Vertices of Lie Modules

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Abstract

Let $\mathrm{Lie}_F(n)$ be the Lie module of the symmetric group $\mathfrak{S}_n$ over a field F of characteristic $p > 0$, that is, $\mathrm{Lie}_F(n)$ is the left ideal of $F\mathfrak{S}_n$ generated by the Dynkin–Specht–Wever element ω_n. We study the problem of parametrizing non-projective indecomposable summands of $\mathrm{Lie}_F(n)$, via describing their vertices and sources. Our main result shows that this can be reduced to the case when n is a power of p. When $n = 9$ and $p = 3$, and when $n = 8$ and $p = 2$, we present a precise answer. This suggests a possible parametrization for arbitrary prime powers.

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1   Introduction

The Lie module of the symmetric group $\mathfrak{S}_n$ occurs in various contexts within algebra and topology, where the name-giving property is its close relation to the free Lie algebra; for more details, see for example the introduction in [14]. In the present paper, letting F be an algebraically closed field of characteristic $p > 0$, we realize the Lie module $\mathrm{Lie}_F(n)$ of $\mathfrak{S}_n$, for $n \geqslant 2$, as the submodule $F\mathfrak{S}_n\omega_n$ of the regular $F\mathfrak{S}_n$-module, where

$$\omega_n := (1 - c_2)(1 - c_3)\cdots(1 - c_n) \in F\mathfrak{S}_n$$

is the Dynkin–Specht–Wever element of $F\mathfrak{S}_n$, where in turn $c_k \in \mathfrak{S}_n$ is the backward cycle $(k, k-1, \ldots, 2, 1)$. Moreover, $\dim(\mathrm{Lie}_F(n)) = (n-1)!$; see A.5.

1.1. It is well known that $\omega_n^2 = n\omega_n$. Hence if p does not divide n, then $\omega_n/n \in F\mathfrak{S}_n$ is an idempotent, so that $\mathrm{Lie}_F(n)$ is then a direct summand of the regular $F\mathfrak{S}_n$-module and is, thus, projective. In the present paper we are interested in the case when p divides n, which we assume from now on in this section. Then $\mathrm{Lie}_F(n)$ cannot be projective; for otherwise $\dim(\mathrm{Lie}_F(n)) = (n-1)!$ would have to be divisible by the p-part of $n!$, which is not the case. Therefore, in this case $\mathrm{Lie}_F(n)$ admits a decomposition

$$\mathrm{Lie}_F(n) = \mathrm{Lie}_F^{\mathrm{pr}}(n) \oplus \mathrm{Lie}_F^{\mathrm{pf}}(n),$$

where $\mathrm{Lie}_F^{\mathrm{pr}}(n)$ is a projective $F\mathfrak{S}_n$-module and where $\mathrm{Lie}_F^{\mathrm{pf}}(n) \neq \{0\}$ is a projective-free $F\mathfrak{S}_n$-module.

The asymptotic behaviour of the quotient $\dim(\mathrm{Lie}_F^{\mathrm{pr}}(n))/\dim(\mathrm{Lie}_F(n))$ has recently been studied by Erdmann–Tan [14], and by Bryant–Lim–Tan [5]. By [5, Thm. 1.2], one has

$$\frac{\dim(\mathrm{Lie}_F^{\mathrm{pr}}(n))}{\dim(\mathrm{Lie}_F(n))} \longrightarrow 1,$$

as $n \longrightarrow \infty$ in $\mathbb{N} \setminus \{p^k \mid k \geqslant 0\}$. Moreover, it is conjectured in [5] that this should remain true when allowing n to vary over all natural numbers. This suggests that $\mathrm{Lie}_F^{\mathrm{pf}}(n)$ should be small, compared with the entire Lie module $\mathrm{Lie}_F(n)$.

Moreover, by work of Erdmann–Tan [15], we also know that the projective-free part $\mathrm{Lie}_F^{\mathrm{pf}}(n)$ of $\mathrm{Lie}_F(n)$ always belongs to the principal block of $F\mathfrak{S}_n$, and Bryant–Erdmann [4] have studied indecomposable direct sum decompositions of the, necessarily projective, part of $\mathrm{Lie}_F(n)$ not contained in the principal block of $F\mathfrak{S}_n$. This leaves open, next to $\mathrm{Lie}_F^{\mathrm{pf}}(n)$, only the direct sum decompositions of the component of $\mathrm{Lie}_F^{\mathrm{pr}}(n)$ belonging to the principal block of $F\mathfrak{S}_n$. We denote the principal block component of $\mathrm{Lie}_F(n)$ by $\mathrm{Lie}_F^{\mathrm{pbl}}(n)$.

1.2. One key ingredient of our approach is a decomposition theorem, expressing $\mathrm{Lie}_F(n)$ as a direct sum of pieces related to Lie modules $\mathrm{Lie}_F(p^d)$, for various d such that p^d divides n. This is obtained by translating the Bryant–Schocker decomposition theorem [6] for Lie powers to Lie modules, using work of Lim–Tan [24]. This paves the way to reduce questions on Lie modules to the case when n is a power of p, and puts the Lie modules $\mathrm{Lie}_F(p^d)$ into the focus of study. In particular, one is tempted to ask whether there is a neat description of the indecomposable direct summands of $\mathrm{Lie}_F(n)$ in terms of those of $\mathrm{Lie}_F(p^d)$, where d varies as indicated above. This has been fully accomplished for the case where p divides n but p^2 does not, with a different line of reasoning, by Erdmann–Schocker [13], while the general case remains a mystery and is subject to further investigations.

Very little information concerning the decomposition of the principal block component of $\mathrm{Lie}_F(p^d)$ is available in the literature, and the projective-free part $\mathrm{Lie}_F^{\mathrm{pf}}(p^d)$ is very poorly understood, even for very small exponents d: to our knowledge, the only cases dealt with systematically are the modules $\mathrm{Lie}_F^{\mathrm{pf}}(p)$, that is, the case $d = 1$, by Erdmann–Schocker [13]; and, apart from the easy case $\mathrm{Lie}_F(4) = \mathrm{Lie}_F^{\mathrm{pf}}(4)$, there are just partial results for $\mathrm{Lie}_F(8)$, by Selick–Wu [33]. The aim of this paper now is to investigate indecomposable direct summands of $\mathrm{Lie}_F(p^d)$, for a few further small values of p and d.

The major obstacle here is that, due to the exponential growth of the dimension of Lie modules in terms of n, these modules quickly become very large. Hence, to proceed further in this direction, we apply computational techniques. More precisely, by this approach we are now able to give a complete description of the Lie modules $\mathrm{Lie}_F(8)$ of dimension 5040, and $\mathrm{Lie}_F(9)$ of dimension 40320.

Actually, in both cases it turns out that the projective-free part of the Lie module is already indecomposable, where $\mathrm{Lie}_F^{\mathrm{pf}}(8)$ has dimension 816, and $\mathrm{Lie}_F^{\mathrm{pf}}(9)$ has dimension 1683. In view of these results, and those on $\mathrm{Lie}_F^{\mathrm{pf}}(4)$ and $\mathrm{Lie}_F^{\mathrm{pf}}(p)$ mentioned above, the question arises whether $\mathrm{Lie}_F^{\mathrm{pf}}(p^d)$ is always indecomposable.

1.3. To analyze the projective-free part of $\mathrm{Lie}_F(n)$, we are, in particular, interested in the Green vertices and sources of the indecomposable direct summands of $\mathrm{Lie}_F^{\mathrm{pf}}(n)$. Using the reduction result mentioned above, to some extent we are able to reduce this problem for arbitrary n to the case where n is a p-power.

Moreover, we are able to compute vertices and sources of $\mathrm{Lie}_F^{\mathrm{pf}}(8)$ and $\mathrm{Lie}_F^{\mathrm{pf}}(9)$. It turns out that both modules are endo-p-permutation modules, in the sense of Urfer [36], their vertices are regular elementary abelian subgroups of $\mathfrak{S}_8$ and $\mathfrak{S}_9$, respectively, and their sources are endo-permutation modules, in the sense of Dade [10], whose class in the Dade group we are able to determine. It is surprising to us to see the class of endo-permutation modules appear in this context.

Hence, in view of these results, and those concerning $\mathrm{Lie}_F^{\mathrm{pf}}(4)$ and $\mathrm{Lie}_F^{\mathrm{pf}}(p)$, one may wonder whether $\mathrm{Lie}_F^{\mathrm{pf}}(p^d)$, assumed to be indecomposable, always is an endo-p-permutation module having regular elementary abelian vertices and endo-permutation sources, and, if so, what the

class of a source in the Dade group looks like. If this holds true, then, by our reduction results, any indecomposable direct summand of any Lie module will have vertices and sources sharing the same properties.

1.4. This paper is organized as follows: in Section 2 we provide the necessary prerequisites; in particular, we recall the notions of Green vertices and sources, endo-permutation and endo-p-permutation modules, and the Dade group.

In Section 3 we present a reduction, eventually showing that vertices and sources of indecomposable direct summands of Lie modules in general can be described in terms of the results in the p-power case. In order to do so, in Theorem 3.4 we provide a description of vertices and sources of indecomposable direct summands of modules for wreath products, based on results by Külshammer [23]. In Theorem 3.5 we present the decomposition theorem for Lie modules mentioned above, and in Theorem 3.7 these are combined to prove the reduction result.

In Section 4 we collect the explicit computational results we have obtained for specific examples; in particular, we revisit $\mathrm{Lie}_F(p)$ and $\mathrm{Lie}_F(4)$. We also discuss the major examples $\mathrm{Lie}_F(8)$ and $\mathrm{Lie}_F(9)$, whose indecomposable direct sum decomposition we determine, together with vertices and sources of the non-projective indecomposable direct summands occurring. We then apply Theorem 3.7 to our computational data to derive Corollaries 4.7 and 4.8.

In order to make this paper sufficiently self-contained, in Appendix A we collect some properties of Lie modules, some variations on the construction, as well as Lie powers and their relation to Lie modules via the Schur functor. Many of these observations are certainly well known to the experts, but explicit references are not too easy to find.

Finally, Appendix B is devoted to presenting some details of the computational ideas and tools we have been using to deduce our results in Section 4.

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2   Prerequisites

2.1. Generalities. (a) Throughout this paper we assume that F is an algebraically closed field of characteristic $p > 0$. Whenever G is a finite group, an FG-module is understood to be a left FG-module of finite F-dimension, unless stated otherwise. If M and N are FG-modules such that N is isomorphic to a direct summand of M then we write $N \mid M$.

Whenever G is a finite group, H is a normal subgroup of G, and M is an $F[G/H]$-module, we denote by $\mathrm{Inf}_{G/H}^G(M)$ the FG-module obtained from M via inflation. More generally, by abuse of notation, given a fixed epimorphism of groups $G \twoheadrightarrow K$ and an FK-module M, we denote the FG-module obtained via inflation with respect to this epimorphism by $\mathrm{Inf}_K^G(M)$ as well.

(b) By $\mathfrak{S}_n$ we denote the symmetric group of degree $n \geqslant 1$. We view permutations in $\mathfrak{S}_n$ as maps, applied to the left, and then the product is composition of maps. So they are multiplied from right to left; for instance, we have $(1,2)(2,3) = (1,2,3)$.

We assume the reader to be familiar with the basic notions of the representation theory of the symmetric group. For detailed background information, we refer to [20, 21]. The Specht modules S^λ of the group algebra $F\mathfrak{S}_n$ will, as usual, be labelled by the partitions λ of n, and the simple $F\mathfrak{S}_n$-modules D^λ by the p-regular partitions of n. Furthermore, we denote by P^λ a projective cover of D^λ.

(c) Whenever G is a finite group with subgroups H and K, we write $H \leqslant_G K$ if H is G-conjugate to a subgroup of K, and we write $H =_G K$ if H is G-conjugate to K.

Next we recall the notions of vertices and sources of indecomposable modules over group algebras, and we summarize some basic properties of endo-permutation modules over finite p-groups. The latter class of modules has been introduced by Dade [10], as generalizations of permutation modules. They have proved to play an important role in the modular representation theory of finite groups, and, as we shall see in subsequent sections, also appear naturally in the context of Lie modules. For a detailed account on the theory of vertices and sources we refer the reader to [30, Chap. 4.3]. Background information concerning endo-permutation modules can be found in [10, 11] and in [34, §28].

2.2. Vertices and sources. (a) Let G be a finite group, and let M be an indecomposable FG-module. By Green's Theorem [17], we can assign to M a G-conjugacy class of p-subgroups of G, the *vertices of M*. A vertex Q of M is characterized by the property that Q is minimal such that M is *relatively Q-projective*, that is, M is isomorphic to a direct summand of the induced module $\mathrm{Ind}_Q^G(N)$, for some indecomposable FQ-module N. In particular, M is projective if and only if $Q = \{1\}$.

Given a vertex Q of M, an indecomposable FQ-module L such that M is isomorphic to a direct summand of $\mathrm{Ind}_Q^G(L)$ is called a *Q-source* of M, and is unique up to isomorphism and conjugation with elements in $N_G(Q)$. Moreover, Q is also a vertex of L.

(b) Let B be the block of FG containing M. If Q is a vertex of M then there are a defect group R of B and a Sylow p-subgroup P of G such that $Q \leqslant R \leqslant P$. Moreover, as a consequence of Green's Indecomposability Theorem [17], $|P : Q|$ divides $\dim(M)$.

(c) Suppose that $H \leqslant G$ is any subgroup of G and that N is an indecomposable direct summand of the restriction $\mathrm{Res}_H^G(M)$ with vertex R and R-source L'. Then there are a vertex Q of M and a Q-source L of M such that $R \leqslant Q$ and $L' \mid \mathrm{Res}_R^Q(L)$. This is seen as follows:

Let Q be any vertex of M, and let L be any Q-source of M. Then we have $L' \mid \mathrm{Res}_R^H(N) \mid \mathrm{Res}_R^G(M)$ and $M \mid \mathrm{Ind}_Q^G(L)$, thus $L' \mid \mathrm{Res}_R^G(\mathrm{Ind}_Q^G(L))$. Now, by Mackey's Theorem and the fact that L' has vertex R, this implies $L' \mid \mathrm{Res}_R^{{}^g Q}({}^g L)$, for some $g \in G$ such that $R \leqslant {}^g Q$. But ${}^g Q$ is also a vertex of M, and ${}^g L$ is a ${}^g Q$-source of M, whence the claim.

