

# Broué’s abelian defect group conjecture and 3-decomposition numbers of the sporadic simple Conway group $\text{Co}_1$

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## Abstract

In the representation theory of finite groups, Broué’s abelian defect group conjecture says that for any prime  $p$ , if a  $p$ -block  $A$  of a finite group  $G$  has an abelian defect group  $P$ , then  $A$  and its Brauer corresponding block  $B$  of the normaliser  $N_G(P)$  of  $P$  in  $G$  are derived equivalent. We prove that Broué’s conjecture, and even Rickard’s splendid equivalence conjecture, are true for the unique 3-block  $A$  of defect 2 of the sporadic simple Conway group  $\text{Co}_1$ , implying that both conjectures hold for all 3-blocks of  $\text{Co}_1$ . To do so, we determine the 3-decomposition numbers of  $A$ , and we actually show that  $A$  is Puig equivalent to the principal 3-block of the symmetric group  $\mathfrak{S}_6$  of degree 6.

*Keywords:* Broué’s conjecture; abelian defect group; splendid derived equivalence; sporadic simple Conway group; 3-decomposition numbers.

*MSC:* 20C, 20D

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## 1. INTRODUCTION AND NOTATION

In the representation theory of finite groups, one of the most important and interesting problems is to give an affirmative answer to a conjecture which was introduced by Broué around 1988 [4]. He actually conjectures the following, where the various notions of equivalences used are recalled more precisely in **1.8**:

**Conjecture 1.1** (Broué’s Abelian Defect Group Conjecture [4]). *Let  $(\mathcal{K}, \mathcal{O}, k)$  be a splitting  $p$ -modular system, where  $p$  is a prime, for all subgroups of a finite group  $G$ . Assume that  $A$  is a block algebra of  $\mathcal{O}G$  with a defect group  $P$  and that  $B$  is a block algebra of  $\mathcal{O}N_G(P)$  such that  $B$  is the Brauer correspondent of  $A$ , where  $N_G(P)$  is the normaliser of  $P$  in  $G$ . Then  $A$  and  $B$  should be derived equivalent provided  $P$  is abelian.*

In fact, a stronger conclusion is expected:

**Conjecture 1.2** (Rickard’s Splendid Equivalence Conjecture [52, 53]). *Keeping the notation of **1.1**, and still supposing  $P$  to be abelian, then there should be a splendid derived equivalence between the block algebras  $A$  of  $\mathcal{O}G$  and  $B$  of  $\mathcal{O}N_G(P)$ .*

Conjectures **1.1** and **1.2** have been verified for several cases, albeit the general conjecture is widely open still. For an overview see [6]; in particular, by [32, 51, 55, 56] **1.1** and **1.2** are proved for blocks with cyclic defect groups in arbitrary characteristic.

Moreover, it is shown in [18, (0.2)Theorem] that conjectures **1.1** and **1.2** hold for the principal block algebra of an arbitrary finite group when the defect group is elementary abelian of order 9. In view of the strategy used in [18], and of a possible future theory reducing conjectures **1.1** and **1.2** to the quasi-simple groups, it seems worthwhile to proceed with this class of groups, as far as non-principal 3-blocks with elementary abelian defect group of order 9 are concerned. Indeed, for these cases there are partial results already known, see [14, 21, 22, 23, 25, 30, 27, 28, 41] for

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instance; all the non-principal 3-blocks with elementary abelian defect group of order 9 for the sporadic simple groups and their covering groups are given in [43].

The present paper is another step in that programme, our main result being the following:

**Theorem 1.3.** *Let  $G$  be the sporadic simple Conway group  $\text{Co}_1$ , and let  $(\mathcal{K}, \mathcal{O}, k)$  be a splitting 3-modular system for all subgroups of  $G$ . Suppose that  $A$  is the unique block algebra of  $\mathcal{O}G$  with elementary abelian defect group  $P = C_3 \times C_3$  of order 9, and that  $B$  is the block algebra of  $\mathcal{O}N_G(P)$  such that  $B$  is the Brauer correspondent of  $A$ . Then,  $A$  and  $B$  are splendidly derived equivalent, hence Conjectures 1.1 and 1.2 of Broué and Rickard hold for the block algebra  $A$ .*

As an immediate corollary we get:

**Corollary 1.4.** *Broué’s abelian defect group conjecture 1.1, and Rickard’s splendid equivalence conjecture 1.2 are true for the prime  $p = 3$  and for all block algebras of  $\mathcal{O}G$ .*

Actually, we prove 1.3 by proving the following:

**Theorem 1.5.** *The block algebra  $A$  of  $G$  and the principal 3-block algebra  $A'$  of the symmetric group  $\mathfrak{S}_6$  of degree 6 are Puig equivalent.*

Hence, recalling the notion from [27, Remark 1.6(c)], this shows that  $A$  is pseudo-principal. Moreover, as an immediate corollary we get:

**Corollary 1.6.** *The block algebra  $A$  of  $G$  and the principal 3-block algebra of the symplectic group  $\text{Sp}_4(q)$ , for any prime power  $q$  such that  $q \equiv 2$  or  $5 \pmod{9}$ , are Puig equivalent.*

Typically, the starting point to proving theorems like those above is the decomposition matrix of the block in question. Alone, for the block algebra  $A$  these have not been known before. So we first set out to tackle this problem, and arrived at a fairly good approximation to the decomposition matrix of  $A$ , but we have not been able to determine it completely. Still, the approximation was good enough to compare the result with the data in [18] to arrive at the sensible guess that  $A$  might be closely related to the principal block algebra  $A'$  of the symmetric group  $\mathfrak{S}_6$ , whose decomposition matrix is well-known of course. Now, using the partial results on decomposition numbers, it was possible to activate the sophisticated block-theoretic machinery needed to actually compare the block algebras  $A$  and  $A'$ , which in turn finally paved the way to complete the decomposition matrix of  $A$ . Here it is:

**Theorem 1.7.** *The 3-decomposition matrix of  $A$  and of  $A'$  is the following, where we denote ordinary characters by their degrees, but also give the ATLAS notation for the characters in  $A$ :*

	$A$	$A'$	$S_1$	$S_2$	$S_3$	$S_4$	$S_5$	
$\chi_{29}$	2 816 856	1	1	.	.	.	.	
$\chi_{38}$	16 347 825	$5'$	1	1	.	.	.	$\dim S_1 = 2\,816\,856$
$\chi_{51}$	44 013 375	$1^-$	.	.	1	.	.	$\dim S_2 = 13\,530\,969$
$\chi_{55}$	57 544 344	$5^-$	.	1	1	.	.	$\dim S_3 = 44\,013\,375$
$\chi_{62}$	91 547 820	$10^-$	.	1	.	1	.	$\dim S_4 = 78\,016\,851$
$\chi_{80}$	251 756 505	5	1	.	.	.	1	$\dim S_5 = 248\,939\,649$
$\chi_{85}$	292 953 024	$5'^-$	.	.	1	.	1	
$\chi_{89}$	326 956 500	10	.	.	.	1	1	
$\chi_{91}$	387 317 700	16	1	1	1	1	1	

We just remark that it is an ongoing project, see [60], to determine all the decomposition numbers of all the finite simple and closely related groups occurring in the Atlas [7]. The above result is also a contribution to this project.  $\square$

Following the general recipe indicated above, the present paper is organised as follows: In §2 we set out to find an approximation to the decomposition matrix of  $A$ , by using techniques from

computational modular character theory. The result, where only a parameter  $\alpha \in \{0, 1, 2\}$  is yet undetermined, is given in **2.7**. In §3 we collect the necessary facts for the symmetric group  $\mathfrak{S}_6$  and its principal block  $A'$ . In particular, we determine the trivial-source modules in  $A'$ , which play a particularly important role in the sequel. In §4 we prove that the block  $A$  and its Brauer correspondent  $B$  in  $N_G(P)$  are splendidly stably equivalent of Morita type. This immediately implies that  $A$  and  $A'$  also are splendidly stably equivalent of Morita type. In §5 we finally show that the stable equivalence between  $A$  and  $A'$  respects simple modules, by a thorough consideration of trivial-source modules in  $A$ . This implies that  $A$  and  $A'$  actually are Puig equivalent, from which the other assertions follow immediately.

In order to facilitate the necessary computations, we make use of the computer algebra system GAP [9], to deal with finite groups, in particular permutation and matrix groups, and with ordinary and Brauer characters of finite groups. In particular, we make use of the character table library [3], which provides electronic access to the data collected in the Atlas [7] and in the ModularAtlas [13, 60], of the interface [59] to the database [61], and of the package [40] providing the necessary tools from computational modular character theory.

Moreover, we use the computer algebra system MeatAxe [54], and its extensions [36, 37, 38, 39] to deal with matrix representations over finite fields. Here, we use ‘small’ finite fields, but we always make sure, silently, that these are chosen such that the computational results thus obtained remain valid without change after scalar extension to the fixed field of positive characteristic which is ‘large enough’ in the sense of **1.8** below.

