oplus (\mathfrak{Y}(G', P, H')\text{-proj}), \end{aligned}$$

see [40, Chap.4, §4]. On the other hand, we get from **6.6** that

$$\begin{aligned} S_4{\downarrow}_{H'} &= S_4{\downarrow}_N{\downarrow}_{H'} = \Big(f(S_4) \oplus (\mathfrak{Y}(G, P, N)\text{-proj})\Big){\downarrow}_{H'} \\ &= \Big(2\delta \oplus (\mathfrak{Y}(G, P, N)\text{-proj})\Big){\downarrow}_{H'} \\ &= (1\delta \oplus 1\delta) \oplus \Big(\mathfrak{Y}(G, P, N)\text{-proj}\Big){\downarrow}_{H'}. \end{aligned}$$

Therefore, by comparing the vertices of the indecomposables showing up above and by Krull-Schmidt's theorem, we finally know that $1\delta = 1d$. □

Lemma 7.2. *The blocks A and A' are Morita equivalent induced by an (A, A')-bimodule which is ΔP-projective and is a trivial source $k[G \times G']$-module.*

Proof. We are now able to specify functors as envisaged in **5.1**: Indeed A and A' are splendidly stable equivalent of Morita type by either of the (A, A')-bimodules

$$\widetilde{\mathcal{M}}_i \;=\; {}_A(\mathcal{M} \otimes_{B_i} B_i \otimes_{B'} \mathcal{M}'^{\vee})_{A'},$$

where $\mathcal{M}$ and $\mathcal{M}'$ are the same as in **5.5** and **3.7**, respectively, and $i = 1, 3$ by **4.14**(i). Hence, the following holds from **3.8**, **5.7**(ii) as well as **6.5**, **6.6**, **6.7**, **4.14**(ii)–(iv) and **7.1**:

$$
\begin{array}{ccccccc}
\text{mod-}A & \xrightarrow{\mathcal{F}} & \text{mod-}B_i & \xrightarrow{\mathrm{Res}\downarrow^{H}_{H'}} & \text{mod-}B' & \xrightarrow{\mathcal{F}'^{-1}} & \text{mod-}A' \\
\hline
S_1 & \mapsto & \boxed{1\alpha_i} & \mapsto & \left\{\begin{matrix}\boxed{1a}\\ \boxed{1b}\end{matrix}\right. & \mapsto & \left.\begin{matrix}k_{G'}\\ 7\end{matrix}\right\} \\
S_2 = S_1^* & \mapsto & \boxed{1\beta_i} & \mapsto & \left\{\begin{matrix}\boxed{1b}\\ \boxed{1a}\end{matrix}\right. & \mapsto & \left.\begin{matrix}7\\ k_{G'}\end{matrix}\right\} \\
S_3 & \mapsto & \boxed{\begin{matrix}1\gamma_i\\ 2_{B_i}\\ 1\gamma_i\end{matrix}} & \mapsto & \boxed{\begin{matrix}1c\\ 2\\ 1c\end{matrix}} & \mapsto & 13 \\
S_4 & \mapsto & \boxed{1\delta_i} & \mapsto & \boxed{1d} & \mapsto & 28 \\
S_5 & \mapsto & \boxed{2_{B_i}} & \mapsto & \boxed{2} & \mapsto & 35
\end{array}
$$

Note that, by **3.6**(i), the B'-module $\mathcal{F}'(13)$ is uniquely (up to isomorphism) determined by its Loewy series; hence we indeed have

$$\mathrm{Res}\downarrow^{H}_{H'}(\mathcal{F}(S_3)) \cong \mathcal{F}'(13).$$

Therefore we finally get that A and A' are Morita equivalent by **2.2**(ii). More precisely, we know also that the Morita equivalence is given by either of the bimodules $\widetilde{\mathcal{M}}_i$, satisfying the properties desired. □