In particular, if $H \leqslant G$ is such that M is relatively H-projective, then $M \mid \mathrm{Ind}_H^G(\mathrm{Res}_H^G(M))$ implies that there is an indecomposable direct summand of $\mathrm{Res}_H^G(M)$ sharing a vertex and a source with M.

(d) Suppose, conversely, that $H \geqslant G$ is a finite overgroup of G and that N is an indecomposable direct summand of $\mathrm{Ind}_G^H(M)$. Then, given a vertex Q of M and a Q-source L of M, there is a vertex P of N such that $P \leqslant Q$, and there is some P-source of N that is isomorphic to a direct summand of $\mathrm{Res}_P^Q(L)$. This is seen as follows:

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Let P be any vertex of N, and let L' be any P-source of N. Then, by Mackey's Theorem again, we get $L' \mid \mathrm{Res}_P^{{}^h Q}({}^h L)$, for some $h \in H$ such that $P \leqslant {}^h Q$. In other words, we have ${}^{h^{-1}} L' \mid \mathrm{Res}_{h^{-1}P}^{Q}(L)$, where ${}^{h^{-1}} P$ is also a vertex of N, and ${}^{h^{-1}} L'$ is a ${}^{h^{-1}} P$-source of N, whence the claim. (Note that we cannot conclude that ${}^{h^{-1}} L'$ is an arbitrary ${}^{h^{-1}} P$-source of N, since the conjugating element h might depend on the choice of L'.)

In particular, since $M \mid \mathrm{Res}_G^H(\mathrm{Ind}_G^H(M))$, there is some indecomposable direct summand of $\mathrm{Ind}_G^H(M)$ sharing a vertex and a source with M.

2.3. Endo-permutation modules. An FG-module M, where G is a finite group, is called an *endo-permutation module* if its F-endomorphism ring $\mathrm{End}_F(M) \cong M \otimes M^*$ is a permutation FG-module. We list some properties of endo-permutation modules that we shall need later in this paper:

(a) Permutation modules are endo-permutation modules. The class of endo-permutation modules over finite p-groups is closed under taking F-linear duals, direct summands, tensor products, restriction to subgroups, inflation from factor groups, and taking Heller translates Ω and Ω^{-1}, but it is neither closed under taking direct sums, nor under induction to finite overgroups. In particular, any indecomposable endo-permutation module has endo-permutation sources.

(b) The problem of classifying the indecomposable endo-permutation modules for finite p-groups P has been worked on by various people. The final classification result was obtained by Bouc [1], but when P is abelian, the following classification result is already due to Dade [11]. We shall describe this result below; this is the version we shall need.

Note that it is indeed sufficient to classify the indecomposable endo-permutation FP-modules with vertex P, since if M is an indecomposable endo-permutation FP-module with vertex $Q < P$ then any Q-source S of M is an endo-permutation FQ-module with vertex Q, and by Green's Indecomposability Theorem [17] we have $M \cong \mathrm{Ind}_Q^P(S)$.

2.4 Theorem ([11, Thm. 12.5])**.** *Let P be an abelian p-group, and let M be an indecomposable endo-permutation FP-module with vertex P. Then M is, up to isomorphism, the unique indecomposable direct summand of*

$$\bigotimes_{|P:Q| \geqslant 3} \mathrm{Inf}_{P/Q}^P(\Omega^{n_Q}(F_{P/Q}))$$

having vertex P. Here $n_Q \in \mathbb{Z}$, for $Q < P$, is uniquely determined by M if P/Q is non-cyclic; otherwise n_Q is uniquely determined modulo 2.

In other words, the isomorphism types of indecomposable endo-permutation FP-modules with *abelian* vertex P are in bijection with the elements of the *Dade group*

$$\mathfrak{D}(P) \cong \left(\sum_{|P:Q| \geqslant 3,\, P/Q \text{ non-cyclic}} \mathbb{Z} \right) \oplus \left(\sum_{|P:Q| \geqslant 3,\, P/Q \text{ cyclic}} \mathbb{Z}/2\mathbb{Z} \right).$$

For a precise definition of the Dade group of an arbitrary p-group P and further details, see [10] and [34, §29]. Whenever S is an indecomposable endo-permutation FP-module with vertex P, its image in $\mathfrak{D}(P)$ will be denoted by $[S]$. If S and S' are indecomposable endo-permutation FP-modules with vertex P then their tensor product $S \otimes S'$ has a unique (up

to isomorphism) indecomposable direct summand T with vertex P, and the multiplication in $\mathfrak{D}(P)$ is then given by $[S] \cdot [S'] := [T]$.

2.5. Endo-p-permutation modules. According to Urfer [36, 35], one can weaken the notion of endo-permutation modules as follows: an FG-module M, where G is a finite group, is called an *endo-p-permutation module*, if its F-endomorphism ring $\mathrm{End}_F(M) \cong M \otimes M^*$ is a p-permutation FG-module, that is, all its indecomposable direct summands are trivial-source modules.

(a) Again, one has the following standard properties of endo-p-permutation modules: endo-permutation modules are endo-p-permutation modules. The class of endo-p-permutation modules is closed under taking F-linear duals, Heller translates, direct summands, tensor products, restriction to subgroups, and inflation from factor groups, but it is neither closed under taking direct sums, nor under induction to finite overgroups. Note that for finite p-groups the classes of endo-p-permutation modules and of endo-permutation modules coincide, thus any indecomposable endo-p-permutation module has endo-permutation sources.

(b) A characterization of indecomposable endo-p-permutation module in terms of vertices and sources is given in [36, Thm. 1.5]. For the cases of interest in the present paper it can be rephrased as follows:

Let $P \leqslant G$ be a p-group. As before, for any indecomposable endo-permutation FP-module S with vertex P let $[S] \in \mathfrak{D}(P)$ be the associated element of the Dade group. Then $[S] \in \mathfrak{D}(P)$ is called *G-stable* if

$$\mathrm{Res}^P_{P \cap {}^gP}([S]) = \mathrm{Res}^{gP}_{P \cap {}^gP}([^gS]) \in \mathfrak{D}(P \cap {}^gP), \quad \text{for all} \quad g \in G.$$

Then, by [36, Thm. 1.5], an indecomposable FG-module with vertex P and P-source S is an endo-p-permutation module if and only if S is an endo-permutation module such that $[S]$ is G-stable.

(c) As in part (b), let $P \leqslant G$ be a p-group. In good situations the G-stable elements of $\mathfrak{D}(P)$ are described by a Burnside-type fusion argument as follows:

Suppose $N_G(P)$ *controls fusion* in P, that is, whenever $Q \leqslant P$ and $g \in G$ are such that ${}^gQ \leqslant P$, there are some $h \in N_G(P)$ and $z \in C_G(Q)$ such that $g = hz$. Then, by [36, Lemma 1.8, Prop. 1.9], an element $[S] \in \mathfrak{D}(P)$ is G-stable if and only if it is fixed by the conjugation action of $N_G(P)$ on $\mathfrak{D}(P)$. (Note that we do not require an additional saturation condition here, as is done in [36]: an inspection of the proofs of [36, Lemma 1.8, Prop. 1.9] shows that they are valid under the assumptions made here.)

Moreover, if P is *abelian* then an element $[S] \in \mathfrak{D}(P)$ is an $N_G(P)$-fixed point if and only if the associated function $Q \longrightarrow n_Q$ is constant on the $N_G(P)$-orbits on $\{Q < P \mid |P : Q| \geqslant 3\}$.

3 A Reduction Theorem

The aim of this section is to establish Theorem 3.7, which will allow for a partial reduction of the question concerning vertices and sources of indecomposable direct summands of $\mathrm{Lie}^{\mathrm{pf}}_F(n)$ to the case where n is a p-power. The key ingredients will be Theorem 3.5, where we translate properties of Lie powers into the language of Lie modules, and Theorem 3.4 on vertices of indecomposable modules of wreath products. Therefore, we start out by collecting a number of general facts on wreath products and their representations, which we shall then apply in the context of Lie modules.

3.1. Wreath products and their modules. (a) Let G be a finite group, and consider the wreath product

$$G \wr \mathfrak{S}_n := \{(g_1, \ldots, g_n; \sigma) \mid g_1, \ldots, g_n \in G, \sigma \in \mathfrak{S}_n\}.$$

Recall that the multiplication in $G \wr \mathfrak{S}_n$ is given by

$$(g_1, \ldots, g_n; \sigma)(h_1, \ldots, h_n; \pi) = (g_1 h_{\sigma^{-1}(1)}, \ldots, g_n h_{\sigma^{-1}(n)}; \sigma\pi), \tag{1}$$

for $g_1, \ldots, g_n, h_1, \ldots, h_n \in G$ and $\sigma, \pi \in \mathfrak{S}_n$. Hence we have the natural epimorphism

$$^- : G \wr \mathfrak{S}_n \longrightarrow \mathfrak{S}_n, \ (g_1, \ldots, g_n; \sigma) \longmapsto \sigma.$$

We denote by G^n the *base group* of $G \wr \mathfrak{S}_n$, that is,

$$G^n = \{(g_1, \ldots, g_n; 1) \mid g_1, \ldots, g_n \in G\} \trianglelefteq G \wr \mathfrak{S}_n.$$

Moreover, letting $\sigma^\sharp := (1, \ldots, 1; \sigma) \in G \wr \mathfrak{S}_n$, for $\sigma \in \mathfrak{S}_n$, we get an isomorphism

$$\mathfrak{S}_n^\sharp := \{\sigma^\sharp \mid \sigma \in \mathfrak{S}_n\} \cong \mathfrak{S}_n;$$

note that the map $(-)^\sharp : \mathfrak{S}_n \longrightarrow G \wr \mathfrak{S}_n$ is a section for the natural epimorphism $^- : G \wr \mathfrak{S}_n \longrightarrow \mathfrak{S}_n$. More generally, if $H \leqslant G$ and $U \leqslant \mathfrak{S}_n$ then we further set $U^\sharp := \{\sigma^\sharp \mid \sigma \in U\} \leqslant \mathfrak{S}_n^\sharp$, as well as $H^n := \{(g_1, \ldots, g_n; 1) \mid g_1, \ldots, g_n \in H\} \leqslant G^n$, and

$$H \wr U := \{(g_1, \ldots, g_n; \sigma) \mid g_1, \ldots, g_n \in H, \sigma \in U\} \leqslant G \wr \mathfrak{S}_n.$$

(b) Let M be an FG-module. Then the (outer) tensor product $M^{\otimes n} = M \otimes \cdots \otimes M$ becomes an $F[G \wr \mathfrak{S}_n]$-module via

$$(g_1, \ldots, g_n; \sigma)(x_1 \otimes \cdots \otimes x_n) := g_1 x_{\sigma^{-1}(1)} \otimes \cdots \otimes g_n x_{\sigma^{-1}(n)},$$

for $g_1, \ldots, g_n \in G$, $\sigma \in \mathfrak{S}_n$, and $x_1, \ldots, x_n \in M$. This module is called a *tensor-induced* module.

From now on, we denote by $\Lambda(m, n)$ the set of compositions of n with at most m non-zero parts. If $\lambda = (\lambda_1, \ldots, \lambda_m) \in \Lambda(m, n)$ then we denote by $\mathfrak{S}_\lambda$ the corresponding (standard) Young subgroup $\mathfrak{S}_{\lambda_1} \times \cdots \times \mathfrak{S}_{\lambda_m}$ of $\mathfrak{S}_n$. With this notation,

$$(G \wr \mathfrak{S}_{\lambda_1}) \times \cdots \times (G \wr \mathfrak{S}_{\lambda_m}) \cong G \wr \mathfrak{S}_\lambda \leqslant G \wr \mathfrak{S}_n.$$

Thus, if $M_1, \ldots, M_m$ are FG-modules, the (outer) tensor product

$$M^{\otimes \lambda} := M_1^{\otimes \lambda_1} \otimes \cdots \otimes M_m^{\otimes \lambda_m}$$

carries an $F[G \wr \mathfrak{S}_\lambda]$-module structure.

Moreover, suppose again that $H \leqslant G$ and $U \leqslant \mathfrak{S}_n$, and let L be an FU-module. Then, via the map $(-)^\sharp$, the FU-module L can be viewed as an $FU^\sharp$-module, which we denote by $L^\sharp$. Via inflation along the natural epimorphism $^-$, the FU-module L becomes also an $F[H \wr U]$-module, which we denote by $\widehat{L} := \mathrm{Inf}_U^{H \wr U}(L)$. Thus we have $\mathrm{Res}_{U^\sharp}^{H \wr U}(\widehat{L}) = L^\sharp$.

(c) Let N be an $F\mathfrak{S}_n$-module, and again let M be an FG-module. In this section, we shall describe vertices and sources of indecomposable direct summands of the $F[G \wr \mathfrak{S}_n]$-module

$M^{\otimes n} \otimes \widehat{N}$ in terms of those of the indecomposable direct summands of M and N. We, therefore, recall the structure of the indecomposable direct summands of the $F[G \wr \mathfrak{S}_n]$-modules $M^{\otimes n}$ and $\widehat{N}$, respectively:

Let $N_1, \ldots, N_l$ be pairwise non-isomorphic indecomposable $F\mathfrak{S}_n$-modules, and $b_1, \ldots, b_l \in \mathbb{N}$ be such that $N \cong b_1 N_1 \oplus \cdots \oplus b_l N_l$. Then we get

$$\widehat{N} = \operatorname{Inf}_{\mathfrak{S}_n}^{G \wr \mathfrak{S}_n}(N) \cong \bigoplus_{i=1}^{l} b_i \operatorname{Inf}_{\mathfrak{S}_n}^{G \wr \mathfrak{S}_n}(N_i) = \bigoplus_{i=1}^{l} b_i \widehat{N}_i,$$

where the $F[G \wr \mathfrak{S}_n]$-modules $\widehat{N}_i := \operatorname{Inf}_{\mathfrak{S}_n}^{G \wr \mathfrak{S}_n}(N_i)$ are pairwise non-isomorphic and indecomposable. Thus, the indecomposable direct summmands of the $F\mathfrak{S}_n$-module N and those of the $F[G \wr \mathfrak{S}_n]$-module $\widehat{N}$ are in natural bijection, and hence in the sequel we may assume that N is indecomposable.