**Notation/Definition 1.8.** Throughout this paper, we use the standard notation and terminology as is used in the Atlas [7] and textbooks like [42, 57]. We recall a few for convenience:

(i) If  $A$  and  $B$  are finite dimensional  $k$ -algebras, where  $k$  is a field, we denote by  $\text{mod-}A$ ,  $A\text{-mod}$  and  $A\text{-mod-}B$  the categories of finitely generated right  $A$ -modules, left  $A$ -modules and  $(A, B)$ -bimodules, respectively. A module always refers to a finitely generated right module, unless stated otherwise. Let  $M$  be an  $A$ -module. We let  $M^\vee = \text{Hom}_A(M_A, A_A)$  be the  $A$ -dual of  $M$ , so that  $M^\vee$  becomes a left  $A$ -module. We denote by  $\text{soc}(M)$  and  $\text{rad}(M)$  the socle and the radical of  $M$ , respectively. For non-isomorphic simple  $A$ -modules  $S_1, \dots, S_n$ , and positive integers  $a_1, \dots, a_n$ , we write that “ $M = a_1 \cdot S_1 + \dots + a_n \cdot S_n$ , as composition factors” when the composition factors are  $a_1$  times  $S_1, \dots, a_n$  times  $S_n$ . For another  $A$ -module  $N$ , we write  $M|N$  if  $M$  is isomorphic to a direct summand of  $N$  as an  $A$ -module; and we set  $[M, N]^A = \dim_k(\text{Hom}_A(M, N))$ .

(ii) By  $G$  we always denote a finite group, and we fix a prime number  $p$ . Assume that  $(\mathcal{K}, \mathcal{O}, k)$  is a splitting  $p$ -modular system for all subgroups of  $G$ , that is,  $\mathcal{O}$  is a complete discrete valuation ring of rank one such that its quotient field  $\mathcal{K}$  is of characteristic zero, and its residue field  $k = \mathcal{O}/\text{rad}(\mathcal{O})$  is of characteristic  $p$ , and that  $\mathcal{K}$  and  $k$  are splitting fields for all subgroups of  $G$ .

We denote by  $\text{Irr}(G)$  and  $\text{IBr}(G)$  the sets of all irreducible ordinary and Brauer characters of  $G$ , respectively. If  $A$  is a block algebra of  $\mathcal{O}G$ , then we write  $\text{Irr}(A)$  and  $\text{IBr}(A)$  for the sets of all characters in  $\text{Irr}(G)$  and  $\text{IBr}(G)$  which belong to  $A$ , respectively.

We say that a  $kG$ -module  $M$  is a *trivial-source* module if  $M$  is indecomposable and  $M$  has a trivial source, see [31, II Definition 12.1]; note that the definition here is slightly different from [57, §27 p.218] where indecomposability is not assumed. Thus, in particular, projective indecomposable modules are trivial-source modules. We recall the following facts, see [31, II Theorem 12.4, I Proposition 14.8]: If  $M$  is a trivial-source module, then  $M$  lifts uniquely (up to isomorphism) to a trivial source  $\mathcal{O}G$ -lattice  $\widehat{M}$ , in particular this associates an ordinary character  $\chi_M$  to  $M$ . Moreover, if  $N$  is another trivial-source  $kG$ -module, then  $[M, N]^G = \langle \chi_M, \chi_N \rangle_G$ , where  $\langle -, - \rangle_G$  denotes the usual scalar product on ordinary characters.

(iii) Let  $G'$  be another finite group, and let  $V$  be an  $(\mathcal{O}G, \mathcal{O}G')$ -bimodule. Then we can regard  $V$  as a right  $\mathcal{O}[G \times G']$ -module via  $v \cdot (g, g') = g^{-1}vg'$  for  $v \in V$  and  $g, g' \in G$ . Let  $A$  and  $A'$  be block algebras of  $\mathcal{O}G$  and  $\mathcal{O}G'$ , respectively, such that  $A$  and  $A'$  have a defect group  $P$  in common. Then we say that  $A$  and  $A'$  are *Puig equivalent* if there is a Morita equivalence between  $A$  and  $A'$

which is induced by an  $(A, A')$ -bimodule  $\mathfrak{M}$  such that, as a right  $\mathcal{O}[G \times G']$ -module,  $\mathfrak{M}$  is a trivial-source module and  $\Delta P$ -projective. This is equivalent to a condition that  $A$  and  $A'$  have source algebras which are isomorphic as interior  $P$ -algebras, see [50, Remark 7.5] and [35, Theorem 4.1]. We say that  $A$  and  $A'$  are *stably equivalent of Morita type* if there exists an  $(A, A')$ -bimodule  $\mathfrak{M}$  such that both  ${}_A\mathfrak{M}$  and  $\mathfrak{M}_{A'}$  are projective and that  ${}_A(\mathfrak{M} \otimes_{A'} \mathfrak{M}^\vee)_A \cong {}_A A_A \oplus (\text{proj}(A, A)\text{-bimod})$  and  ${}_{A'}(\mathfrak{M}^\vee \otimes_A \mathfrak{M})_{A'} \cong {}_{A'} A'_{A'} \oplus (\text{proj}(A', A')\text{-bimod})$ . We say that  $A$  and  $A'$  are *splendidly stably equivalent of Morita type* if the stable equivalence of Morita type is induced by an indecomposable  $(A, A')$ -bimodule  $\mathfrak{M}$  which is a trivial-source  $\mathcal{O}[G \times G']$ -module and is  $\Delta P$ -projective, see [35, Theorem 3.1].

We say that  $A$  and  $A'$  are *derived equivalent* if  $D^b(\text{mod-}A)$  and  $D^b(\text{mod-}A')$  are equivalent as triangulated categories, where  $D^b(\text{mod-}A)$  is the bounded derived category of  $\text{mod-}A$ . In that case, there even is a *Rickard complex*  $M^\bullet \in C^b(A\text{-mod-}A')$ , where the latter is the category of bounded complexes of finitely generated  $(A, A')$ -bimodules, all of whose terms are projective both as left  $A$ -modules and as right  $A'$ -modules, such that  $M^\bullet \otimes_{A'} (M^\bullet)^\vee \cong A$  in  $K^b(A\text{-mod-}A)$  and  $(M^\bullet)^\vee \otimes_A M^\bullet \cong A'$  in  $K^b(A'\text{-mod-}A')$ , where  $K^b(A\text{-mod-}A)$  is the homotopy category associated with  $C^b(A\text{-mod-}A)$ . We say that  $A$  and  $A'$  are *splendidly derived equivalent* if  $K^b(\text{mod-}A)$  and  $K^b(\text{mod-}A')$  are equivalent via a Rickard complex  $M^\bullet \in C^b(A\text{-mod-}A')$  as above, such that additionally each of its terms is a direct sum of  $\Delta P$ -projective trivial-source modules as an  $\mathcal{O}[G \times G']$ -module; see [34, 35]. Note that a Morita equivalence entails a derived equivalence, and that a Puig equivalence entails a splendid derived equivalence.

## 2. DECOMPOSITION NUMBERS OF $\text{Co}_1$

In this section we use methods from computational modular character theory to find an approximation to the decomposition matrix of the block in question. To do so, we make heavy use of the data on decomposition numbers known for various maximal subgroups of  $G$ , as is contained in [13, 60]. We stress that, to find the approximate decomposition matrix exhibited below in the first place, we make use of the full machinery laid out in [11], in combination with substantial computation, using GAP [9] and the package [40], while the proof presented here is subsequently derived from the computer-based one by straightening it out by hand.

For convenience we recall a few of the basic notions used in this approach, while for full details we refer the reader to [11]:

**Definition 2.1.** We keep the notation in 1.8(ii). Let  $\mathbb{Z}\text{IBr}(A)$  be the lattice of generalised Brauer characters belonging to  $A$ , and  $\mathbb{N}_0\text{IBr}(A) \subset \mathbb{Z}\text{IBr}(A)$  be the non-negative cone of genuine Brauer characters spanned by  $\text{IBr}(A)$ . Then a  $\mathbb{Z}$ -basis  $\mathbf{BS}$  of  $\mathbb{Z}\text{IBr}(A)$  which is contained in  $\mathbb{N}_0\text{IBr}(A)$  is called a *basic set* of Brauer characters; in particular,  $\text{IBr}(A)$  is such a basic set.

Moreover, any projective  $A$ -module has an ordinary character associated with it. Thus running through the projective indecomposable  $A$ -modules, this gives rise to the set  $\text{IPr}(A) \subset \mathbb{N}_0\text{Irr}(A)$ , spanning the lattice  $\mathbb{Z}\text{IPr}(A)$  of generalised projective characters belonging to  $A$ . Then, similarly, a  $\mathbb{Z}$ -basis  $\mathbf{PS}$  of  $\mathbb{Z}\text{IPr}(A)$  which is contained in  $\mathbb{N}_0\text{IPr}(A)$  is called a *basic set* of projective characters; in particular,  $\text{IPr}(A)$  is such a basic set.