Remark 7.3. A remark on the strategy employed in the proofs of **7.1** and **7.2** is in order:

(a) To derive **7.1** we use the full strength of **4.12**: Indeed, using an embedding $H' \leqslant G'$ and the Green correspondence f', we have defined $1d$, see **3.8**, and similarly, using an embedding $H \leqslant N \leqslant G$ and the Green correspondence f, we have defined 1δ see **6.6**. But from that alone we would only be able to conclude that $\{1\gamma, 1\delta\} = \{1c, 1d\}$, see **4.14**. Now only additionally using an embedding $G' \leqslant G$, entailing a compatible embedding $H' \leqslant H$, we are able to conclude as in the proof of **7.1**, whose starting point is restricting S_4 from G to G'.

(b) In order to be able to proceed as in the proof of **7.2** we have to ensure that the functor induced by $\widetilde{\mathcal{M}}_i$ maps simple A-modules to simple A'-modules, which happens if and only if

$$\mathrm{Res}\downarrow^{H}_{H'}(\mathcal{F}(S_4)) = \mathrm{Res}\downarrow^{H}_{H'}(1\delta_i) = 1\delta \overset{!}{=} 1d = \mathcal{F}'(28),$$

which is proved by the full strength of **7.1**. Without using the explicit configuration of groups in **4.12** we only know $1\delta \in \{1c, 1d\}$. (Note that this phenomenon has also been observed in [23, 6.14.Question].) As an alternative we would have to proceed as follows:

By **3.4**(ii) there is an outer automorphism of H', hence inducing a Morita self-equivalence of of $kH' = k[P \rtimes D_8]$, interchanging $1c \leftrightarrow 1d$. Twisting the bimodule $\widetilde{\mathcal{M}}_i$ accordingly then still yields a Morita equivalence between A and A'. But the outer automorphism applied necessarily changes the structure of kH', which is its own source algebra, as an interior P-algebra; in other words, the twisted bimodule then no longer is ΔP-projective, hence it does no longer induce a Puig equivalence between A and A'. Thus, as already indicated in **1.6**(a) we would end up with the weaker statement in **7.2** only saying that A and A' are Morita equivalent.

Remark 7.4. In the proofs of **1.5**, **1.3** and **1.4** below, we note that a Puig equivalence lifts from k to $\mathcal{O}$ by a result of Puig [44, 7.8.Lemma] (see [54, (38.8)Proposition]), and so does a splendid derived equivalence by a result of Rickard [48, Theorem 5.2], see also [11, p.75, lines $-17\sim-13$]). Hence, it is enough to consider all blocks only over k instead of $\mathcal{O}$.

Proof of 1.5. By **1.8**, the assertion of **7.2** is equivalent to saying that A and A' are Puig equivalent. □

Remark 7.5. In the proof of **1.5** the two choices of $\widetilde{\mathcal{M}}_i$, for $i = 1, 3$, account precisely for the two bijections between the simple A- and A'-modules as described in **1.6**(b).

Proof of 1.3. This follows from **1.5**, since by [42, Example 4.3] and [43, Theorem 3] the block algebras A' and B' are splendidly derived equivalent, and by **4.14**(i) and **2.3** the block algebras B' and B are Puig equivalent. □

Proof of 1.4. This follows from **1.3** and **4.2**. □

Acknowledgements

The authors would like to thank the referee for her/his careful reading of the first draft and her/his useful comments. A part of this work was done while the first author was staying in RWTH Aachen University in 2010, 2011 and 2012. He is grateful to Gerhard Hiss for his kind hospitality. For this research the first author was partially supported by the Japan Society for Promotion of Science (JSPS), Grant-in-Aid for Scientific Research (C)20540008, 2008–2010, (B)21340003, 2009–2011, and also (C)23540007, 2011–2014. The second author is grateful for financial support in the framework of the DFG (German Science Foundation) Priority Programme SPP-1388 'Representation Theory', to which this research is a contribution.

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