As for $M^{\otimes n}$, let $M_1, \ldots, M_m$ be pairwise non-isomorphic indecomposable FG-modules, and let $a_1, \ldots, a_m \in \mathbb{N}$ be such that

$$M \cong a_1 M_1 \oplus \cdots \oplus a_m M_m.$$

Then we have the following well-known result; we include a proof for the readers' convenience.

3.2 Lemma. *With the notation as in 3.1(c),*

$$M^{\otimes n} \cong \bigoplus_{\lambda \in \Lambda(m,n)} c_\lambda \cdot \operatorname{Ind}_{G \wr \mathfrak{S}_\lambda}^{G \wr \mathfrak{S}_n}(M_1^{\otimes \lambda_1} \otimes \cdots \otimes M_m^{\otimes \lambda_m})$$

is an indecomposable direct sum decomposition of the $F[G \wr \mathfrak{S}_n]$-module $M^{\otimes n}$, for suitable $c_\lambda \in \mathbb{N}$.

Proof. We have an isomorphism of $F[G \wr \mathfrak{S}_n]$-modules

$$M^{\otimes n} \cong \bigoplus_{\lambda = (\lambda_1, \ldots, \lambda_m) \in \Lambda(m,n)} c_\lambda \cdot (\bigoplus \widetilde{M}_1 \otimes \cdots \otimes \widetilde{M}_n),$$

the inner sum being taken over all n-tuples $(\widetilde{M}_1, \ldots, \widetilde{M}_n)$ of FG-modules satisfying

$$|\{1 \leqslant j \leqslant n \mid \widetilde{M}_j = M_i\}| = \lambda_i, \quad \text{for} \quad i = 1, \ldots, m.$$

The respective coefficient c_λ equals $a_1^{\lambda_1} \cdots a_m^{\lambda_m}$.

Given $\lambda = (\lambda_1, \ldots, \lambda_m) \in \Lambda(m,n)$, the sum $\bigoplus(\widetilde{M}_1 \otimes \cdots \otimes \widetilde{M}_n)$ is a transitive imprimitive $F[G \wr \mathfrak{S}_n]$-module, and the direct summands $\widetilde{M}_1 \otimes \cdots \otimes \widetilde{M}_n$ form a system of imprimitivity. One of these direct summands equals $M^{\otimes \lambda} = M_1^{\otimes \lambda_1} \otimes \cdots \otimes M_m^{\otimes \lambda_m}$. Its restriction to the base group G^n of $G \wr \mathfrak{S}_n$ is indecomposable, and its inertial group in $G \wr \mathfrak{S}_n$ equals $G \wr \mathfrak{S}_\lambda$. Thus, by [9, 50.2], we deduce that

$$\bigoplus \widetilde{M}_1 \otimes \cdots \otimes \widetilde{M}_n \cong \operatorname{Ind}_{G \wr \mathfrak{S}_\lambda}^{G \wr \mathfrak{S}_n}(M^{\otimes \lambda})$$

and, by [23, Prop. 4.1], $\operatorname{Ind}_{G \wr \mathfrak{S}_\lambda}^{G \wr \mathfrak{S}_n}(M^{\otimes \lambda})$ is an indecomposable $F[G \wr \mathfrak{S}_n]$-module. $\qquad\square$

3.3. Wreath products and vertices. We retain the notation from 3.1(c). In particular, we suppose that N is an indecomposable $F\mathfrak{S}_n$-module. We now want to examine the vertices and sources of the indecomposable direct summands of the $F[G \wr \mathfrak{S}_n]$-module $M^{\otimes n} \otimes \widehat{N}$; a description is given in Theorem 3.4 below. To this end, let P be a Sylow p-subgroup of G, and for $j = 1, \ldots, m$ let R_j be a vertex of the FG-module M_j.

(a) Now let L be an indecomposable direct summand of $M^{\otimes n} \otimes \widehat{N}$. Then, by 3.1(c) and Lemma 3.2, there is some $\lambda = (\lambda_1, \ldots, \lambda_m) \in \Lambda(m, n)$ such that L is isomorphic to a direct summand of

$$\operatorname{Ind}_{G \wr \mathfrak{S}_\lambda}^{G \wr \mathfrak{S}_n}(M^{\otimes \lambda}) \otimes \widehat{N} \cong \operatorname{Ind}_{G \wr \mathfrak{S}_\lambda}^{G \wr \mathfrak{S}_n}(M^{\otimes \lambda} \otimes \operatorname{Res}_{G \wr \mathfrak{S}_\lambda}^{G \wr \mathfrak{S}_n}(\widehat{N}))$$
$$\cong \operatorname{Ind}_{G \wr \mathfrak{S}_\lambda}^{G \wr \mathfrak{S}_n}(M^{\otimes \lambda} \otimes \operatorname{Inf}_{\mathfrak{S}_\lambda}^{G \wr \mathfrak{S}_\lambda}(\operatorname{Res}_{\mathfrak{S}_\lambda}^{\mathfrak{S}_n}(N))).$$

Then, by [23, Prop. 5.1] and the discussion preceding it, there is an indecomposable direct summand N' of $\operatorname{Res}_{\mathfrak{S}_\lambda}^{\mathfrak{S}_n}(N)$ such that

$$L \cong \operatorname{Ind}_{G \wr \mathfrak{S}_\lambda}^{G \wr \mathfrak{S}_n}(L'), \quad \text{where} \quad L' := M^{\otimes \lambda} \otimes \widehat{N'}.$$

In particular, L' is an indecomposable $F[G \wr \mathfrak{S}_\lambda]$-module. Now, if $Q' \leqslant \mathfrak{S}_\lambda$ is a vertex of N', then

$$Q := (R_1^{\lambda_1} \times \cdots \times R_m^{\lambda_m}) \rtimes (Q')^\sharp \leqslant G \wr \mathfrak{S}_\lambda \leqslant G \wr \mathfrak{S}_n \tag{2}$$

is a common vertex of L and L'.

(b) We consider a common Q-source S of the $F[G \wr \mathfrak{S}_n]$-module L and the $F[G \wr \mathfrak{S}_\lambda]$-module L'. To this end, we from now on additionally suppose that each of the FG-modules $M_1, \ldots, M_m$ has trivial sources. Note that this, in particular, includes the case that all these modules are projective.

Let $P_\lambda = P_{\lambda_1} \times \cdots \times P_{\lambda_m}$ be a Sylow p-subgroup of the Young subgroup $\mathfrak{S}_\lambda = \mathfrak{S}_{\lambda_1} \times \cdots \times \mathfrak{S}_{\lambda_m}$. Then, in consequence of [23, Prop. 1.2, Prop. 3.1],

$$R_\lambda := (R_1^{\lambda_1} \times \cdots \times R_m^{\lambda_m}) \rtimes P_\lambda^\sharp = (R_1 \wr P_{\lambda_1}) \times \cdots \times (R_m \wr P_{\lambda_m}) \leqslant G \wr \mathfrak{S}_\lambda$$

is a vertex of the indecomposable $F[G \wr \mathfrak{S}_\lambda]$-module $M^{\otimes \lambda}$, and $M^{\otimes \lambda}$ is a trivial-source module, that is, $M^{\otimes \lambda} \mid \operatorname{Ind}_{R_\lambda}^{G \wr \mathfrak{S}_\lambda}(F)$. (Note that the assertion on vertices is just a special case of (2).)

Suppose that S' is a Q'-source of the $F\mathfrak{S}_\lambda$-module N'. From [19, Prop. 2] we deduce that the $F[G \wr \mathfrak{S}_\lambda]$-module $\widehat{N'} := \operatorname{Inf}_{\mathfrak{S}_\lambda}^{G \wr \mathfrak{S}_\lambda}(N')$ has vertex $P \wr Q'$, and $\widehat{S'} := \operatorname{Inf}_{Q'}^{P \wr Q'}(S')$ is a $(P \wr Q')$-source of $\widehat{N'}$. Thus we have $\widehat{N'} \mid \operatorname{Ind}_{P \wr Q'}^{G \wr \mathfrak{S}_\lambda}(\widehat{S'})$. Hence Mackey's Tensor Product Theorem shows that there is some $g \in G \wr \mathfrak{S}_\lambda$ such that L' is a direct summand of $\operatorname{Ind}_{\widetilde{Q}}^{G \wr \mathfrak{S}_\lambda}(\operatorname{Res}_{\widetilde{Q}}^{g(P \wr Q')}({}^g\widehat{S'}))$, where

$$\widetilde{Q} := R_\lambda \cap {}^g(P \wr Q') \leqslant G \wr \mathfrak{S}_\lambda.$$

Hence S is a direct summand of $\operatorname{Res}_Q^{G \wr \mathfrak{S}_\lambda}(\operatorname{Ind}_{\widetilde{Q}}^{G \wr \mathfrak{S}_\lambda}(\operatorname{Res}_{\widetilde{Q}}^{g(P \wr Q')}({}^g\widehat{S'})))$, thus, by Mackey's Theorem, there is some $h \in G \wr \mathfrak{S}_\lambda$ such that S is a direct summand of

$$\operatorname{Ind}_{Q \cap {}^h\widetilde{Q}}^Q(\operatorname{Res}_{Q \cap {}^h\widetilde{Q}}^{h\widetilde{Q}}{}^h(\operatorname{Res}_{\widetilde{Q}}^{g(P \wr Q')}({}^g\widehat{S'}))) = \operatorname{Ind}_{Q \cap {}^h\widetilde{Q}}^Q(\operatorname{Res}_{Q \cap {}^h\widetilde{Q}}^{hg(P \wr Q')}({}^{hg}\widehat{S'})).$$

Since S has vertex Q, we infer $Q \cap {}^h\widetilde{Q} = Q$, so S is a direct summand of $\operatorname{Res}_Q^{hg(P \wr Q')}({}^{hg}\widehat{S'})$.

Now we consider the natural epimorphism $^- : G \wr \mathfrak{S}_\lambda \longrightarrow \mathfrak{S}_\lambda$, and let $\sigma := \overline{hg} \in \mathfrak{S}_\lambda$. Then

$$Q' = \overline{(R_1^{\lambda_1} \times \cdots \times R_m^{\lambda_m}) \rtimes (Q')^\sharp} = \overline{Q} \leqslant \overline{hg(P \wr Q')} = {}^\sigma Q'\,.$$

Hence we have $\sigma \in N_{\mathfrak{S}_\lambda}(Q')$. Moreover, since the base group G^n acts trivially on $\widehat{S'}$, we infer that $Q \cap G^n = R_1^{\lambda_1} \times \cdots \times R_m^{\lambda_m}$ acts trivially on S, and since S' is an indecomposable FQ'-module, we finally conclude that

$$S \cong \mathrm{Res}_Q^{hg(P \wr Q')}({}^{hg}\widehat{S'}) \cong \mathrm{Inf}_{Q'}^Q({}^\sigma S')\,.$$

Recall from 2.2 that, since S' is a Q'-source of N', so is ${}^\sigma S'$.

(c) Keep the notation as in part (b), and suppose additionally that the FG-module M is projective, that is, $R_1 = \cdots = R_m = \{1\}$, thus $Q = (Q')^\sharp$. Furthermore, we now get $\sigma^\sharp = (1, \ldots, 1; \sigma) \in N_{G \wr \mathfrak{S}_\lambda}((Q')^\sharp)$, since $\sigma \in N_{\mathfrak{S}_\lambda}(Q')$. Hence we have $S \cong ({}^\sigma S')^\sharp = {}^{\sigma^\sharp}((S')^\sharp)$. Since S' was an arbitrary Q'-source of N', this shows that indeed every Q'-source of N', in the way just described, yields a common Q-source of L and L'.

Altogether we have, in particular, now proved the following:

3.4 Theorem. *Let M be an FG-module, let N be an indecomposable $F\mathfrak{S}_n$-module, and let L be an indecomposable direct summand of the $F[G \wr \mathfrak{S}_n]$-module $M^{\otimes n} \otimes \widehat{N}$. Suppose that $M \cong \bigoplus_{j=1}^m a_j M_j$ is an indecomposable direct sum decomposition of the FG-module M. For $j = 1, \ldots, m$, let R_j be a vertex of M_j.*

(a) There are some $\lambda = (\lambda_1, \ldots, \lambda_m) \in \Lambda(m, n)$ and an indecomposable direct summand N' of $\mathrm{Res}_{\mathfrak{S}_\lambda}^{\mathfrak{S}_n}(N)$ such that $L \cong \mathrm{Ind}_{G \wr \mathfrak{S}_\lambda}^{G \wr \mathfrak{S}_n}(M^{\otimes \lambda} \otimes \widehat{N'})$. For every vertex $Q' \leqslant \mathfrak{S}_\lambda$ of N', the group

$$Q := (R_1^{\lambda_1} \times \cdots \times R_m^{\lambda_m}) \rtimes (Q')^\sharp \leqslant G \wr \mathfrak{S}_\lambda \leqslant G \wr \mathfrak{S}_n$$

is a vertex of L.

(b) Suppose in addition that $M_1, \ldots, M_m$ are trivial-source modules. Then there is, moreover, a Q'-source S' of N' such that $\mathrm{Inf}_{Q'}^Q(S')$ is a Q-source of L. Here the inflation is taken with respect to the natural epimorphism $Q \longrightarrow Q'$.

(c) If M is a projective FG-module, and if λ and N' are as is part (a), then $(Q')^\sharp$ is a vertex of L. If S' is any Q'-source of N' then $(S')^\sharp$ is also a $(Q')^\sharp$-source of L.

We now come back to Lie modules and their direct sum decompositions. The next theorem is a consequence of the results in [3], [6] and [24]. The latter are concerned with Lie powers of the general linear group, where for more details and the notation we are using we refer the reader to Appendix A.