Now let  $\widehat{\text{Irr}}(A)$  be the set of restrictions of the characters in  $\text{Irr}(A)$  to the  $p$ -regular conjugacy classes of  $G$ , thus we have  $\widehat{\text{Irr}}(A) \subset \mathbb{N}_0\text{IBr}(A)$ . Similarly we obtain  $\widehat{\text{IPr}}(A)$ , and thus we get  $\mathbb{Z}\widehat{\text{IPr}}(A) \subseteq \mathbb{Z}\text{IBr}(A)$  such that  $\widehat{\text{IPr}}(A) \subset \langle \widehat{\text{Irr}}(A) \rangle_{\mathbb{N}_0} \subseteq \mathbb{N}_0\text{IBr}(A)$ . The restriction  $\langle -, - \rangle'_G$  of the character-theoretic scalar product to the  $p$ -regular classes induces a non-degenerate pairing between  $\mathbb{Z}\text{IBr}(A)$  and  $\mathbb{Z}\widehat{\text{IPr}}(A)$ , such that  $\text{IBr}(A)$  and  $\widehat{\text{IPr}}(A)$  are a pair of mutually dual  $\mathbb{Z}$ -bases.

More generally, given basic sets  $\mathbf{BS}$  and  $\mathbf{PS}$  as above, the associated matrix of mutual scalar products is unimodular and has non-negative entries. Now the general strategy is to use character theory to find sequences of basic sets of Brauer characters and of projective characters, which better and better approximate the sets  $\text{IBr}(A)$  and  $\text{IPr}(A)$ , respectively, the guiding principle being the above-mentioned positivity properties.

**Notation 2.2.** From now on we use the following notation. Let  $G := \text{Co}_1$  be the first sporadic simple Conway group, and hence  $|G| = 2^{21} \cdot 3^9 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 23$ , see [7, p.180]. Assume  $p = 3$ , and let  $(\mathcal{K}, \mathcal{O}, k)$  be a splitting 3-modular system for all finite groups we shall treat with. It follows from the character table of  $G$ , see [7, p.180], that there is a unique block algebra  $A$  of  $kG$  with elementary abelian defect group  $P$  of order 9; see also [43]. (We will specify  $P$  as a subgroup of  $G$  more precisely in 4.1 and 4.2 below.)

Moreover, it turns out that  $A$  contains  $k(A) = 9$  irreducible ordinary characters and  $l(A) = 5$  irreducible Brauer characters. According to the ordering in [7, p.180], the ordinary characters in  $A$  are  $\{\chi_{29}, \chi_{38}, \chi_{51}, \chi_{55}, \chi_{62}, \chi_{80}, \chi_{85}, \chi_{89}, \chi_{91}\}$ , for convenience their degrees are indicated in 1.7.

**2.3.** We begin by choosing basic sets of Brauer characters and projective characters.

Let  $\hat{\chi}_i$  denote the  $i$ -th irreducible character of  $G$ , with respect to the ordering given in [7, p.180], restricted to its 3-regular conjugacy classes. Moreover, for a subgroup  $H \leq G$  let  $\psi(H)_i$  denote the  $i$ -th irreducible Brauer character of  $H$  with respect to the ordering given in the databases mentioned, and let  $\Psi(H)_i$  denote the ordinary character of the corresponding projective indecomposable module. We remark that the irreducible Brauer characters and the projective indecomposable characters of the 2-local maximal subgroup  $2^{11} : \text{M}_{24}$  are easily computed using Fong-Reynolds correspondence, but we only need those of the Mathieu group  $\text{M}_{24}$ , inflated along the natural map  $2^{11} : \text{M}_{24} \rightarrow \text{M}_{24}$ .

Then our initial basic set  $\mathbf{BS} = \{\varphi_1, \dots, \varphi_5\}$  of Brauer characters constitutes of the  $A$ -parts of the Brauer characters

$$\hat{\chi}_{29}, \psi(3.\text{Suz}.2)_{7\uparrow^G}, \hat{\chi}_{51}, \psi(3.\text{Suz}.2)_{11\uparrow^G}, \hat{\chi}_{80},$$

and as an initial basic set  $\mathbf{PS} = \{\Phi_1, \dots, \Phi_5\}$  of projective characters we choose the  $A$ -parts of

$$\Psi(\text{Co}_2)_{18\uparrow^G}, \Psi(\text{Co}_2)_{22\uparrow^G}, \Psi(\text{Co}_2)_{29\uparrow^G}, \Psi(2^{11} : \text{M}_{24})_{12\uparrow^G},$$

These indeed are suitable basic sets, as the matrix  $(\langle \varphi_i, \Phi_j \rangle'_G)_{1 \leq i, j \leq 5}$  shows:

$\langle -, - \rangle'_G$	$\Phi_1$	$\Phi_2$	$\Phi_3$	$\Phi_4$	$\Phi_5$
$\varphi_1$	.	.	.	.	1
$\varphi_2$	.	.	.	1	1
$\varphi_3$	.	.	1	.	.
$\varphi_4$	.	1	.	2	1
$\varphi_5$	1	1	3	1	3

Note that in particular this initial matrix already turns out to be unitriangular, which will simplify the arguments to follow.

**2.4.** it is now immediate that  $\varphi_1$  and  $\varphi_3$  are irreducible Brauer characters, and that  $\Phi_1$  is the character of a projective indecomposable module. We denote the known irreducible Brauer characters and projective indecomposable characters in bold face in

Next we prove that  $\varphi_2$  and  $\varphi_4$  actually are irreducible Brauer characters as well. To do so, we consider the  $A$ -parts of the projective characters  $\Psi(3.\text{Suz}.2)_{27\uparrow^G}$  and  $\Psi(2^{11} : \text{M}_{24})_{2\uparrow^G}$ , whose decompositions into  $\mathbf{PS}$  are given as follows:

	$\Phi_1$	$\Phi_2$	$\Phi_3$	$\Phi_4$	$\Phi_5$
$\Psi(3.\text{Suz}.2)_{27\uparrow^G}$	-9	21	16	-1	1
$\Psi(2^{11} : \text{M}_{24})_{2\uparrow^G}$	-4	-1	0	0	4

Now assume that  $\varphi_2$  is reducible. Then  $\varphi_2 - \varphi_1$  is a Brauer character, whose scalar product with  $\Psi(3.\text{Suz}.2)_{27\uparrow^G}$  equals  $[0, 0, 0, 1, 0] \cdot [-9, 21, 16, -1, 1]^\top = -1$ , a contradiction. Hence  $\varphi_2$  is an irreducible Brauer character.

Next, assume that  $\varphi_4$  is reducible. Then either  $\varphi_4 - \varphi_2$  or  $\varphi_4 - \varphi_1$  is a Brauer character. Taking scalar products with  $\Psi(2^{11} : \text{M}_{24})_{2\uparrow^G}$  yields  $[0, 1, 0, 1, 0] \cdot [-4, -1, 0, 0, 4]^\top = -1$  and  $[0, 1, 0, 2, 0] \cdot [-4, -1, 0, 0, 4]^\top = -1$ , respectively, a contradiction. Thus  $\varphi_4$  is an irreducible Brauer character.

**2.5.** We now set out to improve our basic sets **BS** and **PS**.

that  $\varphi_4$  is not contained in  $\varphi_5$ , hence we conclude that  $\Phi'_2 := \Phi_2 - \Phi_1$  is a projective character, and then as such is indecomposable.

Next, using the facts that  $\varphi_4$  and  $\varphi_2$  are irreducible, we conclude that  $\Phi'_4 := \Phi_4 - 2(\Phi_2 - \Phi_1)$  and  $\Phi'_5 := \Phi_5 - (\Phi_2 - \Phi_1) - (\Phi_4 - 2\Phi_2 + \Phi_1) = \Phi_5 - \Phi_4 + \Phi_2$  are projective characters.

Finally, it turns out that  $\widehat{\chi}_{85}$  decomposes into **BS** as follows:

	$\varphi_1$	$\varphi_2$	$\varphi_3$	$\varphi_4$	$\varphi_5$
$\widehat{\chi}_{85}$	-1	.	1	.	1

As  $\varphi_3$  is irreducible it follows that  $\varphi'_5 := \varphi_5 - \varphi_1$  is a Brauer character.

Thus, we have obtained new basic sets **BS'** :=  $\{\varphi_1, \dots, \varphi_4, \varphi'_5\}$  and **PS'** :=  $\{\Phi_1, \Phi'_2, \Phi_3, \Phi'_4, \Phi'_5\}$ , whose matrix of mutual scalar products is given as follows:

$\langle -, - \rangle'_G$	$\Phi_1$	$\Phi'_2$	$\Phi_3$	$\Phi'_4$	$\Phi'_5$
$\varphi_1$	.	.	.	.	1
$\varphi_2$	.	.	.	1	.
$\varphi_3$	.	.	1	.	.
$\varphi_4$	.	1	.	.	.
$\varphi'_5$	1	.	3	1	2

**2.6.** With respect to **PS'** we now have:

	$\Phi_1$	$\Phi'_2$	$\Phi_3$	$\Phi'_4$	$\Phi'_5$
$\Psi(2^{11} : M_{24})_2 \uparrow^G$	-9	3	0	4	4

Since  $\Phi_1$  is already known to be indecomposable, show that it is contained at most once in  $\Phi'_4$  and at most twice in  $\Phi'_5$ , we from this conclude that  $\Phi''_5 := \Phi'_5 - 2\Phi_1$  and  $\Phi''_4 := \Phi'_4 - \Phi_1$  are projective characters. Then, as such, they both are indecomposable.