3.5 Theorem. *Let $k > 1$ with $p \nmid k$. Then, for every $s \geqslant 0$, there is a projective $F\mathfrak{S}_{kp^s}$-module X_{kp^s} such that, for all $d \geqslant 0$, one has*

$$\mathrm{Lie}_F(kp^d) \cong \bigoplus_{t=0}^d \mathrm{Ind}_{\mathfrak{S}_{kp^t} \wr \mathfrak{S}_{p^{d-t}}}^{\mathfrak{S}_{kp^d}} (X_{kp^t}^{\otimes p^{d-t}} \otimes \widehat{\mathrm{Lie}_F(p^{d-t})})\,. \tag{3}$$

Proof. We fix $n := kp^d$, for some $d \geqslant 0$, and consider the natural $F[\mathrm{GL}_n(F)]$-module $E := F^n$. In accordance with (8) in Appendix A, we denote by $L^n(E)$ the n-th Lie power of E. Then, as explained in A.2(b), the Lie module $\mathrm{Lie}_F(n)$ is the image of the $F[\mathrm{GL}_n(F)]$-module $L^n(E)$ under the Schur functor $\mathfrak{W}^n$, taking n-homogeneous polynomial $F[\mathrm{GL}_n(F)]$-modules to $F\mathfrak{S}_n$-modules.

By [3, Thm. 3.4], for all $t \geqslant 0$ there are idempotents f_{kp^t} in the group algebra $F\mathfrak{S}_{kp^t}$, only depending on p, k and t, but independent of n, such that there is an isomorphism of $F[\mathrm{GL}_n(F)]$-modules

$$L^n(E) \cong \bigoplus_{t=0}^{d} L^{p^{d-t}}(E^{\otimes kp^t} * f_{kp^t}) ; \tag{4}$$

here, as detailed in A.1(a), the group $\mathfrak{S}_{kp^t}$ acts from the right on $E^{\otimes kp^t}$ by place permutations, and we denote this action by $*$.

Now fix some $0 \leqslant t \leqslant d$, and set $m := kp^t$ and $q := p^{d-t}$. Suppose that V is any polynomial $F[\mathrm{GL}_n(F)]$-module that is homogeneous of degree m, so that the Schur functor can be applied to V, yielding the left $F\mathfrak{S}_m$-module $\mathfrak{W}^m(V)$. Then, by [24, Cor. 3.2], there is an isomorphism of $F\mathfrak{S}_n$-modules

$$\mathfrak{W}^n(L^q(V)) \cong \mathrm{Ind}_{\mathfrak{S}_m \wr \mathfrak{S}_q}^{\mathfrak{S}_n} (\mathfrak{W}^m(V)^{\otimes q} \otimes \widehat{\mathrm{Lie}_F(q)}) .$$

We apply this to our fixed direct summand on the right-hand side of (4), with $V := E^{\otimes m} * f_m$. Thus letting

$$X_m := \mathfrak{W}^m(E^{\otimes m} * f_m)$$

yields the decomposition (3), and it remains to show that X_m is a projective $F\mathfrak{S}_m$-module, and does not depend on the $F[\mathrm{GL}_n(F)]$-module E used to define it.

But, since $n \geqslant m$, the Schur functor $\mathfrak{W}^m$ takes $E^{\otimes m}$ to the regular module $F\mathfrak{S}_m$, and the isomorphism $\mathfrak{W}^m(E^{\otimes m}) \cong F\mathfrak{S}_m$ translates the place permutation action into right multiplication. Thus we deduce

$$X_m = \mathfrak{W}^m(E^{\otimes m} * f_m) = \mathfrak{W}^m(E^{\otimes m}) * f_m \cong F\mathfrak{S}_m \cdot f_m ,$$

which is of course a projective $F\mathfrak{S}_m$-module, and independent of n. $\qquad\square$

Note that also for the case $k = 1$, which is excluded from the present discussion, we have a decomposition similar to (3), but in this case becoming trivial inasmuch as for $t = 0$ we get the trivial $F\mathfrak{S}_1$-module $X_1 \cong F$, and $X_{p^t} = \{0\}$ for $t \geqslant 1$. Thus the crucial question arising now is whether the projective modules X_{kp^t} in the decomposition (3) could possibly be $\{0\}$. The next lemma, based on the calculations in [5], shows that this is not the case, a result which will be essential for Theorem 3.7 below.

3.6 Lemma. *Keep the notation as in Theorem 3.5. Then $X_{kp^t} \neq \{0\}$, for all $t \geqslant 0$.*

Proof. For $t \geqslant 0$, write

$$x_{kp^t} := \frac{\dim(X_{kp^t})}{\dim(\mathrm{Lie}_F(kp^t))} = \frac{\dim(X_{kp^t})}{(kp^t - 1)!} .$$

Note that, taking $d = t$ in Theorem 3.5, we see that X_{kp^t} is isomorphic to a submodule of the Lie module $\mathrm{Lie}_F(kp^t)$. Thus $0 \leqslant x_{kp^t} \leqslant 1$ and it suffices to show that $x_{kp^t} > 0$. Now observe

that, in the notation of [5, page 851], we have $X_{kp^t} = C(kp^t)$. Hence, by [5, (10)],

$$x_{kp^t} = 1 - \sum_{i=1}^{t} a_i'(x_{kp^{t-i}})^{p^i},$$

where $a_i' = (kp^{t-i})^{-(p^i-1)}$. Since $k > 1$, we have

$$\sum_{i=1}^{t} a_i'(x_{kp^{t-i}})^{p^i} \leqslant \sum_{i=1}^{t} a_i' \leqslant \sum_{i=1}^{t} k^{-(p^i-1)} < \sum_{j=1}^{\infty} k^{-j} = (k-1)^{-1} \leqslant 1.$$

Therefore, $x_{kp^t} > 0$. $\hfill\square$

We are now in a position to formulate the following result, reducing the problem of determining vertices of indecomposable direct summands of $\mathrm{Lie}_F^{\mathrm{pf}}(n)$ to the case where n is a p-power.

3.7 Theorem. *Let $n = k \cdot p^d$, for some $d \geqslant 0$ and some $k > 1$ with $p \nmid k$.*

(a) *Let L be an indecomposable direct summand of $\mathrm{Lie}_F(n)$ with vertex Q. There is an integer $t \in \{0, \ldots, d\}$, a composition λ of p^{d-t}, and an indecomposable direct summand L' of $\mathrm{Res}_{\mathfrak{S}_\lambda}^{\mathfrak{S}_{p^{d-t}}}(\mathrm{Lie}_F(p^{d-t}))$ such that*

$$Q \leqslant_{\mathfrak{S}_n} (Q')^\sharp \leqslant \mathfrak{S}_\lambda^\sharp \leqslant \mathfrak{S}_{kp^t} \wr \mathfrak{S}_{p^{d-t}} \leqslant \mathfrak{S}_n, \quad \text{for every vertex } Q' \text{ of } L'.$$

(b) *In the situation of part (a), if S' is a Q'-source of L', then there is a Q-source S of L such that $S \mid \mathrm{Res}_Q^{(Q')^\sharp}((S')^\sharp)$. Moreover, there is an indecomposable direct summand K of $\mathrm{Lie}_F(p^{d-t})$ having a vertex R with $Q' \leqslant R$. Furthermore, there is an R-source T of K such that $S' \mid \mathrm{Res}_{Q'}^R(T)$.*

(c) *Conversely, let $0 \leqslant t \leqslant d$, and let K be any indecomposable direct summand of $\mathrm{Lie}_F(p^{d-t})$ with vertex R. Then there is an indecomposable direct summand L of $\mathrm{Lie}_F(n)$ with vertex $R^\sharp$, and every R-source of K is then also an $R^\sharp$-source of L.*

Proof. Parts (a) and (b) are immediate consequences of 2.2, Theorem 3.4 and Theorem 3.5. Note that here we need the fact that the $F\mathfrak{S}_{kp^t}$-modules X_{kp^t} in Theorem 3.5 are projective.

It remains to prove (c). So let $t \in \{0, \ldots, d\}$. Let further X be any indecomposable direct summand of the $F\mathfrak{S}_{kp^t}$-module X_{kp^t}; note that here we need Lemma 3.6 to ensure that all the projective modules in Theorem 3.5 are indeed non-zero. Now consider the one-part partition $\lambda = (p^{d-t})$ of p^{d-t}. Then, by 3.3, we get the indecomposable $F[\mathfrak{S}_{kp^t} \wr \mathfrak{S}_{p^{d-t}}]$-module $L' := X^{\otimes p^{d-t}} \otimes \widehat{K}$. By Theorem 3.4(c), L' has vertex $R^\sharp$, and every R-source T of K yields the $R^\sharp$-source $T^\sharp$ of L'. As we have remarked in 2.2(d), there is an indecomposable direct summand L of $\mathrm{Ind}_{\mathfrak{S}_{kp^t} \wr \mathfrak{S}_{p^{d-t}}}^{\mathfrak{S}_n}(L')$ with vertex $R^\sharp$ and $R^\sharp$-source $T^\sharp$. By Theorem 3.5, we have $\mathrm{Ind}_{\mathfrak{S}_{kp^t} \wr \mathfrak{S}_{p^{d-t}}}^{\mathfrak{S}_n}(L') \mid \mathrm{Lie}_F(n)$, and hence assertion (c) follows. $\hfill\square$

4 Computational Data

In this section we present our computational results concerning the Lie modules for some symmetric groups. Before doing so, we need a few preparations. More details on specific computational techniques and implementations used to obtain the results of this section are given in Appendix B.

4.1. Some subgroups of symmetric groups. (a) We shall use the following convention for denoting the Sylow p-subgroups of the symmetric group $\mathfrak{S}_n$: suppose first that $n = p^d$, for some $d \geqslant 0$. Moreover, we set $P_1 := 1$ and $P_p := C_p$, where $C_p := \langle (1, 2, \ldots, p) \rangle$, as well as

$$P_{p^{i+1}} := P_{p^i} \wr C_p = \{(g_1, \ldots, g_p; \sigma) \mid g_1, \ldots, g_p \in P_{p^i},\, \sigma \in C_p\} \quad \text{for} \quad i \geqslant 1\,.$$

Note that the multiplication in $P_{p^{i+1}}$ is as explained in (1). For $i \geqslant 0$, we view P_{p^i} as a subgroup of $\mathfrak{S}_{p^i}$ in the obvious way. Then, by [21, 4.1.22, 4.1.24], P_n is a Sylow p-subgroup of $\mathfrak{S}_n$, which can be generated by the following elements in $\mathfrak{S}_n$:

$$g_j := \prod_{k=1}^{p^{j-1}} (k, k + p^{j-1}, k + 2p^{j-1}, \ldots, k + (p-1)p^{j-1}) \quad \text{with} \quad j = 1, \ldots, d\,.$$

Next suppose that $p \mid n$, but n is not necessarily a p-power. Consider the p-adic expansion $n = \sum_{j=1}^s \alpha_j p^{i_j}$, for some $s \geqslant 1$, $i_1 > \ldots > i_s \geqslant 1$, and $1 \leqslant \alpha_j \leqslant p - 1$ for $j = 1, \ldots, s$. By [21, 4.1.22, 4.1.24], $P_n := \prod_{j=1}^s \prod_{l_j=1}^{\alpha_j} P_{p^{i_j}, l_j}$ is then a Sylow p-subgroup of $\mathfrak{S}_n$. Here, the direct factor $P_{p^{i_1}, 1}$ is acting on $\{1, \ldots, p^{i_1}\}$, $P_{p^{i_1}, 2}$ is acting on $\{p^{i_1} + 1, \ldots, 2p^{i_1}\}$, and so on.

If n is not divisible by p, let $m < n$ be maximal such that $p \mid m$, and set $P_n := P_m$, so that P_n is a Sylow p-subgroup of $\mathfrak{S}_n$ also in this case.

(b) For $d \geqslant 1$ we denote by E_{p^d} the following maximal elementary abelian subgroup of P_{p^d} that acts regularly on $\{1, \ldots, p^d\}$: recall the generators $g_1, \ldots, g_d$ of P_{p^d} from (a) above. For $j = 1, \ldots, d-1$, let $g_{j,j+1} := \prod_{i=0}^{p-1} g_{j+1}^i g_j g_{j+1}^{-i}$, and for $l = 1, \ldots, d - j - 1$, we inductively set

$$g_{j,j+1,\ldots,j+l+1} := \prod_{i=0}^{p-1} g_{j+l+1}^i \cdot g_{j,j+1,\ldots,j+l} \cdot g_{j+l+1}^{-i}\,.$$

Then $E_{p^d} := \langle g_{1,\ldots,d}, g_{2,\ldots,d}, \ldots, g_{d-1,d}, g_d \rangle$, and we get $|E_{p^d}| = p^d$. Letting $n := p^d$ and $Q := E_{p^d}$, we determine the structure of $N_{\mathfrak{S}_n}(Q)$:

Since $Q \leqslant \mathfrak{S}_n$ is an abelian transitive subgroup, it is self-centralizing, and thus $N_{\mathfrak{S}_n}(Q)/Q$ is isomorphic to a subgroup of $\mathrm{GL}_d(p)$. Moreover, since the affine linear group $\mathrm{AGL}_d(p) \cong C_p^d \rtimes \mathrm{GL}_d(p)$ acts transitively and faithfully on its elementary abelian subgroup C_p^d, there is an embedding $\mathrm{AGL}_d(p) \longrightarrow \mathfrak{S}_n$, mapping C_p^d to Q. Hence we conclude that

$$N_{\mathfrak{S}_n}(Q) \cong Q \rtimes \mathrm{GL}_d(p)\,.$$

Thus $N_{\mathfrak{S}_n}(Q)$ acts transitively on each of the sets $\{R \leqslant Q \mid |R| = p^i\}$, for $0 \leqslant i \leqslant d$, and the stabilizer $N_{N_{\mathfrak{S}_n}(Q)}(R)$ induces the full automorphism group on any subgroup $R \leqslant Q$. Hence $N_{\mathfrak{S}_n}(Q)$ controls fusion in Q, in the sense of 2.5.

4.2 Example. If $p = 2$ and $n = 4$ then

$$P_4 = \langle (1,2), (1,3)(2,4) \rangle \quad \text{and}$$
$$E_4 = \langle (1,2)(3,4), (1,3)(2,4) \rangle.$$

If $p = 2$ and $n = 8$ then

$$P_8 = \langle (1,2), (1,3)(2,4), (1,5)(2,6)(3,7)(4,8) \rangle \quad \text{and}$$
$$E_8 = \langle (1,2)(3,4)(5,6)(7,8), (1,3)(2,4)(5,7)(6,8), (1,5)(2,6)(3,7)(4,8) \rangle.$$

If $p = 3$ and $n = 9$ then

$$P_9 = \langle (1,2,3), (1,4,7)(2,5,8)(3,6,9) \rangle \quad \text{and}$$
$$E_9 = \langle (1,2,3)(4,5,6)(7,8,9), (1,4,7)(2,5,8)(3,6,9) \rangle.$$

4.3 Remark. (a) Before proceeding to computationally substantial examples, for later use we recall the following well-known case: by [13, Cor. 9] the projective-free part $\operatorname{Lie}_F^{\mathrm{pf}}(p)$ of the Lie module $\operatorname{Lie}_F(p)$ is indecomposable, and actually isomorphic to the Specht module $S^{(p-1,1)}$ of dimension $p - 1$, hence

$$\operatorname{Lie}_F^{\mathrm{pf}}(p) \cong \Omega(F) \quad \text{as } F\mathfrak{S}_p\text{-modules}.$$

Hence $\operatorname{Lie}_F^{\mathrm{pf}}(p)$ has vertex $E_p = C_p$, of course, and $\Omega(F)$ is an E_p-source, having dimension $p - 1$. Note that the FE_p-module $\Omega(F)$ is an endo-permutation module. Thus, from Theorem 2.4 and the remarks in 2.5 and 4.1(b) we conclude that $\operatorname{Lie}_F^{\mathrm{pf}}(p)$ is an endo-p-permutation module.

(b) In view of the subsequent results, we ask ourselves whether $\operatorname{Lie}_F^{\mathrm{pf}}(p)$ itself possibly is an endo-permutation $F\mathfrak{S}_p$-module. Indeed, for $p = 2$ we have $\operatorname{Lie}_F(2) \cong \operatorname{Lie}_F^{\mathrm{pf}}(2) \cong F$, hence $\operatorname{Lie}_F^{\mathrm{pf}}(2)$ is even a permutation $F\mathfrak{S}_2$-module.

For $p = 3$ we have $\operatorname{Lie}_F(3) \cong \operatorname{Lie}_F^{\mathrm{pf}}(3) \cong \Omega(F)$, and from the theory of blocks of cyclic defect it is immediate that

$$\operatorname{Lie}_F^{\mathrm{pf}}(3) \otimes \operatorname{Lie}_F^{\mathrm{pf}}(3)^* \cong \Omega(F) \otimes \Omega(F)^* \cong F \oplus P^{(2,1)},$$

where $D^{(2,1)}$ is the sign representation. Note that, in accordance with part (a), all indecomposable direct summands of $\operatorname{Lie}_F^{\mathrm{pf}}(3) \otimes \operatorname{Lie}_F^{\mathrm{pf}}(3)^*$ are indeed trivial-source modules.

To show that $\operatorname{Lie}_F^{\mathrm{pf}}(3) \otimes \operatorname{Lie}_F^{\mathrm{pf}}(3)^*$ is not a permutation $F\mathfrak{S}_3$-module, assume to the contrary that it is. Thus, by dimension reasons we conclude that $P^{(2,1)}$ is an indecomposable transitive permutation $F\mathfrak{S}_3$-module, but $P^{(2,1)}$ does not have the trivial module as an epimorphic image, a contradiction.

We now turn to specific examples, dealt with by computational techniques. However, efficient machine treatment is only feasible for modules over finite fields. Hence, as explained in A.2(b), we pass from the Lie $F\mathfrak{S}_n$-module $\operatorname{Lie}_F(n)$, defined over the algebraically closed field F, to its $\mathbb{F}_p$-form $\operatorname{Lie}_p(n)$, defined over the prime field $\mathbb{F}_p$.

4.4. Examining $\mathrm{Lie}_2(4)$. Let $p = 2$. We examine the $\mathbb{F}_2\mathfrak{S}_4$-module $\mathrm{Lie}_2(4)$.

(a) By A.5, $\dim(\mathrm{Lie}_2(4)) = 3! = 6$. A dimension consideration shows that $\mathrm{Lie}_2(4)$ cannot contain a projective direct summand, hence $\mathrm{Lie}_2(4)$ coincides with its projective-free part $\mathrm{Lie}_2^{\mathrm{pf}}(4)$. Moreover, it is easily checked computationally, that $\mathrm{Lie}_2^{\mathrm{pf}}(4)$ is absolutely indecomposable, more precisely we have

$$\mathrm{Lie}_2^{\mathrm{pf}}(4) \cong \Omega^{-1}(D^{(3,1)}),$$

where $D^{(3,1)} \cong \mathrm{Inf}_{\mathfrak{S}_3}^{\mathfrak{S}_4}(D^{(2,1)})$ is the simple $\mathbb{F}_2\mathfrak{S}_4$-module of dimension 2, and the inflation is along the natural isomorphism $\mathfrak{S}_4/E_4 \cong \mathfrak{S}_3$. Since $D^{(2,1)}$ is a projective simple $\mathbb{F}_2\mathfrak{S}_3$-module, $D^{(3,1)}$ is a trivial-source module with vertex E_4. Thus we conclude that $\mathrm{Lie}_2^{\mathrm{pf}}(4)$ has vertex E_4, and $\Omega^{-1}(\mathbb{F}_2) = \mathbb{F}_2E_4/\mathrm{Soc}(\mathbb{F}_2E_4)$ is an E_4-source, having dimension 3.

Note that $\Omega^{-1}(\mathbb{F}_2)$ is an endo-permutation module. Thus from Theorem 2.4, and the remarks in 2.5 and 4.1(b), we conclude that $\mathrm{Lie}_2^{\mathrm{pf}}(4)$ is an endo-p-permutation module.

(b) In view of the above and the subsequent results, it seems worthwhile to show that $\mathrm{Lie}_2^{\mathrm{pf}}(4)$ is *not* an endo-permutation $\mathbb{F}_2\mathfrak{S}_4$-module. To this end, we compute an explicit indecomposable direct sum decomposition of $\mathrm{Lie}_2^{\mathrm{pf}}(4) \otimes \mathrm{Lie}_2^{\mathrm{pf}}(4)^*$:

$$\mathrm{Lie}_2^{\mathrm{pf}}(4) \otimes \mathrm{Lie}_2^{\mathrm{pf}}(4)^* \cong D^{(3,1)} \oplus \mathrm{Ind}_{\mathfrak{A}_4}^{\mathfrak{S}_4}(\mathbb{F}_2) \oplus P^{(4)} \oplus 3 \cdot P^{(3,1)}, \tag{5}$$

where both projective indecomposable $\mathbb{F}_2\mathfrak{S}_4$-modules $P^{(4)}$ and $P^{(3,1)}$ have dimension 8. Note that, in accordance with part (a), we indeed observe that all indecomposable direct summands of $\mathrm{Lie}_2^{\mathrm{pf}}(4) \otimes \mathrm{Lie}_2^{\mathrm{pf}}(4)^*$ are trivial-source modules.

To show that $\mathrm{Lie}_2^{\mathrm{pf}}(4)\otimes\mathrm{Lie}_2^{\mathrm{pf}}(4)^*$ is not a permutation $\mathbb{F}_2\mathfrak{S}_4$-module, assume to the contrary that it is. Then there is some $H \leqslant \mathfrak{S}_4$ such that $D^{(3,1)}$ is isomorphic to a direct summand of $\mathrm{Ind}_H^{\mathfrak{S}_4}(\mathbb{F}_2)$ and such that $\mathrm{Ind}_H^{\mathfrak{S}_4}(\mathbb{F}_2)$ is isomorphic to a direct summand of $\mathrm{Lie}_2^{\mathrm{pf}}(4) \otimes \mathrm{Lie}_2^{\mathrm{pf}}(4)^*$. In particular, $D^{(3,1)}$ is then relatively H-projective, and since, by (a), $E_4 \trianglelefteq \mathfrak{S}_4$ is a vertex of $D^{(3,1)}$, we infer $E_4 \leqslant H$. On the other hand, H cannot possibly contain a Sylow 2-subgroup of $\mathfrak{S}_4$, since otherwise $\mathrm{Ind}_H^{\mathfrak{S}_4}(\mathbb{F}_2)$ would have the trivial $\mathbb{F}_2\mathfrak{S}_4$-module as a direct summand. This leaves the cases $H \in \{E_4, \mathfrak{A}_4\}$ where $\mathfrak{A}_4$ is the alternating group of degree 4. But if $H = \mathfrak{A}_4$ then $D^{(3,1)} \nmid \mathrm{Ind}_H^{\mathfrak{S}_4}(\mathbb{F}_2)$, and if $H = E_4$ then $2 \cdot D^{(3,1)} \mid \mathrm{Ind}_H^{\mathfrak{S}_4}(\mathbb{F}_2)$. In either case, we obtain a contradiction.

4.5. Examining $\mathrm{Lie}_2(8)$. Let $p = 2$. We examine the $\mathbb{F}_2\mathfrak{S}_8$-module $\mathrm{Lie}_2(8)$.

(a) By A.5(a), $\dim(\mathrm{Lie}_2(8)) = 7! = 5040$. Moreover, using the 2-modular decomposition matrix of $\mathfrak{S}_8$ and [4, Cor. 3.4], we find $\dim(\mathrm{Lie}_2^{\mathrm{pbl}}(8)) = 4016$. By work of Selick–Wu [33], it is known that

$$\mathrm{Lie}_2^{\mathrm{pbl}}(8) \cong \mathrm{Lie}_2^{\mathrm{pf}}(8) \oplus 2 \cdot P^{(6,2)} \oplus P^{(5,3)} \oplus 4 \cdot P^{(4,3,1)}, \tag{6}$$

and we infer that $\dim(\mathrm{Lie}_2^{\mathrm{pf}}(8)) = 816 = 2^4 \cdot 3 \cdot 17$. We have verified the decomposition (6) independently, with the computational techniques described in Appendix B. In addition to the calculations in [33], we have checked explicitly that $\mathrm{Lie}_2^{\mathrm{pf}}(8)$ is actually absolutely indecomposable.

(b) We shall subsequently describe the vertices and sources of the projective-free part $\mathrm{Lie}_2^{\mathrm{pf}}(8)$. In order to do so, we consider the restriction of $\mathrm{Lie}_2^{\mathrm{pf}}(8)$ to the Sylow 2-subgroup P_8

of $\mathfrak{S}_8$; note that, since $|P_8| = 2^7$, from 2.2 we conclude that every vertex of $\mathrm{Lie}_2^{\mathrm{pf}}(8)$ has order at least 8. Our computations yield the following decomposition:

$$\mathrm{Res}_{P_8}^{\mathfrak{S}_8}(\mathrm{Lie}_2^{\mathrm{pf}}(8)) = M_1 \oplus M_2 \oplus (\mathrm{cyc}),$$

where '(cyc)' denotes a direct sum of absolutely indecomposable $\mathbb{F}_2 P_8$-modules with vertex $Z(P_8)$ of order 2, and with trivial sources.

The direct summand M_2 is absolutely indecomposable of dimension 96, and has vertex

$$V := \langle (1,3)(2,4)(5,6)(7,8), (1,4)(2,3)(5,8)(6,7) \rangle \cong C_2 \times C_2$$

of order 4, and a V-source isomorphic to $\mathbb{F}_2 V / \mathrm{Soc}(\mathbb{F}_2 V) \cong \Omega^{-1}(\mathbb{F}_2)$. In particular, the sources of M_2 are endo-permutation modules.

The remaining direct summand, M_1, is absolutely indecomposable of dimension 336, and has vertex E_8 and an E_8-source S of dimension 21 satisfying

$$\mathrm{End}_{\mathbb{F}_2}(S) \cong S \otimes S^* \cong \mathbb{F}_2 \oplus \bigoplus_{Q < E_8, |Q| = 2} 2 \cdot \mathrm{Ind}_Q^{E_8}(\mathbb{F}_2) \oplus (\mathrm{proj}),$$

where '(proj)' denotes a projective $\mathbb{F}_2 E_8$-module. Consequently, $\mathrm{End}_{\mathbb{F}_2}(S)$ is a permutation $\mathbb{F}_2 E_8$-module, that is, S is an endo-permutation $\mathbb{F}_2 E_8$-module. In fact, by Theorem 2.4, the isomorphism type of S is determined by the following isomorphism, which is easily verified computationally, using the techniques in Appendix B:

$$\Omega^3(\mathbb{F}_2) \otimes \bigotimes_{Q < E_8, |Q| = 2} \mathrm{Inf}_{E_8/Q}^{E_8}(\Omega^{-1}((\mathbb{F}_2)_{E_8/Q})) \cong S \oplus (\mathrm{proj}).$$

Note that S is the only non-projective direct summand occurring.

In conclusion, this shows that $\mathrm{Lie}_2^{\mathrm{pf}}(8)$ has vertex E_8 and endo-permutation source S. In particular, by 2.5 and 4.1(b), we conclude that $\mathrm{Lie}_2^{\mathrm{pf}}(8)$ is an endo-p-permutation module.

4.6. Examining $\mathrm{Lie}_3(9)$**.** Next let $p = 3$. We examine the $\mathbb{F}_3 \mathfrak{S}_9$-module $\mathrm{Lie}_3(9)$.

(a) By A.5(a), $\dim(\mathrm{Lie}_3(9)) = 8! = 40320$. Moreover, using the 3-modular decomposition matrix of $\mathfrak{S}_9$ and [4, Cor. 3.4], we find $\dim(\mathrm{Lie}_3^{\mathrm{pbl}}(9)) = 16020$, where $\mathrm{Lie}_3^{\mathrm{pbl}}(9)$ denotes the principal block component of $\mathrm{Lie}_3(9)$. Employing the techniques described in Appendix B, we obtain the following decomposition

$$\mathrm{Lie}_3^{\mathrm{pbl}}(9) \cong 2 \cdot P^{(7,1^2)} \oplus 5 \cdot P^{(6,3)} \oplus 3 \cdot P^{(6,2,1)} \oplus 4 \cdot P^{(5,2^2)} \oplus 2 \cdot P^{(4,3,2)} \oplus P^{(4^2,1)} \oplus 4 \cdot P^{(3^2,2,1)}$$
$$\oplus \mathrm{Lie}_3^{\mathrm{pf}}(9),$$

where hence $\mathrm{Lie}_3^{\mathrm{pf}}(9)$ has dimension $1683 = 3^2 \cdot 11 \cdot 17$, and turns out to be absolutely indecomposable.