Finally, the  $A$ -part of the tensor product  $\Phi''_4 \otimes \widehat{\chi}_2$  decomposes into the newly found projective characters as follows:

	$\Phi_1$	$\Phi'_2$	$\Phi_3$	$\Phi''_4$	$\Phi''_5$
$\Phi''_4 \otimes \widehat{\chi}_2$	-4	2	4	2	0

Hence  $\Phi'_3 := \Phi_3 - \Phi_1$  is a projective character.

Thus we have obtained the new basic set **PS''** :=  $\{\Phi_1, \Phi'_2, \Phi'_3, \Phi''_4, \Phi''_5\}$ , whose matrix of scalar products with **BS'** is given as follows:

$\langle -, - \rangle'_G$	$\Phi_1$	$\Phi'_2$	$\Phi'_3$	$\Phi''_4$	$\Phi''_5$
$\varphi_1$	.	.	.	.	1
$\varphi_2$	.	.	.	1	.
$\varphi_3$	.	.	1	.	.
$\varphi_4$	.	1	.	.	.
$\varphi'_5$	1	.	2	.	.

Hence, only three possible decomposition matrices remain: one of  $\Phi'_3 - \alpha \cdot \Phi_1$  for  $0 \leq \alpha \leq 2$  is a projective indecomposable character. Thus we have proved:

**Lemma 2.7.** *The 3-decomposition matrix of  $A$  is the following:*

	$S_1$	$S_2$	$S_3$	$S_4$	$S_5$
$\chi_{29}$	1	.	.	.	.
$\chi_{38}$	1	1	.	.	.
$\chi_{51}$	.	.	1	.	.
$\chi_{55}$	.	1	1	.	.
$\chi_{62}$	.	1	.	1	.
$\chi_{80}$	1	.	$\alpha$	.	1
$\chi_{85}$	.	.	$1 + \alpha$	.	1
$\chi_{89}$	.	.	$\alpha$	1	1
$\chi_{91}$	1	1	$1 + \alpha$	1	1

where  $\alpha$  is a certain integer such that  $\alpha \in \{0, 1, 2\}$ .

*Proof.* This follows from **2.3–2.6**, where here we reverse the order of the projective characters in the final basic set, and already indicate the notation for the associated simple  $A$ -modules which will be used later.  $\square$

### 3. THE GROUP $\mathfrak{S}_6$

**Notation 3.1.** Set  $G' := \mathfrak{S}_6$ , the symmetric group of degree 6. Then the Sylow 3-subgroup of  $G'$  are elementary abelian of order 9. Hence fixing a block defect group  $P$  of  $A$ , see **2.2**, we may identify  $P$  with a Sylow 3-subgroup of  $G'$ . Doing so, we have  $P = Q \times R$  with  $Q \cong C_3 \cong R$ , where, with respect to the tautological permutation representation of  $G'$ , we may assume that the non-trivial elements of  $Q$  are 3-cycles, while those of  $R$  are fixed-point free.

Let  $A'$  be the principal block algebra of  $kG'$ . Then we have  $\text{IBr}(A') = \{1a, 1b, 4a, 4b, 6\}$  and  $\text{Irr}(A') = \{\chi_1, \chi_{1^-}, \chi_5, \chi_{5^-}, \chi_{5'}, \chi_{5'^-}, \chi_{10}, \chi_{10^-}, \chi_{16}\}$ , where we use the following notation:

As usual, we denote by  $\chi_1 := 1_{A'} \in \text{Irr}(A')$  and  $\chi_{1^-} := 1^- \in \text{Irr}(A')$  the trivial and the sign character, respectively. Moreover, we let  $\chi_5 \in \text{Irr}(A')$  be the non-trivial constituent of the tautological permutation character of  $A'$ , which is characterised by having positive values on both the 3-cycles and the transpositions, and we let  $\chi_{5'} \in \text{Irr}(A')$  be the irreducible character having positive values on both the fixed-point free elements of order 3 and 2; note that  $\chi_5$  and  $\chi_{5'}$  are interchanged by the non-trivial outer automorphism of  $A'$ . Finally, we let  $\chi_{10} \in \text{Irr}(A')$  be the irreducible character having positive value on the transpositions. Then we get  $\chi_{5^-} := \chi_5 \otimes 1^- \in \text{Irr}(A')$  and  $\chi_{5'^-} := \chi_{5'} \otimes 1^- \in \text{Irr}(A')$  as well as  $\chi_{10^-} := \chi_{10} \otimes 1^- \in \text{Irr}(A')$ .

As for the simple modular representations occurring, we let, again as usual,  $1a := k_{G'} \in \text{IBr}(A')$  and  $1b := 1^- \in \text{IBr}(A')$  be the trivial and the sign representation, respectively. Moreover, we let  $4a$  be the non-trivial modular constituent of the tautological permutation representation of  $G'$ , then we get  $4b := 4a \otimes 1^- \in \text{IBr}(A')$ . Note that the outer automorphism of  $G'$  mentioned earlier interchanges  $4a \leftrightarrow 4b$ , and leaves  $1a$ ,  $1b$ , and  $6$  fixed.

**Lemma 3.2.** *The 3-decomposition matrix of  $A'$  is given as follows:*

	$1a$	$1b$	$4a$	$4b$	$6$
$\chi_1$	1	.	.	.	.
$\chi_{1^-}$	.	1	.	.	.
$\chi_5$	1	.	1	.	.
$\chi_{5^-}$	.	1	.	1	.
$\chi_{5'}$	1	.	.	1	.
$\chi_{5'^-}$	.	1	1	.	.
$\chi_{10}$	.	.	1	.	1
$\chi_{10^-}$	.	.	.	1	1
$\chi_{16}$	1	1	1	1	1

*Proof.* This is taken from [13, p. 4].  $\square$

**Lemma 3.3.** *Using the notation introduced in 3.1, the Loewy and socle series of the trivial-source  $kG'$ -modules in  $A'$  are given as follows, where for the non-projective ones we also include their associated ordinary characters:*

(i) *The trivial-source  $kG'$ -modules in  $A'$  with vertex  $P$  are the following:*

$$\begin{array}{ccccccc}
 & & & \boxed{\begin{array}{c} 4a \\ 1a \ 1b \\ 4a \end{array}} & \boxed{\begin{array}{c} 4b \\ 1a \ 1b \\ 4b \end{array}} & \boxed{\begin{array}{c} 6 \\ 4a \ 4b \\ 6 \end{array}} & \\
 1a & 1b & & & & & \\
 \downarrow & \downarrow & & \downarrow & \downarrow & \downarrow & \\
 \chi_1 & \chi_{1^-} & & \chi_5 + \chi_{5'^-} & \chi_{5^-} + \chi_{5'} & \chi_{10} + \chi_{10^-} &
 \end{array}$$

(ii) *The trivial-source  $kG'$ -modules in  $A'$  with vertex  $Q$  are the following:*

$$\begin{array}{ccccccc}
 \boxed{\begin{array}{c} 1a \\ 4a \\ 1a \end{array}} & \boxed{\begin{array}{c} 1b \\ 4b \\ 1b \end{array}} & \boxed{\begin{array}{c} 4a \\ 1a \ 6 \\ 4a \end{array}} & \boxed{\begin{array}{c} 4b \\ 1b \ 6 \\ 4b \end{array}} & & & \\
 \downarrow & \downarrow & \downarrow & \downarrow & & & \\
 \chi_1 + \chi_5 & \chi_{1^-} + \chi_{5'^-} & \chi_5 + \chi_{10} & \chi_{5^-} + \chi_{10^-} & & &
 \end{array}$$

(iii) *The trivial-source  $kG'$ -modules in  $A'$  with vertex  $R$  are the following:*

$$\begin{array}{ccccccc}
 \boxed{\begin{array}{c} 1a \\ 4b \\ 1a \end{array}} & \boxed{\begin{array}{c} 1b \\ 4a \\ 1b \end{array}} & \boxed{\begin{array}{c} 4b \\ 1a \ 6 \\ 4b \end{array}} & \boxed{\begin{array}{c} 4a \\ 1b \ 6 \\ 4a \end{array}} & & & \\
 \downarrow & \downarrow & \downarrow & \downarrow & & & \\
 \chi_1 + \chi_{5'} & \chi_{1^-} + \chi_{5'^-} & \chi_{5'} + \chi_{10^-} & \chi_{5'^-} + \chi_{10} & & &
 \end{array}$$

(iv) *The projective indecomposable  $kG'$ -modules in  $A'$  have the following structure:*

$$\begin{array}{ccccc}
 \boxed{\begin{array}{c} 1a \\ 4a \ 4b \\ 1a \ 1a \ 1b \ 6 \\ 4a \ 4b \\ 1a \end{array}} & \boxed{\begin{array}{c} 1b \\ 4a \ 4b \\ 1a \ 1b \ 1b \ 6 \\ 4a \ 4b \\ 1b \end{array}} & \boxed{\begin{array}{c} 4a \\ 1a \ 1b \ 6 \\ 4a \ 4a \ 4b \\ 1a \ 1b \ 6 \\ 4a \end{array}} & \boxed{\begin{array}{c} 4b \\ 1a \ 1b \ 6 \\ 4a \ 4b \ 4b \\ 1a \ 1b \ 6 \\ 4b \end{array}} & \boxed{\begin{array}{c} 6 \\ 4a \ 4b \\ 1a \ 1b \ 6 \\ 4a \ 4b \\ 6 \end{array}}
 \end{array}$$

*Proof.* (i)–(iii) This is found by explicit computation as follows: By Green correspondence and [42, Chap.4 Problem 10], the modules we are interested in are precisely the indecomposable direct summands with maximal vertex of the permutation  $kG'$ -modules on the cosets of  $P$ ,  $Q$ , and  $R$ , respectively. Now, the relevant permutation actions of  $G'$  can be computed using GAP [9], and going over to permutation  $kG'$ -modules, their indecomposable direct summands, together with their structure, are subsequently found using the MeatAxe [54]. Recall that the number of trivial-source  $kG'$ -modules with vertex as prescribed above can be determined a priority, using the facts that  $N_{G'}(P)/P \cong D_8$  and  $N_{G'}(Q)/Q \cong 2 \times \mathfrak{S}_3 \cong N_{G'}(R)/R$ . Finally, knowing the structure of the trivial-source modules in question, the ordinary characters associated with them are determined using 3.2.