(b) To describe the vertices and sources of the projective-free part $\mathrm{Lie}_3^{\mathrm{pf}}(9)$, we first note that from $|P_9| = 3^4$ and 2.2 we conclude that every vertex of $\mathrm{Lie}_3^{\mathrm{pf}}(9)$ has order at least 9. We determine an indecomposable direct sum decomposition of the restriction of $\mathrm{Lie}_3^{\mathrm{pf}}(9)$ to P_9, and get

$$\mathrm{Res}_{P_9}^{\mathfrak{S}_9}(\mathrm{Lie}_3^{\mathrm{pf}}(9)) \cong N_1 \oplus 2 \cdot N_2 \oplus 4 \cdot N_3 \oplus (\mathrm{proj}),$$

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where $N_2 \ncong N_3$ are absolutely indecomposable of dimension 54 each, having non-conjugate cyclic vertices of order 3, and endo-permutation sources of dimension 2.

The direct summand N_1 is absolutely indecomposable of dimension 144, and has vertex E_9 and an E_9-source S' of dimension 16 satisfying

$$\mathrm{End}_{\mathbb{F}_3}(S') \cong S' \otimes (S')^* \cong \mathbb{F}_3 \oplus \bigoplus_{Q < E_9, |Q|=3} \mathrm{Ind}_Q^{E_9}(\mathbb{F}_3) \oplus (\mathrm{proj}).$$

Consequently, $\mathrm{End}_{\mathbb{F}_3}(S')$ is a permutation $\mathbb{F}_3 E_9$-module, that is, S' is an endo-permutation $\mathbb{F}_3 E_9$-module. Its isomorphism type, in the sense of Theorem 2.4, is determined by the following isomorphism, where again S' is the only non-projective direct summand occurring:

$$\Omega^{-2}(\mathbb{F}_3) \otimes \bigotimes_{Q < E_9, |Q|=3} \mathrm{Inf}_{E_9/Q}^{E_9}(\Omega((\mathbb{F}_3)_{E_9/Q})) \cong S' \oplus (\mathrm{proj}).$$

Also this decomposition is verified computationally by the techniques described in Appendix B.

In conclusion, this shows that $\mathrm{Lie}_3^{\mathrm{pf}}(9)$ has vertex E_9 and endo-permutation source S'. In particular, by 2.5 and 4.1(b), we conclude that $\mathrm{Lie}_3^{\mathrm{pf}}(9)$ is an endo-p-permutation module.

Exploiting the computational data collected above, and applying Theorem 3.7, we now obtain the following results. Note that these, by virtue of B.6 and the property $\mathrm{Lie}_F(n) \cong F \otimes_{\mathbb{F}_p} \mathrm{Lie}_p(n)$, may safely be stated in terms of the Lie modules $\mathrm{Lie}_F(n)$ again.

4.7 Corollary. *Let $p = 2$ and $n = k \cdot 2^d$, where $k \geqslant 1$ is odd and $0 \leqslant d \leqslant 3$, and let L be an indecomposable direct summand of $\mathrm{Lie}_F(n)$.*

(a) *Let $Q \leqslant \mathfrak{S}_n$ be a vertex of L, and let S be a Q-source. Then Q is elementary abelian of order $|Q| \leqslant 2^d$, and S is an endo-permutation module.*

(b) *Suppose that $|Q|$ is maximal amongst the orders of the vertices of all the indecomposable direct summands of $\mathrm{Lie}_F(n)$. Then one has $Q =_{\mathfrak{S}_n} E_{2^d}^\sharp \leqslant \mathfrak{S}_{2^d}^\sharp \leqslant \mathfrak{S}_k \wr \mathfrak{S}_{2^d} \leqslant \mathfrak{S}_n$; in particular, Q is uniquely determined up to $\mathfrak{S}_n$-conjugation. Moreover, every E_{2^d}-source of $\mathrm{Lie}_F^{\mathrm{pf}}(2^d)$ is an $E_{2^d}^\sharp$-source of L.*

Proof. To show (a), by Theorem 3.7 and 2.2(c), there is some integer $t \in \{0, \ldots, d\}$, and there is an indecomposable direct summand L' of $\mathrm{Lie}_F(2^{d-t})$ with vertex R such that $Q \leqslant_{\mathfrak{S}_n} R^\sharp \leqslant \mathfrak{S}_{k \cdot 2^t} \wr \mathfrak{S}_{2^{d-t}} \leqslant \mathfrak{S}_n$. (Note that the image of the map $(-)^\sharp$ depends on the particular choice of t.) Moreover, we observe from the results of 4.4, 4.5 and Remark 4.3 that all the indecomposable direct summands of $\mathrm{Lie}_F(2^{d-t})$ are either projective, or are isomorphic to $\mathrm{Lie}_F^{\mathrm{pf}}(2^{d-t})$ and have elementary abelian vertex of order 2^{d-t} and endo-permutation sources. Finally, recall that the property of being an endo-permutation module is retained under restriction to subgroups and under taking direct summands.

To show (b), recall again that if L is an indecomposable direct summand of $\mathrm{Lie}_F(n)$ then there is some $t \leqslant d$ such that the vertices of L are conjugate to subgroups of $E_{2^{d-t}}^\sharp$. Now note that, by Theorem 3.7(c), there indeed is an indecomposable direct summand of $\mathrm{Lie}_F(n)$ having a vertex that is $\mathfrak{S}_n$-conjugate to $E_{2^d}^\sharp$. Thus, if $|Q|$ is maximal then we have $|Q| = 2^d$, and Q is $\mathfrak{S}_n$-conjugate to $E_{2^d}^\sharp$; hence we may assume that $Q = E_{2^d}^\sharp$. But this forces $t = 0$ and $K = \mathrm{Lie}_F^{\mathrm{pf}}(2^d)$. By Theorem 3.7(c) again, every E_{2^d}-source of $\mathrm{Lie}_F^{\mathrm{pf}}(2^d)$ is an $E_{2^d}^\sharp$-source of L. $\qquad \square$

The following result deals with the case where $p = 3$. The proof is completely analogous to that of Corollary 4.7, and is thus left to the reader.

4.8 Corollary. *Let $p = 3$ and $n = k \cdot 3^d$, where $k \geqslant 1$ is such that $3 \nmid k$, and $0 \leqslant d \leqslant 2$, and let L be an indecomposable direct summand of $\mathrm{Lie}_F(n)$.*

(a) *Let $Q \leqslant \mathfrak{S}_n$ be a vertex of L, and let S be a Q-source. Then Q is elementary abelian of order $|Q| \leqslant 3^d$, and S is an endo-permutation module.*

(b) *Let $|Q|$ be maximal amongst the orders of the vertices of all the indecomposable direct summands of $\mathrm{Lie}_F(n)$. Then one has $Q =_{\mathfrak{S}_n} E_{3^d}^\sharp \leqslant \mathfrak{S}_{3^d}^\sharp \leqslant \mathfrak{S}_k \wr \mathfrak{S}_{3^d} \leqslant \mathfrak{S}_n$; in particular, Q is uniquely determined up to $\mathfrak{S}_n$-conjugation. Moreover, every E_{3^d}-source of $\mathrm{Lie}_F^{\mathrm{pf}}(3^d)$ is an $E_{3^d}^\sharp$-source of L.*

A   Appendix: The Lie Module of the Symmetric Group

In this appendix we list some properties of the Lie $F\mathfrak{S}_n$-module, and briefly discuss variations. Many of these observations are certainly well known to the experts, but explicit references are not too easy to find. Thus we recall them here for the readers' convenience, and to make this paper as self-contained as possible.

A.1. Lie powers. (a) Let $\mathrm{GL}_n(F)$ be the general linear group over F, where $n \geqslant 1$, which acts naturally on E where $E := F^n$, and let $\{e_1, \ldots, e_n\}$ be the standard basis of E. We may view $\mathfrak{S}_n$ as a subgroup of $\mathrm{GL}_n(F)$, by identifying a permutation $\pi \in \mathfrak{S}_n$ with the corresponding permutation matrix in $\mathrm{GL}_n(F)$.

The r-th tensor power $E^{\otimes r}$, where $r \geqslant 1$, is an $F[\mathrm{GL}_n(F)]$-module by way of the diagonal action. Thus, via restriction, $E^{\otimes r}$ becomes an $F\mathfrak{S}_n$-module, where the symmetric group $\mathfrak{S}_n$ acts by

$$\pi : v_1 \otimes \cdots \otimes v_r \longmapsto \pi v_1 \otimes \cdots \otimes \pi v_r, \quad \text{for} \quad v_1, \ldots, v_r \in E, \ \pi \in \mathfrak{S}_n.$$

On the other hand, $E^{\otimes r}$ also carries a right $F\mathfrak{S}_r$-action '$*$' via place permutations, which hence centralizes the $F[\mathrm{GL}_n(F)]$-action:

$$\sigma : v_1 \otimes \cdots \otimes v_r \longmapsto (v_1 \otimes \cdots \otimes v_r) * \sigma = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(r)}, \quad \text{for} \quad v_1, \ldots, v_r \in E, \ \sigma \in \mathfrak{S}_r.$$

(b) We regard the tensor algebra $T(E) = \bigoplus_{r \geqslant 1} E^{\otimes r}$ as a Lie algebra in the usual way, i.e. with bracket $[a, b] = ab - ba$. We consider the *Lie bracket*

$$\kappa_2 : E^{\otimes 2} \longrightarrow E^{\otimes 2}, \ v_1 \otimes v_2 \longmapsto [v_1, v_2] := v_1 \otimes v_2 - v_2 \otimes v_1, \quad \text{for} \quad v_1, v_2 \in E.$$

More generally, we have the (*left-normed*) Lie bracket

$$\kappa_r : E^{\otimes r} \longrightarrow E^{\otimes r}, \ v_1 \otimes \cdots \otimes v_r \longmapsto [[v_1, v_2, \ldots, v_r] := [\cdots [[v_1, v_2], v_3], \ldots, v_r],$$

for all $r \geqslant 2$, and for completeness we also let $\kappa_1 := \mathrm{id}$.

Hence we have $\kappa_r \in \mathrm{End}_F(E^{\otimes r})$, for all $r \geqslant 1$, where we assume $\mathrm{End}_F(E^{\otimes r})$ to act on $E^{\otimes r}$ from the right, the action also being denoted by '$*$'. The image $(v_1 \otimes \cdots \otimes v_r) * \kappa_r \in E^{\otimes r}$ of a pure tensor $v_1 \otimes \cdots \otimes v_r \in E^{\otimes r}$ is called an *(iterated) Lie bracket* of *length r*. Hence, by

definition, for all $r \geqslant 2$ we have $(v_1 \otimes \cdots \otimes v_r) * \kappa_r = [(v_1 \otimes \cdots \otimes v_{r-1}) * \kappa_{r-1}, v_r]$. Moreover, for $r \geqslant 1$, the right adjoint action of $T(E)$ translates into

$$(v \otimes (v_1 \otimes \cdots \otimes v_r) * \kappa_r) * \kappa_{r+1} = [v, (v_1 \otimes \cdots \otimes v_r) * \kappa_r] \in E^{\otimes(r+1)}. \tag{7}$$

The map κ_r centralizes the $F[\mathrm{GL}_n(F)]$-action, hence the image

$$L^r(E) := E^{\otimes r} * \kappa_r \subseteq E^{\otimes r} \tag{8}$$

of κ_r is an $F[\mathrm{GL}_n(F)]$-submodule of $E^{\otimes r}$, called the rth *Lie power* of E, where of course we have $L^1(E) = F^n * \kappa_1 = E * \mathrm{id} = E$. Thus we obtain the free Lie algebra on $\{e_1, \ldots, e_n\}$

$$L(E) := \bigoplus_{r \geqslant 1} L^r(E) \subseteq T(E) \, .$$

The fact that $L(E)$ is free as a Lie algebra is well known, and is due to Witt.

(c) The action of κ_r is induced by the place permutation of some element $\omega_r \in F\mathfrak{S}_r$ which we will now show.

For $r \geqslant 1$ let $c_r := (r, r-1, \ldots, 1) \in \mathfrak{S}_r$. Note that, of course, $c_1 = 1$. Then the place permutation action yields

$$(v_1 \otimes \cdots \otimes v_r) * c_r = (v_r \otimes v_1 \otimes \cdots \otimes v_{r-1}), \quad \text{for} \quad v_1, \ldots, v_r \in E \, .$$

Now, for $r = 2$, we have $(v_1 \otimes v_2) * \kappa_2 = [v_1, v_2] = v_1 \otimes v_2 - v_2 \otimes v_1 = (v_1 \otimes v_2) * (1 - c_2)$, while for $r \geqslant 3$ and $v_1, \ldots, v_r \in E$ we get

$$\begin{aligned}
(v_1 \otimes \cdots \otimes v_r) * \kappa_r &= ((v_1 \otimes \cdots \otimes v_{r-1}) * \kappa_{r-1}) \otimes v_r - v_r \otimes ((v_1 \otimes \cdots \otimes v_{r-1}) * \kappa_{r-1}) \\
&= (v_1 \otimes \cdots \otimes v_r) * (\kappa_{r-1} \otimes \mathrm{id}) - (v_1 \otimes \cdots \otimes v_r) * (\kappa_{r-1} \otimes \mathrm{id}) * c_r \\
&= (v_1 \otimes \cdots \otimes v_r) * (\kappa_{r-1} \otimes \mathrm{id}) * (1 - c_r).
\end{aligned}$$

Thus, by induction on $r \geqslant 2$, this gives $\kappa_r = *(1 - c_2) * (1 - c_3) * \cdots * (1 - c_r) : E^{\otimes r} \longrightarrow E^{\otimes r}$, so that, for $r \geqslant 2$, we have

$$\omega_r := (1 - c_2)(1 - c_3) \cdots (1 - c_r) \in F\mathfrak{S}_r \, ,$$

and $L^r(E) = E^{\otimes r} * \kappa_r = E^{\otimes r} * \omega_r$. The element ω_r is called the *Dynkin–Specht–Wever element* of $F\mathfrak{S}_r$; for completeness, since $\kappa_1 = \mathrm{id}$, we let $\omega_1 := c_1 \in F\mathfrak{S}_1$. Note that we even have $\omega_r \in \mathbb{F}_p \mathfrak{S}_r$, where $\mathbb{F}_p$ is the prime field of F.