(iv) This is contained in [58], and can also be rechecked computationally by applying the MeatAxe [54] to the regular representation of  $kG'$ .  $\square$

**Notation 3.4.** Set  $H' := N_{G'}(P)$ . Then we have  $H' \cong P: D_8$ , where it follows from the description in 3.1 that  $P$  is simple, viewed as an  $\mathbb{F}_3 D_8$ -module, which hence determines  $H'$  up to isomorphism. But note that, by its mere definition,  $H'$  comes with a fixed embedding into  $G'$ .

Let  $B'$  be the principal block algebra of  $kH'$ , being the Brauer correspondent of  $A'$  in  $H'$ ; note that we actually have  $B' = kH'$ .



**Lemma 3.5.** *Let  $\mathcal{M}'$  be the Scott  $k[G' \times H']$ -module with vertex  $\Delta P$ . (Note that this is the up to isomorphism unique indecomposable direct summand of the  $(A', B')$ -bimodule  $A' \downarrow_{G' \times H'}^{G' \times G'} \cdot 1_{B'}$  with vertex  $\Delta P$ .) Then the pair  $(\mathcal{M}', \mathcal{M}'^\vee)$  induces a splendid stable equivalence of Morita type between the block algebras  $A'$  and  $B'$ .*

*Proof.* This follows from [45, Example 4.4] and [46, Corollary 2].  $\square$

#### 4. STABLE EQUIVALENCE BETWEEN $A$ AND $B$

**Lemma 4.1.** *Recall the notation  $G$ ,  $P$ , and  $A$  as in 2.2. Then the following holds:*

- (i) *We have  $H := N_G(P) = P.U_4(3).D_8$  and  $C_G(P) = P.U_4(3)$ .*
- (ii) *The non-trivial elements in  $P$  fall into two conjugacy classes of  $H$ . Thus  $P$  has exactly two  $H$ -conjugacy classes of subgroups of order 3. Moreover, the non-trivial elements of  $P$  belong to conjugacy classes  $3A$  and  $3B$  in [7, p.184].*

*Proof.* We use the smallest faithful permutation representation of  $G$  on 98280 points, available in [61], and GAP [9] to find a defect group  $P \leq G$  of  $A$  explicitly. In order to do so, using the character table of  $G$ , see [7, p.180], it turns out that conjugacy classes  $4D$ ,  $5B$   $7A$ , and  $8A$  are defect classes of  $A$ . By a random search we find an element of order 42 in  $G$ , whose 6-th power,  $x$  say, hence belongs to conjugacy class  $7A$ . Thus we may let  $P$  be a Sylow 3-subgroup of  $C_G(x)$ .

Having  $P$  found explicitly, it turns out that  $H = N_G(P)$  has order 235 146 240, and that  $C_G(P)$  has order 29 393 280 =  $|H|/8$ . Now a consideration of the orders of the maximal subgroups of  $G$ , see [7, p.180], shows that  $H$  necessarily is a maximal subgroup of  $G$ , of shape  $P.U_4(3).D_8$ . Then it is clear that  $C_G(P)$  is of shape  $P.U_4(3)$ . This shows (i). The character table of the maximal subgroup  $H$  of  $G$ , available in [3], shows that  $P - \{1\} \subset H$  consists of two rational conjugacy classes. a consideration of character values shows, together with Brauer's Second Main Theorem, that the latter conjugacy classes fuse into the conjugacy classes  $3A$  and  $3B$  of  $G$ .  $\square$

**Notation 4.2.** Set  $H := N_G(P)$ . Then by 4.1(ii) we have  $P = Q \times R$  with  $Q \cong C_3 \cong R$ , such that  $Q$  and  $R$  are not conjugate in  $G$ , and we may assume that the non-trivial elements in  $Q$  and  $R$  belong to conjugacy classes  $3A$  and  $3B$ , respectively. Recall that we have already chosen an embedding of  $P$  into  $G'$ , see 3.1, but since the automorphism group of  $P$  acts transitively on the minimal generating sets of  $P$ , both choices are consistent, justifying the reuse of earlier notation.

Let  $B$  be a block algebra of  $kH$  which is the Brauer correspondent of  $A$ . Let  $(P, e)$  be a maximal  $A$ -Brauer pair in  $G$ , namely,  $e$  is a block idempotent of  $kC_G(P)$  such that  $\text{Br}_P(1_A) \cdot e = e$ , see [1], [5] and [57, §40]. Let  $i$  and  $j$  respectively be source idempotents of  $A$  and  $B$  with respect to  $P$ . As remarked in [35, pp.821–822], we can take  $i$  and  $j$  such that  $\text{Br}_P(i) \cdot e = \text{Br}_P(i) \neq 0$  and that  $\text{Br}_P(j) \cdot e = \text{Br}_P(j) \neq 0$ .

Set  $G_P = C_G(P) = C_H(P) = H_P$ . We set  $G_Q = C_G(Q)$  and  $H_Q = C_H(Q)$ . By replacing  $e_Q$  and  $f_Q$  (if necessary), we may assume that  $e_Q$  and  $f_Q$  respectively are block idempotents of  $kG_Q$  and  $kH_Q$  such that  $e_Q$  and  $f_Q$  are determined by  $i$  and  $j$ , respectively. Namely,  $\text{Br}_Q(i) \cdot e_Q = \text{Br}_Q(i)$  and  $\text{Br}_Q(j) \cdot f_Q = \text{Br}_Q(j)$ . Let  $A_Q = kG_Q \cdot e_Q$  and  $B_Q = kH_Q \cdot f_Q$ , so that  $e_Q = 1_{A_Q}$  and  $f_Q = 1_{B_Q}$ . Similarly we define  $G_R$ ,  $H_R$ ,  $A_R$ ,  $B_R$ ,  $e_R$  and  $f_R$ .

**Lemma 4.3.** *The following holds:*

- (i) *We have  $H = N_G(P, e) := \{g \in N_G(P) | g^{-1}eg = e\}$ .*
- (ii)  *$B = kHe \cong \text{Mat}_{729}(k[P : D_8])$ , where  $P : D_8 \cong H'$ . Hence we may write  $\text{IBr}(B) := \{729a, 729b, 729c, 729d, 1458\}$ .*
- (iii) *The blocks  $B$  and  $B'$  are canonically Puig equivalent.*

*Proof.* (i) The group  $C_G(P)/P = U_4(3)$ , being a simple group of Lie type in characteristic 3, has a unique irreducible character of defect zero, namely the Steinberg character of degree 729. Hence we conclude that  $C_G(P)$  has a unique block with defect group  $P$ , and thus we have  $N_G(P, e) = N_G(P) = H$ .

(ii) We argue similar to the lines of [27, Proof of Lemma 4.6(iv)]: By [29, A.Theorem] we have  $B \cong \text{Mat}_{729}(k^\alpha[P : D_8])$ , for some cocycle  $\alpha \in Z^2(D_8, k^\times)$ , and where the action of  $D_8 \cong N_G(P)/C_G(P)$  in the semidirect product  $P : D_8$  is determined by the action of  $N_G(P)$  on  $P$ . Now it follows from 4.1(ii) that  $P$  is simple as an  $\mathbb{F}_3 D_8$ -module, hence by 3.4 we have  $P : D_8 \cong H'$ .

Since, by [12, Satz V.25.6] we have  $|\mathbb{H}^2(D_8, k^\times)| = 2$ , it remains to show that  $\alpha \equiv 1 \pmod{B^2(D_8, k^\times)}$ . Indeed, using the character table of  $H$ , GAP [9] shows that  $B$  has nine irreducible characters and five irreducible Brauer characters, from which the last part of (ii) follows using [15, p.34, Tbl.1].

Finally, in (iii), the statement about source algebras follows from [49, Proposition 14.6], see [57, (45.12)Theorem] and [2, Theorem 13].  $\square$

**Lemma 4.4.** *The following holds:*

- (i)  $G_Q = Q.\text{Suz}$ , so that  $G_Q/Q \cong \text{Suz}$ .
- (ii)  $H_Q = P.U_4(3).2'_3$ , so that  $H_Q/Q \cong 3_2.U_4(3).2'_3$ .
- (iii)  $G_R = H_R = P.U_4(3).2'_2$ , so that  $G_R/R = H_R/R \cong 3_1.U_4(3).2'_2$ .

*Proof.* By [7, p.183] we have  $N_G(Q) = Q.\text{Suz}.2$ , a maximal subgroup of  $G$ , implying (i).

To show (ii), a consideration of the  $\mathbb{F}_3 D_8$ -module shows that  $N_H(Q)$  has index 2 in  $H$ , and is of shape  $P.U_4(3).2^2$ . Thus  $H_Q$  is of shape  $P.U_4(3).2$ , and hence  $H_Q/Q$  is of shape  $3.U_4(3).2$ . To find the precise structure, we note that  $H_Q/Q$  is a subgroup of  $G_Q/Q \cong \text{Suz}$ , and a consideration of the orders of the maximal subgroups of  $\text{Suz}$ , see [7, p.131], shows that  $H_Q/Q$  necessarily is a maximal subgroup of  $\text{Suz}$ , namely the normaliser of a cyclic subgroup of order 3 whose non-trivial elements belong to the conjugacy class  $3A$  of  $\text{Suz}$ . Now there is a known typo in [7, p.131], the shape of the maximal subgroup in question being erroneously stated as  $3_2.U_4(3).2_3$ . This has been corrected in the reprint of 2003, but can also be explicitly checked using GAP [9]: the character tables of  $\text{Suz}$ , see [7, p.131], and the various bicyclic extensions of  $U_4(3)$ , see [7, pp.52–59], show that the shape is as asserted above. This implies (ii).

Finally, to show (iii), we use GAP [9] and the explicit choices of subgroups made in the proof of 4.1 and in 4.2, to determine  $N_G(R)$  explicitly. It turns out that  $N_G(R)$  is a subgroup of  $H$  of index 2. Hence we have  $N_G(R) = N_H(R)$ , and as above we conclude that  $N_G(R)$  is of shape  $P.U_4(3).2^2$  and hence  $G_R$  is of shape  $P.U_4(3).2$ ; but note that  $N_H(R) \neq N_H(Q)$ . Computing  $G_R$  and the character table of  $G_R/R$  explicitly, and comparing with those of the various bicyclic extensions of  $U_4(3)$ , we conclude that  $G_R/R$  is of the shape asserted, thus (iii) follows.  $\square$

**Lemma 4.5.** *Set  $\tilde{G} = \text{Suz}$ .*

- (i) *There is a unique block algebra  $\tilde{A}$  of  $k\tilde{G}$  with a defect group  $D \cong C_3$ ; the non-trivial elements of  $D$  belong to the conjugacy class  $3A$  of  $\tilde{G}$ .*
- (ii) *We can write  $\text{Irr}(\tilde{A}) = \{\chi_{16}, \chi_{38}, \chi_{41}\}$  such that  $\chi_{16}(1) = 18954$ ,  $\chi_{38}(1) = 189540$ ,  $\chi_{41}(1) = 208494$  and  $\chi_{16}(u) = \chi_{38}(u) = 729$  for any element  $u$  belonging to the conjugacy class  $3A$ . Moreover, we can write  $\text{IBr}(\tilde{A}) = \{\varphi_1, \varphi_2\}$  such that the 3-decomposition matrix of  $\tilde{A}$  is as follows:*

	$\varphi_1$	$\varphi_2$
$\chi_{16}$	1	.
$\chi_{38}$	.	1
$\chi_{41}$	1	1

*Further, the simple  $k\tilde{G}$ -modules in  $\tilde{A}$  affording  $\varphi_1$  and  $\varphi_2$  are trivial-source  $k\tilde{G}$ -modules.*

- (iii) *Set  $\tilde{H} = N_{\tilde{G}}(D)$ . Then  $\tilde{H} = 3_2.U_4(3).2'_3$ .*
- (iv) *Let  $\tilde{B}$  be the block algebra of  $k\tilde{H}$  that is the Brauer correspondent of  $\tilde{A}$ . Let further  $\tilde{f}$  be the Green correspondence with respect to  $(\tilde{G} \times \tilde{G}, \Delta D, \tilde{G} \times \tilde{H})$ . Then,  $\tilde{f}$  induces a Puig equivalence between  $\tilde{A}$  and  $\tilde{B}$ .*

*Proof.* (i)-(ii) follow from calculations by GAP [9], using the ordinary and Brauer character tables of  $\text{Suz}$ , see [7, p.128ff.] and [13,  $\text{Suz} \pmod{3}$ ].

(iii) follows from [7, p.131], as was already remarked earlier in the proof of 4.4.

(iv) follows from (i)-(iii) and [20, Theorem 1.2].  $\square$

**Lemma 4.6.** *Let  $\mathcal{M}_Q$  be the unique (up to isomorphism) indecomposable direct summand of  $A_Q \downarrow_{G_Q \times H_Q}^{G_Q \times G_Q} \cdot 1_{B_Q}$  with vertex  $\Delta P$ . Then, the pair  $(\mathcal{M}_Q, \mathcal{M}_Q^\vee)$  induces a Puig equivalence between  $A_Q$  and  $B_Q$ .*

*Proof.* Note first that  $\mathcal{M}_Q$  exists by [23, 2.4.Lemma], and also that  $P$  is a defect group of  $A_Q$  and  $B_Q$  by [35, 7.6]. Now we follow the strategy already employed in [23, Proof of 6.2.Lemma]: Using 4.4(i) and (ii), as well as 4.5(iv), the assertion follows by going over to the central quotients  $G_Q/Q$  and  $H_Q/Q$  and their blocks  $\tilde{A}$  and  $\tilde{B}$  dominating  $A_Q$  and  $B_Q$ , respectively, [19, Theorem].  $\square$

**Lemma 4.7.** *Let  $\mathcal{M}$  be the unique (up to isomorphism) indecomposable direct summand of the  $(A, B)$ -bimodule  $A \downarrow_{G \times H}^{G \times G} \cdot 1_B$  with vertex  $\Delta P$ . Then the pair  $(\mathcal{M}, \mathcal{M}^\vee)$  induces a splendid stable equivalence of Morita type between the block algebras  $A$  and  $B$ .*

*Proof.* Note first that  $\mathcal{M}$  exists, again, by [23, 2.4.Lemma]. Then, by [24, Theorem], we have  $\mathcal{M}_Q \cong e_Q \cdot \mathcal{M}(\Delta Q) \cdot f_Q$  as  $(A_Q, B_Q)$ -bimodules. Now by 4.6, and the fact from 4.4(iv) that  $G_R = H_R$ , the gluing theorem [35, 3.1.Theorem] implies the assertion; note that the fusion condition in the latter theorem is automatically satisfied by [22, 1.15.Lemma].  $\square$

**Lemma 4.8.** *The block algebras  $A$  and  $A'$  are splendidly stably equivalent of Morita type.*

*Proof.* This follows immediately by 4.7, 4.3(iii) and 3.5.  $\square$

## 5. IMAGE OF THE STABLE EQUIVALENCE

**Notation 5.1.** Recall the notation in 2.2, 3.1, 3.4, and 4.2. We denote by  $f := f_{(G,P,H)}$  the Green correspondence with respect to  $(G, P, H)$ . Moreover, let  $\text{IBr}(A) := \{S_1, \dots, S_5\}$  and  $\alpha$  be as in 2.7. Finally, let  $\mathcal{M}$  and  $\mathcal{M}'$  be the bimodules from 4.7 and 3.5, respectively.

**Strategy 5.2.** By 4.3(iii) there is a  $(B, B')$ -bimodule  $\mathfrak{M}$  inducing a Puig equivalence between  $B$  and  $B'$ . But there is a little bit of freedom in choosing  $\mathfrak{M}$ , which we are going to exploit. Anyway, for any admissible choice of  $\mathfrak{M}$ , a splendid stable equivalence between  $A$  and  $A'$ , as in 4.8, is afforded by the  $(A, A')$ -bimodule  $\mathcal{M} \otimes_B \mathfrak{M} \otimes_{B'} \mathcal{M}'^\vee$ . This yields the functor

$$F : \text{mod-}A \rightarrow \text{mod-}A' : X \mapsto X \otimes_A \mathcal{M} \otimes_B \mathfrak{M} \otimes_{B'} \mathcal{M}'^\vee,$$

which induces an equivalence  $\underline{\text{mod-}}A \rightarrow \underline{\text{mod-}}A'$  between the respective stable module categories. Recall that hence, by [33, Theorem 2.1(ii)], the functor  $F$  maps each simple  $kG$ -module in  $A$  to an indecomposable  $kG'$ -module in  $A'$ .

Our practical aim in this section now is to prove that  $\mathfrak{M}$  can be chosen such that  $F$  actually transfers each simple  $kG$ -module in  $A$  to a simple  $kG'$ -module. Our choice of  $\mathfrak{M}$  is specified below in 5.6. If this has been achieved, then Linckelmann's Theorem [33, Theorem 2.1(iii)] yields that  $F$  realizes a Morita equivalence between  $A$  and  $A'$ , and thus even a Puig equivalence between these block algebras. This will then also decide the missing entries in the 3-decomposition matrix of  $A$ , see 2.7.