A.2. Lie modules and the Schur functor. (a) Now let $n \geqslant r$. Then the *classical Schur functor* $\mathfrak{W}^r$ takes homogeneous polynomial $F[\mathrm{GL}_n(F)]$-modules of degree r to $F\mathfrak{S}_r$-modules, where, more precisely, an $F[\mathrm{GL}_n(F)]$-module V is mapped to its (1^r)-weight space $\mathfrak{W}^r(V)$. In particular, for the $F[\mathrm{GL}_n(F)]$-module $E^{\otimes r}$ one gets the following:

As mentioned above, the natural $\mathrm{GL}_n(F)$-action on $E^{\otimes r}$ induces a permutation action of $\mathfrak{S}_n$, and thus also a permutation action of $\mathfrak{S}_r$, on $E^{\otimes r}$. The vector $e_1 \otimes \cdots \otimes e_r \in E^{\otimes r}$ affords a regular $\mathfrak{S}_r$-orbit and, hence, induces an embedding of the regular $F\mathfrak{S}_r$-module into $E^{\otimes r}$ via

$$F\mathfrak{S}_r \longrightarrow E^{\otimes r}, \ \pi \longmapsto \pi e_1 \otimes \cdots \otimes \pi e_r = e_{\pi(1)} \otimes \cdots \otimes e_{\pi(r)}, \quad \text{for} \quad \pi \in \mathfrak{S}_r.$$

The image of this embedding equals the (1^r)-weight space

$$\mathfrak{W}^r(E^{\otimes r}) = \mathrm{Span}_F(\{e_{\pi(1)} \otimes \cdots \otimes e_{\pi(r)} \mid \pi \in \mathfrak{S}_r\}) \subseteq E^{\otimes r}$$

of the $F[\mathrm{GL}_n(F)]$-module $E^{\otimes r}$. Moreover, the place permutation action of $\mathfrak{S}_r$ on $E^{\otimes r}$ restricts to $\mathfrak{W}^r(E^{\otimes r})$, and via the above isomorphism $\mathfrak{W}^r(E^{\otimes r}) \cong F\mathfrak{S}_r$ translates into right multiplication on $F\mathfrak{S}_r$.

From now on, suppose that $n = r$, which is the case most relevant to us.

(b) Now one defines the *Lie module* $\mathrm{Lie}_F(n)$ of $F\mathfrak{S}_n$ as the (1^n)-weight space of the nth Lie power $L^n(E)$, that is,

$$\mathrm{Lie}_F(n) := \mathfrak{W}^n(L^n(E)) \subseteq E^{\otimes n}.$$

From $L^n(E) = E^{\otimes n} * \kappa_n \subseteq (E)^{\otimes n}$ one thus gets

$$\mathrm{Lie}_F(n) = \mathfrak{W}^n(E^{\otimes n} * \kappa_n) = \mathfrak{W}^n(E^{\otimes n}) * \kappa_n = \mathrm{Span}_F(\{[[e_{\pi(1)}, \ldots, e_{\pi(n)}] \mid \pi \in \mathfrak{S}_n\}).$$

Via the isomorphism $\mathfrak{W}^n(E^{\otimes n}) \cong F\mathfrak{S}_n$ of $F\mathfrak{S}_n$-modules, $\mathrm{Lie}_F(n)$ can be regarded as a submodule of the regular $F\mathfrak{S}_n$-module $F\mathfrak{S}_n$. Since the action of κ_n is induced by the place permutation action of ω_n, we get

$$\mathrm{Lie}_F(n) \cong F\mathfrak{S}_n \cdot \omega_n \subseteq F\mathfrak{S}_n. \tag{9}$$

Note that, in particular, $\mathrm{Lie}_F(1) \cong F$, the trivial $F\mathfrak{S}_1$-module. We shall sometimes write a $\cdot$ in a product to make it easier to read.

Moreover, we observe that $\mathrm{Lie}_F(n)$ is already realized over the prime field $\mathbb{F}_p$ of F, that is, letting

$$\mathrm{Lie}_p(n) := \mathbb{F}_p\mathfrak{S}_n \cdot \omega_n \subseteq \mathbb{F}_p\mathfrak{S}_n \quad \text{as } \mathbb{F}_p\mathfrak{S}_n\text{-modules,}$$

we get $\mathrm{Lie}_F(n) \cong F \otimes_{\mathbb{F}_p} \mathrm{Lie}_p(n)$ as $F\mathfrak{S}_n$-modules. We have made use of this in order to facilitate explicit computations, see B.2.

A.3. Variations on Lie modules. Since there also exist slight modifications of the above modules in the literature, we briefly comment on variations of the construction:

(a) Firstly, starting with another vector $e_{\pi(1)} \otimes \cdots \otimes e_{\pi(n)} \in E^{\otimes n}$, where $\pi \in \mathfrak{S}_n$, leads to a different identification of $\mathrm{Lie}_F(n)$ with a submodule of $F\mathfrak{S}_n$, namely to the $F\mathfrak{S}_n$-module $F\mathfrak{S}_n \cdot \pi\omega_n\pi^{-1}$, that is, amounts to relabelling.

(b) Secondly, taking right-normed Lie brackets instead, for $r \geqslant 2$ one gets

$$\kappa_r' : E^{\otimes r} \longrightarrow E^{\otimes r}, \; v_1 \otimes \cdots \otimes v_r \longmapsto [v_1, v_2, \ldots, v_r]] := [v_1, [v_2, \ldots, [v_{r-1}, v_r] \cdots]];$$

we again let $\kappa_1' := \mathrm{id}$. Since $[v_1, v_2, \ldots, v_r]] = (-1)^{r-1} \cdot [[v_r, v_{r-1}, \ldots, v_1]$, for $r \geqslant 1$, we get $\kappa_r' = (-1)^{r-1} * w_r * \kappa_r : E^{\otimes r} \longrightarrow E^{\otimes r}$, where $w_r \in \mathfrak{S}_r$ is the longest element of $\mathfrak{S}_r$ in the Coxeter sense, that is,

$$w_r = (1, r)(2, r-1) \cdots$$

Thus this construction yields $\mathrm{Lie}_F'(n) = \mathfrak{W}^n(E^{\otimes n}) * \kappa_n' = \mathfrak{W}^n(E^{\otimes n}) * w_n * \kappa_n = \mathfrak{W}^n(E^{\otimes n}) * \kappa_n = \mathrm{Lie}_F(n)$.

(c) Lastly, we analyze the $F\mathfrak{S}_n$-module $F\mathfrak{S}_n \cdot \omega_n^\iota \subseteq F\mathfrak{S}_n$, where

$$\omega_n^\iota := (1 - c_n^{-1})(1 - c_{n-1}^{-1}) \cdots (1 - c_2^{-1}) \in F\mathfrak{S}_n$$

is the image of ω_n under the F-algebra anti-automorphism $\iota : F\mathfrak{S}_n \longrightarrow F\mathfrak{S}_n$ defined by $\iota : \pi \longmapsto \pi^{-1}$, for $\pi \in \mathfrak{S}_n$. Then we have an isomorphism of left $F\mathfrak{S}_n$-modules

$$F\mathfrak{S}_n \cdot \omega_n^\iota \cong (F\mathfrak{S}_n \cdot \omega_n)^*,$$

where the latter $F\mathfrak{S}_n$-module denotes the contragredient dual of $F\mathfrak{S}_n \cdot \omega_n$. This fact is most elegantly established by recalling that group algebras are, in particular, symmetric algebras, and using the general isomorphism (10) in Remark A.4 below. Thus we briefly deviate to establish this:

A.4 Remark. Let A be a finite-dimensional symmetric F-algebra with symmetrizing F-bilinear form $\langle \cdot | \cdot \rangle$. That is, $\langle \cdot | \cdot \rangle$ is associative, symmetric, and non-degenerate. Hence

$$A \longrightarrow \operatorname{Hom}_F(A, F),\ a \longmapsto (b \longmapsto \langle b | a \rangle), \quad \text{for} \quad a, b \in A,$$

is an isomorphism of (A, A)-bimodules. Letting $\omega \in A$, this induces an isomorphism

$$\operatorname{Hom}_F(A\omega, F) \cong A/(A\omega)^\perp$$

of right A-modules, where $^\perp$ denotes taking orthogonal spaces with respect to $\langle \cdot | \cdot \rangle$. Moreover,

$$(A\omega)^\perp = \{a \in A \mid \langle A\omega | a \rangle = 0\} = \{a \in A \mid \langle A | \omega a \rangle = 0\} = \{a \in A \mid \omega a = 0\} =: \ker(\omega \cdot).$$

Since $A/\ker(\omega \cdot) \cong \operatorname{im}(\omega \cdot) = \omega A$, this yields an isomorphism of right A-modules

$$\operatorname{Hom}_F(A\omega, F) \cong A/(A\omega)^\perp = A/\ker(\omega \cdot) \cong \operatorname{im}(\omega \cdot) = \omega A.$$

Finally, suppose that there is an involutory F-algebra anti-automorphism $\iota : A \longrightarrow A$, $a \longmapsto a^\iota$. Then, whenever M is a right A-module, one can define a left A-module structure on M by $a \cdot x := xa^\iota$, for $x \in M$, $a \in A$. Denoting the resulting module by M^ι, one, in particular, gets $(\omega A)^\iota \cong A\omega^\iota$ as left A-modules. Thus one has an isomorphism of left A-modules

$$\operatorname{Hom}_F(A\omega, F)^\iota \cong (\omega A)^\iota \cong A\omega^\iota. \tag{10}$$

A.5. Properties of Lie modules. Lastly, we collect a couple of properties of Lie modules.

(a) We exhibit an explicit F-basis of $\operatorname{Lie}_F(n)$. Firstly, any element of $\operatorname{Lie}_F(n)$ is an F-linear combination of Lie brackets of the form $[[e_n, e_{\pi(1)}, \ldots, e_{\pi(n-1)}]$, where $\pi \in \mathfrak{S}_{n-1}$. This is clear for $n \leqslant 2$ anyway, and for $n \geqslant 3$ is seen as follows: letting w be a Lie bracket involving a subset of $\{e_1, \ldots, e_{i-1}, e_{i+1}, \ldots, e_{n-1}\}$, where $1 \leqslant i \leqslant n-1$, we have $[[w, e_i, e_n] = [e_n, [e_i, w]]$, where, by (7), the latter can be written as a sum of Lie brackets having e_n as their first component.

Now, since expanding $[[e_n, e_{\pi(1)}, \ldots, e_{\pi(n-1)}]$ into the standard F-basis of $E^{\otimes n}$ yields a unique summand having e_n as its first component, namely $e_n \otimes e_{\pi(1)} \otimes \cdots \otimes e_{\pi(n-1)}$, we conclude that

$$\{[[e_n, e_{\pi(1)}, \ldots, e_{\pi(n-1)}] \mid \pi \in \mathfrak{S}_{n-1}\} \subseteq \operatorname{Lie}_F(n)$$

is F-linearly independent, thus is an F-basis. Moreover, since

$$[[e_n, e_{\pi(1)}, \ldots, e_{\pi(n-1)}] = \pi \cdot [[e_n, e_1, \ldots, e_{n-1}] = \pi \cdot c_n \cdot [[e_1, e_2, \ldots, e_n] \quad \text{for} \quad \pi \in \mathfrak{S}_{n-1},$$

the above F-basis can also be written as

$$\{\pi \cdot c_n \cdot [[e_1, \ldots, e_n]] \mid \pi \in \mathfrak{S}_{n-1}\} \subseteq \mathrm{Lie}_F(n).$$

Thus $\mathrm{Res}_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n}(\mathrm{Lie}_F(n))$ is isomorphic to the regular module $F\mathfrak{S}_{n-1}$, in particular saying that $\dim_F(\mathrm{Lie}_F(n)) = (n-1)!$. Moreover, since $[[e_1, \ldots, e_n]] = (e_1 \otimes \cdots \otimes e_n) * \omega_n \in \mathrm{Lie}_F(n)$ is sent to $\omega_n \in F\mathfrak{S}_n$ via the isomorphism (9), this means that an F-basis of $F\mathfrak{S}_n \cdot \omega_n$ is obtained as

$$\{\pi \cdot c_n \cdot \omega_n \mid \pi \in \mathfrak{S}_{n-1}\}.$$

As we shall explain in B.3, this F-basis has been particularly useful to facilitate the explicit computations; note that, by the observations in A.2, this is even an $\mathbb{F}_p$-basis of $\mathrm{Lie}_p(n) = \mathbb{F}_p\mathfrak{S}_n \cdot \omega_n$.