**Lemma 5.3.** *The following characters are afforded by direct sums of of trivial-source  $A$ -modules:*

- (i)  $\chi_{29}$
- (ii)  $\chi_{29} + \chi_{38}$
- (iii)  $\chi_{38} + \chi_{62}$
- (iv)  $\chi_{29} + \chi_{38} + \chi_{55}$
- (v)  $\chi_{29} + \chi_{62} + \chi_{89}$

*Proof.* Using GAP [9], and the character tables of  $G$  and its maximal subgroups, inducing suitable linear characters we determine various permutation characters of  $G$ , and their components belonging to the block  $A$ . Doing so we find

$$(1_{2_1^{1+s}.O_8^+(2)}) \uparrow^G \cdot 1_A = \chi_{29} \quad \text{and} \quad (1_{U_6(2).\mathfrak{S}_3}) \uparrow^G \cdot 1_A = \chi_{29} + \chi_{38},$$

where 1 denotes the respective trivial character, verifying (i) and (ii). Moreover, (iii) follows from

$$(\lambda_{2^{4+12}.(\mathfrak{S}_3 \times 3\mathfrak{S}_6)}) \uparrow^G \cdot 1_A = \chi_{38} + \chi_{62},$$

where  $\lambda$  is the inflation to  $2^{4+12}.(\mathfrak{S}_3 \times 3\mathfrak{S}_6)$  of the unique linear character of  $\mathfrak{S}_3 \times 3\mathfrak{S}_6$  having kernel  $3 \times 3\mathfrak{S}_6$ . Finally, for (iv) and (v) we observe

$$(1_{2^{2+12}.(\mathfrak{A}_8 \times \mathfrak{S}_3)}) \uparrow^G \cdot 1_A = \chi_{29} + \chi_{38} + \chi_{55} \quad \text{and} \quad (1_{2^{2+12}.(\mathfrak{A}_8 \times \mathfrak{S}_3)}) \uparrow^G \cdot 1_A = \chi_{29} + \chi_{62} + \chi_{89},$$

where  $1^-$  denotes the unique non-trivial linear character of  $2^{2+12} : (\mathfrak{A}_8 \times \mathfrak{S}_3)$ .  $\square$

**Lemma 5.4.** *The characters in  $A$  have the following values at elements of order 3:*

	3A	3B
$\chi_{29}$	18954	729
$\chi_{38}$	-18954	1458
$\chi_{51}$	189540	729
$\chi_{55}$	208494	-729
$\chi_{62}$	18954	729
$\chi_{80}$	208494	-729
$\chi_{85}$	-189540	1458
$\chi_{89}$	189540	729
$\chi_{91}$	-208494	-1458

*Proof.* This follows from [7, pp.184–186].  $\square$

**Lemma 5.5.** *The following holds:*

- (i)  $S_1$  is a trivial-source  $kG$ -module with vertex  $P$  and associated character  $\chi_{29}$ .
- (ii) Using the notation in 4.3(ii) we have  $f(S_1) \in \{729a, 729b, 729c, 729d\}$ .

*Proof.* (i) From 5.3(i), we know that  $S_1$  is a trivial-source module with associated character  $\chi_{29}$ , and, by Knörr's result [16, 3.7.Corollary],  $S_1$  has  $P$  as its vertex.

(ii) Setting  $T_1 := f(S_1)$ , by (i) and [44, Lemma 2.2] we have  $T_1 \in \text{IBr}(B)$ . Set  $\dim_k(T_1) = 729 \cdot d$ , where  $d \in \{1, 2\}$ . By the definition of Green correspondence, we have  $T_1 \uparrow^G = S_1 \oplus X$ , for a  $C_3$ -projective  $kG$ -module  $X$ . Then we have  $3^{9-1} \mid \dim_k(X)$  by [42, Theorem 4.7.5], implying that  $\dim_k(T_1 \uparrow^G) \equiv \dim_k(S_1) \pmod{3^8}$ . Now by (i) we have  $\dim_k(S_1) = \chi_{29}(1) = 2816586 \equiv 2187 \pmod{3^8}$ , and  $\dim_k(T_1 \uparrow^G) = |G : H| \cdot \dim_k(T_1) = 17681664000 \cdot 729 \cdot d \equiv 2187 \cdot d \pmod{3^8}$ , implying that  $d = 1$ .  $\square$

**Lemma 5.6.** *The following holds:*

- (i) There is an  $(B, B')$ -bimodule  $\mathfrak{M}$  inducing a Puig equivalence between  $B$  and  $B'$ , such that  $S_1 \otimes_A \mathcal{M} \otimes_B \mathfrak{M} = 1a$ .
- (ii) Choosing  $\mathfrak{M}$  like this we have  $F(S_1) = 1a$ .

*Proof.* (i) By 4.3(iii) there is a  $(B, B')$ -bimodule  $\mathfrak{M}$  inducing a Puig equivalence between  $B$  and  $B'$ . Hence the functor  $-\otimes_B \mathfrak{M}$  induces a bijection from  $\{729a, 729b, 729c, 729d\}$  to  $\{1a, 1b, 1c, 1d\}$ . Thus by 5.5(ii), using [26, Lemma A.3], we have  $S_1 \otimes_A \mathcal{M} \otimes_B \mathfrak{M} = f(S_1) \otimes_B \mathfrak{M} = 1x$ , for some  $x \in \{a, b, c, d\}$ . Now tensoring with  $1x$  induces a Puig auto-equivalence of  $B'$ , see [23, Lemma 2.8]. Thus by replacing  $\mathfrak{M}$  by  $\mathfrak{M} \otimes 1x$ , the assertion follows from observing that  $1x \otimes 1x = 1a$  as  $kH'$ -modules.

(ii) follows from  $F(S_1) = 1a \otimes_{B'} \mathcal{M}'^\vee = 1a$ , using [26, Lemma A.3], and the fact that Green correspondence maps the trivial module to the trivial module.  $\square$

**Notation 5.7.** From now on let  $\mathfrak{M}$  be chosen as in 5.6, and let  $F$  be the associated functor as described in 5.2.

**Lemma 5.8.** *The following holds:*

- (i) *There is a trivial-source  $kG$ -module  $T$  in  $A$  with associated character  $\chi_{29} + \chi_{38}$  and vertex  $R$ , so that  $T$  is uniserial with Loewy and socle series*

$$T = \begin{array}{|c|} \hline S_1 \\ \hline S_2 \\ \hline S_1 \\ \hline \end{array}.$$

- (ii) *We have*

$$F(T) = \begin{array}{|c|} \hline 1a \\ \hline 4b \\ \hline 1a \\ \hline \end{array} \oplus (\text{proj}) \quad \text{and} \quad F(S_2) = 4b.$$

*Proof.* (i) From 5.3(ii) we know that there is a self-dual  $kG$ -module  $T$  in  $A$  which is a direct sum of trivial-source modules and has character  $\chi_{29} + \chi_{38}$ . Thus, by 2.7,  $T = 2 \cdot S_1 + S_2$  as composition factors, and hence from  $[T, T]^G = 2$  we conclude that  $T$  is indecomposable, and thus of shape as asserted. Finally,  $R$  is a vertex of  $T$  by making use of 5.4 and [31, II Lemma 12.6(ii)].

(ii) be the projective-free part of  $F(T)$ . Then, by 5.6(ii), [26, Lemma A.3], and 3.3(iii) we conclude that  $X$  is of the shape asserted. Now the splitting-off method, see [26, Lemma A.1], yields  $F(S_2) = 4b$ .  $\square$

**Lemma 5.9.** *The following holds:*

- (i) *There is a trivial-source  $kG$ -module  $U$  in  $A$  with associated character  $\chi_{38} + \chi_{55}$  and vertex  $P$ , so that  $U$  has Loewy and socle series*

$$U = \begin{array}{|c|} \hline S_2 \\ \hline S_1 \ S_3 \\ \hline S_2 \\ \hline \end{array}.$$

- (ii) *We have*

$$F(U) = \begin{array}{|c|} \hline 4b \\ \hline 1a \ 1b \\ \hline 4b \\ \hline \end{array} \oplus (\text{proj}) \quad \text{and} \quad F(S_3) = 1b.$$

- (iii)  *$S_3$  is a trivial-source  $kG$ -module in  $A$  with associated character  $\chi_{51}$  and vertex  $P$ .*

*Proof.* (i) By 5.3(iv), there is a self-dual  $kG$ -module  $X$  being the direct sum of trivial-source  $kG$ -modules and having character  $\chi_{29} + \chi_{38} + \chi_{55}$ . Hence, by 2.7, we have  $X = 2 \cdot S_1 + 2 \cdot S_2 + S_3$  as composition factors.