(b) We now show that $\omega_n^2 = n \cdot \omega_n \in F\mathfrak{S}_n$, in particular implying that $\frac{1}{n} \cdot \omega_n \in F\mathfrak{S}_n$ is an idempotent whenever $p \nmid n$. We proceed in various steps:

Firstly, we show that for $n \geqslant 1$ we have $\omega_n \cdot (e_1 \otimes \cdots \otimes e_n) = [[e_1, \ldots, e_n]] \in E^{\otimes n}$: this is clear for $n = 1$ anyway, and for $n = 2$ we have $\omega_2 \cdot (e_1 \otimes e_2) = (1 - c_2) \cdot (e_1 \otimes e_2) = e_1 \otimes e_2 - e_2 \otimes e_1 = (e_1 \otimes e_2) * \kappa_2$. For $n \geqslant 3$, arguing by induction and using $\omega_n = \omega_{n-1} \cdot (1 - c_n) \in F\mathfrak{S}_n$, we get

$$\begin{aligned}
\omega_n \cdot (e_1 \otimes \cdots \otimes e_n) &= \omega_{n-1} \cdot (e_1 \otimes \cdots \otimes e_n - e_n \otimes e_1 \otimes \cdots \otimes e_{n-1}) \\
&= ((e_1 \otimes \cdots \otimes e_{n-1}) * \kappa_{n-1}) \otimes e_n - e_n \otimes ((e_1 \otimes \cdots \otimes e_{n-1}) * \kappa_{n-1}) \\
&= [(e_1 \otimes \cdots \otimes e_{n-1}) * \kappa_{n-1}, e_n] \\
&= (e_1 \otimes \cdots \otimes e_n) * \kappa_n = [[e_1, \ldots, e_n]].
\end{aligned}$$

Secondly, we show that for $n \geqslant 2$ we have $\omega_{n-1} c_n \cdot [[e_1, \ldots, e_n]] = -[[e_1, \ldots, e_n]] \in E^{\otimes n}$: Recall that $\kappa_n \in \mathrm{End}_F(E^{\otimes n})$ centralizes the $F[\mathrm{GL}_n(F)]$-action. Then, using (7) we get

$$\begin{aligned}
\omega_{n-1} c_n \cdot (e_1 \otimes \cdots \otimes e_n) * \kappa_n &= (e_n \otimes \omega_{n-1} \cdot (e_1 \otimes \cdots \otimes e_{n-1})) * \kappa_n \\
&= (e_n \otimes (e_1 \otimes \cdots \otimes e_{n-1}) * \kappa_{n-1}) * \kappa_n \\
&= [e_n, (e_1 \otimes \cdots \otimes e_{n-1}) * \kappa_{n-1}] \\
&= -[(e_1 \otimes \cdots \otimes e_{n-1}) * \kappa_{n-1}, e_n] \\
&= -(e_1 \otimes \cdots \otimes e_n) * \kappa_n.
\end{aligned}$$

Combining these computations we get

$$\omega_{n-1} c_n \cdot \omega_n \cdot (e_1 \otimes \cdots \otimes e_n) = -\omega_n \cdot (e_1 \otimes \cdots \otimes e_n),$$

thus translating via the isomorphism $\mathfrak{W}^n(E^{\otimes n}) \cong F\mathfrak{S}_n$ yields $\omega_{n-1} c_n \cdot \omega_n = -\omega_n \in F\mathfrak{S}_n$.

Thirdly, and finally, we show that $\omega_n^2 = n \cdot \omega_n \in F\mathfrak{S}_n$, for $n \geqslant 1$: this is clear for $n = 1$ anyway, and for $n = 2$ we have $\omega_2^2 = (1 - c_2)^2 = \omega_2 - c_2\omega_2 = 2\omega_2$. Then, for $n \geqslant 3$ we have $\omega_n^2 = \omega_{n-1}(1 - c_n)\omega_n = \omega_{n-1}\omega_n - \omega_{n-1}c_n\omega_n$, where, by induction, the first summand equals

$$\omega_{n-1}^2(1 - c_n) = (n-1)\omega_{n-1}(1 - c_n) = (n-1)\omega_n.$$

The second summand being $-\omega_{n-1}c_n\omega_n = \omega_n$, this yields $\omega_n^2 = (n-1)\omega_n + \omega_n = n\omega_n$.

B Appendix: Computational Tools

We now give a description of the tools from computational group theory and computational representation theory we have employed, and indicate the computational ideas used to obtain the explicit results concerning some larger Lie modules in Section 4. As a general background reference, see [25].

B.1. To deal with finite groups, in particular permutation groups, matrix groups and p-groups, and to examine their structure, for example determining subgroup lattices and finding subgroup coset representatives, we use the general purpose computer algebra systems MAGMA [7] and GAP [16]. Moreover, we make use of the character table library CTblLib [2] of GAP, which provides electronic access to the data collected in the Atlas [8] and in the ModularAtlas [22, 37]; these databases, in particular, contain the explicit 2- and 3-modular decomposition matrices for various symmetric groups given in [20, App.] or [21, App. I].

Additionally, we have used the more specialized computer algebra system MeatAxe [31, 32], and its extensions [26, 27, 28, 29], to deal with various aspects concerning matrix representations over (small) finite fields. Apart from general linear algebra, these tools, in particular, allow us to find composition series and direct sum decompositions, including isomorphism checks of simple and indecomposable modules, respectively, and to find splitting fields and to check absolute indecomposability; moreover, they enable us to compute homomorphism spaces and endomorphism rings, to determine radical and socle series, and to compute submodule lattices; apart from these analytic capabilities, they also provide the constructions needed below, such as Kronecker products and the computation of Heller translates.

B.2. To facilitate explicit computations, we make use of the observation in A.2, saying that $\mathrm{Lie}_F(n) \cong F \otimes_{\mathbb{F}_p} \mathrm{Lie}_p(n)$ as $F\mathfrak{S}_n$-modules, where

$$\mathrm{Lie}_p(n) := \mathbb{F}_p\mathfrak{S}_n \cdot \omega_n \subseteq \mathbb{F}_p\mathfrak{S}_n \quad \text{as } \mathbb{F}_p\mathfrak{S}_n\text{-modules}.$$

Thus we are indeed reduced to considerations of permutation representations, and matrix representations over finite (prime) fields.

Having got our hands on the $\mathbb{F}_p\mathfrak{S}_n$-module $\mathrm{Lie}_p(n)$, the task then is to find a decomposition $\mathrm{Lie}_p(n) \cong \mathrm{Lie}_p^{\mathrm{pf}}(n) \oplus \mathrm{Lie}_p^{\mathrm{pr}}(n)$ into its projective-free and projective part, respectively, to determine how $\mathrm{Lie}_p^{\mathrm{pr}}(n)$ decomposes into projective indecomposable modules, and what the indecomposable direct summands of $\mathrm{Lie}_p^{\mathrm{pf}}(n)$ look like. However, the examples of Lie modules $\mathrm{Lie}_p(n)$ to be dealt with here are too large to simply apply to them the general techniques available to compute direct sum decompositions. Hence we have to proceed otherwise to find $\mathrm{Lie}_p^{\mathrm{pf}}(n)$ in the first place; after all, by the asymptotic results mentioned in the introduction, we expect $\mathrm{Lie}_p^{\mathrm{pf}}(n)$ to be small compared with $\mathrm{Lie}_p(n)$, small enough to allow for a detailed analysis.

By [15], we know that, in order to detect $\mathrm{Lie}_p^{\mathrm{pf}}(n)$, we only need to consider the component $\mathrm{Lie}_p^{\mathrm{pbl}}(n)$ of $\mathrm{Lie}_p(n)$ belonging to the principal p-block of $\mathbb{F}_p\mathfrak{S}_n$. Using the p-modular decomposition matrix of $\mathfrak{S}_n$, which is available for all cases considered here, and [4, Cor. 3.4], we may determine $\dim(\mathrm{Lie}_p^{\mathrm{pbl}}(n))$ in advance, and [4, Thm. 3.1] also tells us the projective indecomposable direct summands of $\mathrm{Lie}_p(n)/\mathrm{Lie}_p^{\mathrm{pbl}}(n)$, so that next to $\mathrm{Lie}_p^{\mathrm{pf}}(n)$ only the projective indecomposable direct summands of $\mathrm{Lie}_p^{\mathrm{pbl}}(n)$ have to be determined.

B.3. To find an $\mathbb{F}_p$-basis of $\mathrm{Lie}_p(n)$ or $\mathrm{Lie}_p^{\mathrm{pbl}}(n)$ in the first place, let $\epsilon_n \in \mathbb{F}_p\mathfrak{S}_n$ be the centrally primitive idempotent belonging to the principal p-block of $\mathbb{F}_p\mathfrak{S}_n$; recall that ϵ_n can be computed from the ordinary character table of $\mathfrak{S}_n$. We now use the observation in A.5, saying that an $\mathbb{F}_p$-basis of $\mathrm{Lie}_p(n)$ is

$$\{\pi \cdot c_n \cdot \omega_n \mid \pi \in \mathfrak{S}_{n-1}\} \subseteq \mathrm{Lie}_p(n) \subseteq \mathbb{F}_p\mathfrak{S}_n \,,$$

implying that an $\mathbb{F}_p$-spanning set of $\mathrm{Lie}_p^{\mathrm{pbl}}(n)$ is given as

$$\{\pi \cdot c_n \cdot \omega_n \cdot \epsilon_n \mid \pi \in \mathfrak{S}_{n-1}\} \subseteq \mathrm{Lie}_p(n) \cdot \epsilon_n = \mathrm{Lie}_p^{\mathrm{pbl}}(n) \subseteq \mathbb{F}_p\mathfrak{S}_n \,.$$

Hence our starting point is the regular representation $\mathbb{F}_p\mathfrak{S}_n$ of $\mathfrak{S}_n$, equipped with its natural $\mathbb{F}_p$-basis. Determining the permutation action of elements of $\mathfrak{S}_n$ on coordinate vectors with respect to this basis essentially amounts to computing with permutations in $\mathfrak{S}_n$. This allows us to apply successively all elements of $\mathfrak{S}_{n-1}$ to $c_n \cdot \omega_n \in \mathbb{F}_p\mathfrak{S}_n$ and $c_n \cdot \omega_n \cdot \epsilon_n \in \mathbb{F}_p\mathfrak{S}_n$, respectively; to do this efficiently, we first find a Schreier tree of $\mathfrak{S}_{n-1}$ in terms of some generating set, our favourite one being $\{(1,\ldots,n-1),(1,2)\}$.

Thus having found an $\mathbb{F}_p$-basis of $\mathrm{Lie}_p(n)$, we directly determine the action of a generating set of $\mathfrak{S}_n$, our favourite one again being $\{(1,\ldots,n),(1,2)\}$. For $\mathrm{Lie}_p^{\mathrm{pbl}}(n)$, before doing so, we pick an $\mathbb{F}_p$-basis out of the $\mathbb{F}_p$-spanning set obtained. These tasks are efficiently solved using the linear algebra routines available in the `MeatAxe`.

B.4. Next we proceed to find the projective indecomposable direct summands of $\mathrm{Lie}_p^{\mathrm{pbl}}(n)$. To do so, we apply a technique based on the considerations in [26]. In order to describe this we first recall the relevant notions:

Let A be a finite-dimensional K-algebra, where K is any field, and let S be a simple A-module. Then, the endomorphism algebra $\mathrm{End}_A(S)$ is a skew field, and for $a \in A$ letting $\ker(a_S)$ denote the kernel of the K-endomorphism of S induced by the action of a, we have $\dim(\mathrm{End}_A(S)) \mid \dim(\ker(a_S))$. Now $a \in A$ is called an S-*peakword*, if $\dim(\ker(a_S^2)) = \dim(\mathrm{End}_A(S))$, and $\ker(a_T) = \{0\}$ for all simple A-modules T not isomorphic to S. In particular, if K is a splitting field of A, then the first condition just becomes $\dim(\ker(a_S^2)) = \dim(\ker(a_S)) = 1$. In practice, peakwords are found by a random search, yielding a Monte Carlo method, which for the case of K being a (small) finite field is available in the `MeatAxe`.

Let now M be an A-module, and let $a \in A$ be an S-peakword. Then, by [26, Thm.2.5], the set of all submodules L of M such that $L/\mathrm{Rad}(L) \cong S$ concides with the set of all cyclic submodules of M generated by some $v \in \bigcup_{i \geqslant 1} \ker(a_M^i) \smallsetminus \{0\}$. Thus, in particular, all submodules of M isomorphic to the projective cover P_S of S are found this way, and for a cyclic submodule L as above we have $L \cong P_S$ if and only if $\dim(L) = \dim(P_S)$. Thus, if A is a self-injective algebra, a random search through $\bigcup_{i \geqslant 1} \ker(a_M^i)$ yields a Monte Carlo method to find a largest direct summand of M being the direct sum of copies of P_S. Note that $\dim(P_S)$ is indeed known in advance in all explicit cases considered here, and that, if K is a (small) finite field, then techniques to compute cyclic submodules are available in the `MeatAxe`.

B.5. Thus, quotienting out the projective direct summands found, we now have $\mathrm{Lie}_p^{\mathrm{pf}}(n)$ in our hands, at least with high probability. In all cases considered here this module turns out to be small enough to apply to it the general techniques available in the `MeatAxe` to find direct sum decompositions. The latter techniques would also find a projective direct summand left over, thus providing a verification of the above Monte Carlo results. Actually, for the examples considered here, this even shows that $\mathrm{Lie}_p^{\mathrm{pf}}(n)$ is indecomposable and non-projective.

Hence we may now assume that we have got a non-projective indecomposable $\mathbb{F}_p\mathfrak{S}_n$-module M, for which we have to find a vertex and a source. In order to do so, we consider the restriction $\mathrm{Res}^{\mathfrak{S}_n}_{P_n}(M)$ of M to the Sylow p-subgroup P_n of $\mathfrak{S}_n$; recall from 2.2 that, since M is relatively P_n-projective, $\mathrm{Res}^{\mathfrak{S}_n}_{P_n}(M)$ has an indecomposable direct summand sharing a vertex and a source with M.

Hence we may assume that M is an $\mathbb{F}_pP$-module, where P is a p-group. Again, we have to find direct sum decompositions, which can be speeded up by detecting particular indecomposable direct summands beforehand. Namely, in case of an $\mathbb{F}_pP$-module the strategy described in B.4 specializes to the following: the set of all submodules L of M such that $L/\mathrm{Rad}(L)$ is simple is precisely the set of all non-zero cyclic submodules of M. (Note that, in terms of the language used above, since the trivial module is the only simple $\mathbb{F}_pP$-module, the zero element in $\mathbb{F}_pP$ is a peakword.) This leads to a straightforward Monte Carlo method to find a largest direct summand of M that is the direct sum of copies of the regular module $\mathbb{F}_pP$; see [12, Sect. 3.2].

Quotienting out projective direct summands we again, in all cases considered here, end up with a module whose direct sum decomposition can be computed with the general techniques available in the MeatAxe.

Hence we may finally assume that M is an indecomposable non-projective $\mathbb{F}_pP$-module such that we are in a position to use the techniques described in [12, Sect. 3.1], whose basic ingredient is Higman's Criterion for relative projectivity; an implementation is available in MAGMA.

B.6. Finally, to make sure that computational results are still valid when going over to the algebraically closed field F again, we always check that the indecomposable modules found are actually absolutely indecomposable; techniques to achieve that are available in the MeatAxe. Recall that it is well known that $\mathbb{F}_p$ is a splitting field of $\mathbb{F}_p\mathfrak{S}_n$, hence absolute indecomposability is automatic anyway for the simple modules and the projective indecomposable modules found.

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