Assume first that  $X$  is indecomposable. Then from self-duality and  $[U, U]^G = 3$  we conclude that  $U$  has Loewy and socle series

$$U = \begin{array}{|c|} \hline S_1 \ S_2 \\ \hline S_3 \\ \hline S_1 \ S_2 \\ \hline \end{array}.$$

But from  $\langle \chi_{29} + \chi_{38}, \chi_{29} + \chi_{38} + \chi_{55} \rangle_G = 2$  and the structure of  $T$  in 5.8(i) we conclude that there is an embedding of  $T$  into  $X$ , and hence  $X$  has  $T \oplus S_2$  as submodule, implying that  $S_3$  is an epimorphic image, and thus a direct summand of  $X$ , a contradiction.

Hence  $X$  is decomposable. Assume now that there is a direct summand with character  $\chi_{38}$ , or a direct summand with character  $\chi_{55}$ . Then in either case it follows from self-duality that  $X$  has a direct summand isomorphic to  $S_2$ . But by 2.7 the module  $S_2$  is not liftable, hence is not a trivial-source module, a contradiction.

Thus we conclude that there is a trivial-source  $kG$ -module  $U$  with character  $\chi_{38} + \chi_{55}$ , and again by self-duality  $U$  has shape asserted. Moreover, by **5.4** and [31, II, Lemma 12.6(ii)],  $U$  has  $P$  as its vertex.

(ii) Let  $X$  be the projective-free part of  $F(U)$ . Then, by **5.8(ii)**, [26, A.3 Lemma], [26, A.1 Lemma], and **3.3(i)**, we get that  $X$  is a trivial-source  $kG'$ -module in  $A'$  with vertex  $P$  such that  $[4b, X]^{G'} \neq 0$ . Hence, **3.3(i)** yields the shape of  $X$  as asserted. This, by **5.6(ii)** and using stripping-off again, implies the statement on  $F(S_3)$ .

(iii) follows from (ii), **3.3(i)**, [26, A.3 Lemma], and **2.7**; the statement on the vertex also follows from Knörr's result [16, 3.7.Corollary].  $\square$

**Lemma 5.10.** *The following holds:*

- (i) *There is a trivial source  $kG$ -module  $V$  in  $A$  with associated character  $\chi_{38} + \chi_{62}$  and vertex  $R$ , so that  $V$  has Loewy and socle series*

$$V = \begin{array}{|c|} \hline S_2 \\ \hline S_1 \quad S_4 \\ \hline S_2 \\ \hline \end{array}.$$

- (ii) *We have*

$$F(V) = \begin{array}{|c|} \hline 4b \\ \hline 1a \quad 6 \\ \hline 4b \\ \hline \end{array} \oplus (\text{proj}) \quad \text{and} \quad F(S_4) = 6.$$

*Proof.* (i) By **5.3(iii)**, there is a self-dual  $kG$ -module  $V$  which is the direct sum of trivial-source  $kG$ -modules and has character  $\chi_{38} + \chi_{62}$ . Hence, by **2.7**, we have  $V = S_1 + 2 \cdot S_2 + S_4$  as composition factors. Thus from  $[V, V]^G = 2$  we conclude that  $V$  is indecomposable, and of shape as asserted. Finally,  $R$  is a vertex of  $T$  by making use of **5.4** and [31, II Lemma 12.6(ii)].

(ii) Let  $X$  be the projective-free part of  $F(V)$ . Then, by **5.8(ii)**, [26, A.3 Lemma], [26, A.1 Lemma], and **3.3(iii)**, we get that  $X$  is a trivial-source  $kG'$ -module in  $A'$  with vertex  $R$  such that  $[4b, X]^{G'} \neq 0$ . Hence, **3.3(iii)** yields the shape of  $X$  as asserted. This, by **5.6(ii)** and using stripping-off again, implies the statement on  $F(S_4)$ .  $\square$

**Lemma 5.11.** *We have  $\text{Ext}_{kG}^1(S_4, S_3) = 0 = \text{Ext}_{kG}^1(S_3, S_4)$ .*

*Proof.* **5.10(ii)**, **5.9(ii)**, and **3.3(iv)** that  $\text{Ext}_A^1(S_4, S_3) \cong \text{Ext}_{A'}^1(F(S_4), F(S_3)) \cong \text{Ext}_{A'}^1(6, 1b) = 0$ . The second statement follows similarly.  $\square$

**Lemma 5.12.** *The following holds:*

- (i) *There is a trivial source  $kG$ -module  $W$  in  $A$  with associated character  $\chi_{62} + \chi_{89}$  and vertex  $P$ , such that  $W$  has Loewy and socle series*

$$W = \begin{array}{|c|} \hline S_4 \\ \hline S_2 \quad S_5 \\ \hline S_4 \\ \hline \end{array}.$$

- (ii) *We have  $\alpha = 0$ .*

- (iii) *We have*

$$F(W) = \begin{array}{|c|} \hline 6 \\ \hline 4a \quad 4b \\ \hline 6 \\ \hline \end{array} \oplus (\text{proj}) \quad \text{and} \quad F(S_5) = 4a.$$

*Proof.* (i)–(ii) By **5.3(v)**, there is a self-dual  $kG$ -module  $X$  being the direct sum of trivial-source  $kG$ -modules and having character  $\chi_{29} + \chi_{62} + \chi_{89}$ . Hence, by **2.7**, we have  $X = S_1 + S_2 + \alpha \cdot S_3 + 2 \cdot S_4 + S_5$  as composition factors. Now, from  $\langle \chi_{29}, \chi_{29} + \chi_{62} + \chi_{89} \rangle_G = 1$ , using **5.5(i)**, we infer

that  $[S_1, X]^G = 1 = [X, S_1]^G$ , thus  $S_1$  is a direct summand of  $X$ , so that there is a trivial-source module  $W$  with character  $\chi_{62} + \chi_{89}$ , and thus  $W = S_2 + \alpha \cdot S_3 + 2 \cdot S_4 + S_5$  as composition factors.

Next, from  $\langle \chi_{51}, \chi_{62} + \chi_{89} \rangle_G = 0$  and **5.9**(iii) we conclude that  $[S_3, W]^G = 0 = [W, S_3]^G$ , that is  $S_3$  does not occur neither in the socle nor the head of  $W$ . Moreover, since both  $S_2$  and  $S_5$  are not liftable, by **2.7**, neither of them is a trivial-source module, hence we infer that  $[S_2, W]^G = 0 = [W, S_2]^G$  and  $[S_5, W]^G = 0 = [W, S_5]^G$ . Thus we conclude that  $W/\text{rad}(W) \cong \text{soc}(W) \cong S_4$ , in particular  $W$  is indecomposable. By making use of **5.4** and [31, II Lemma 12.6(ii)] we get that  $P$  is a vertex of  $W$ .

Thus for the heart  $Y := \text{rad}(W)/\text{soc}(W)$  of  $W$  we have  $Y = S_2 + S_5 + \alpha \cdot S_3$  as composition factors. Now assume that  $\alpha \neq 0$ . Then, by self-duality, we infer that  $[S_3, Y]^G = 1 = [Y, S_3]^G$ . Hence, since  $W/\text{rad}(W) \cong \text{soc}(W) \cong S_4$ , we infer that there are uniserial modules of shape

$$\begin{array}{|c|} \hline S_3 \\ \hline S_4 \\ \hline \end{array} \quad \text{and} \quad \begin{array}{|c|} \hline S_4 \\ \hline S_3 \\ \hline \end{array}$$

contradicting **5.11**. Hence we infer  $\alpha = 0$ , and  $W$  has shape as asserted.

(iii) Let  $X$  be the projective-free part of  $F(W)$ . Then, by **5.10**(ii), [26, A.3 Lemma], [26, A.1 Lemma], and **3.3**(i), we get that  $X$  is a trivial-source  $kG'$ -module in  $A'$  with vertex  $P$  such that  $[6, X]^{G'} \neq 0$ . Hence, **3.3**(i) yields the shape of  $X$  as asserted. This, by **5.8**(ii) and using stripping-off again, implies the statement on  $F(S_5)$ .  $\square$

**Proof of Theorem 1.5.** First,  $A$  and  $A'$  are splendidly stably equivalent of Morita type via the functor  $F$  by **4.8** and **5.2**. Moreover, by the choice in **5.7**, as well as **5.6**(ii), **5.8**(ii), **5.9**(ii), **5.10**(ii), and **5.12**(iii), all simple  $A$ -modules are sent to simple  $A'$ -modules via the functor  $F$ . Hence, [33, Theorem 2.1(iii)] yields that  $A$  and  $A'$  are Morita equivalent, and hence are Puig equivalent.  $\square$

**Proof of Theorem 1.3.** Just as in [23, (6.13)], we conclude that  $A$  and  $B$  are splendidly derived equivalent, by using the splendid derived equivalence between  $A'$  and  $B'$  given in [45, Example 4.4] and [46, Corollary 2].  $\square$

**Proof of Corollary 1.4.** Using the character table of  $G$ , GAP [9] shows that  $G$  has five 3-blocks, of defect 9, 3, 2, 1, and 0. While the former two, according to [43], have non-abelian defect groups, the latter two have cyclic and trivial defect groups, respectively, for which both Conjectures **1.1** and **1.2** are well-known to hold. Hence in this respect the only block of interest is the one of defect 2 under consideration.  $\square$

**Proof of 1.6.** This follows from **1.5** and [47, Theorem 2.3], see [45, Remark 3.7].  $\square$

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