Is the projective cover of the trivial module in characteristic 11 for the sporadic simple Janko group J_4 a permutation module?

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Dedicated to the memory of Richard Parker.

Abstract

We determine the ordinary character of the projective cover of the trivial module in characteristic 11 for the sporadic simple Janko group J_4, and answer the question posed in the title.

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1 Introduction

(1.1) Let p be a prime, let $\mathbb{F}$ be a field of characteristic p, let G be a finite group, and let $P_{\mathbb{F}_G}$ be the projective cover of the trivial $\mathbb{F}[G]$-module $\mathbb{F}_G$.

The present article is motivated by the recent paper [12], which deals with the question when $P_{\mathbb{F}_G}$ is a permutation module. This amounts to asking whether there is a subgroup $H \leq G$ such that $P_{\mathbb{F}_G}$ is isomorphic to the induced module $\mathbb{F}_H^G$. Obviously, this is the case whenever G is a p'-group, so that we may assume that $p \mid |G|$. Moreover, by [12, Cor.2.6], if G has the above property, then so has any composition factor of G, which shifts focus to non-abelian simple groups.

Now, in [12], the non-abelian simple groups having the above property are classified, apart from the groups of Lie type in defining characteristic and, amongst the sporadic simple groups, the largest Janko group J_4 in characteristic 11.

The latter escapes all purely character theoretic attacks. Fortunately, I have been able to provide computational help to answer the above question: It turns out that $P_{\mathbb{F}_{J_4}}$ is *not* a permutation module, where $\mathbb{F}$ is a field of characteristic 11. This is already reported in [12, Thm.4.1], albeit without proof. It is the purpose of the present article to provide the details of the computations I made.

Actually, these computations even reveal the projective indecomposable (ordinary) character $\Psi_{\mathbb{F}_{J_4}}$ afforded by $P_{\mathbb{F}_{J_4}}$, and a few more projective indecomposable characters belonging to the principal 11-block of J_4. About the latter virtually nothing is known so far, so that the results presented here might be the first steps towards finding its decomposition matrix.

(1.2) We describe the general approach, see also [12, Ch.2]: Note first that, given G, the above property only depends on p, but not on the particular choice of $\mathbb{F}$, so that later on we will let $\mathbb{F}$ be the prime field $\mathbb{F}_p$.

Now, if G has the above property, since $P_{\mathbb{F}_G}$ is indecomposable, it necessarily is a transitive permutation module. Thus it has shape $\mathbb{F}_H^G$, for some $H \leq G$, where since $P_{\mathbb{F}_G}$ is projective H necessarily is a p'-subgroup. Moreover, since for any p'-subgroup $U \leq G$ the module $\mathbb{F}_U^G$ has $P_{\mathbb{F}_G}$ as a direct summand, we conclude that H has maximal order amongst all p'-subgroups of G.

If the p-modular decomposition matrix of G is known, then by the theory of trivial-source modules, see [8, Ch.II.12], we may just check whether the permutation character 1_H^G associated with the action of G on the cosets of H coincides with the projective indecomposable character $\Psi_{\mathbb{F}_G}$. But, as mentioned above, for the case of interest to us we are not at all in this comfortable position.

Hence we have to proceed otherwise: We pick any p'-subgroup $H \leq G$ of maximal order, and check whether $\mathbb{F}_H^G$ is an indecomposable $\mathbb{F}[G]$-module, where the latter property is equivalent to the endomorphism algebra $\mathrm{End}_{\mathbb{F}[G]}(\mathbb{F}_H^G)$ being a local $\mathbb{F}$-algebra. Thus, to pursue this, we have to analyze the structure of the endomorphism algebra of a permutation module, which for the group in question due to sheer size is not too easy to handle.

(1.3) The present article is organized as follows: In Section 2 we describe the background concerning endomorphism algebras and the orbit enumeration techniques used; in Section 3 we provide some character theoretic data on the principal 11-block of $G := J_4$, and specify the subgroup H to consider; in Section 4 we consider the G-action on the set $\mathcal{O}$ of cosets of H, and apply ORB to find the H-orbits in $\mathcal{O}$; in Section 5 we consider the endomorphism algebra E of the permutation module afforded by $\mathcal{O}$, and determine the character table of E; and finally in Section 6 we compute the decomposition matrix of E, and answer the question we started with.

(1.4) Acknowledgments. It is a great honor to have this opportunity to thank Richard Parker for a wealth of mathematical ideas he was always keen to share with everybody. In particular, the present article owes much to his work, as a glance into the list of references reveals. Notably, proving the sheer existence of J_4 was Richard Parker's original motivation to invent the MeatAxe.

Moreover, I would also like to thank Thomas Breuer for inspiring discussions (not only) about the topic of this article, and the referee for their careful reading.

2 Prerequisites

(2.1) Endomorphism algebras. We recall the necessary facts about the structure of endomorphism algebras of permutation modules, thereby fixing the notation used later; as a general reference see [8, Ch.II.12], while the background of the particular approach we follow is described in detail in [16]:

a) Let G be a finite group, let $H \leq G$ be a subgroup, let $\mathcal{O}$ be the set of (right) cosets of H in G, and let $n := |\mathcal{O}|$. Moreover, if R is an integral domain, let R_H^G be the permutation $R[G]$-module associated with $\mathcal{O}$, and let $E_R := \mathrm{End}_{R[G]}(R_H^G)$ be its endomorphism R-algebra. Then E_R is R-free of rank $r := \langle 1_H^G, 1_H^G \rangle_G$, where 1_H^G is the induced character from the trivial character 1_H of H, and $\langle \cdot, \cdot \rangle_G$ denotes the usual scalar product on the characters of G.

Let $\{v_1, \ldots, v_r\}$ be a set of representatives of the H-orbits $\mathcal{O}_j := (v_j)^H \subseteq \mathcal{O}$, where v_1 denotes the coset H itself, and let $g_j \in G$ such that $v_1 g_j = v_j$. Moreover, let $H_j := \mathrm{Stab}_H(v_j)$, and let $n_j := |\mathcal{O}_j|$. The *paired orbit* $\mathcal{O}_{j^*}$ of $\mathcal{O}_j$ is defined to be the H-orbit containing $v_1 g_j^{-1}$.

Let $\mathcal{O}_j^+ := \sum_{v \in \mathcal{O}_j} v \in R_H^G$ be the associated *orbit sum*. Then E_R has a distinguished R-basis $\{A_1, \ldots, A_r\}$, being called its *Schur basis*, where A_j is defined by $v_1 \mapsto \mathcal{O}_j^+$, and subsequent extension by G-transitivity to all of $\mathcal{O}$; in particular A_1 is the identity map. Thus, abbreviating $E = E_{\mathbb{Z}}$, we have $E_R = E \otimes_{\mathbb{Z}} R$.

Writing $A_i A_j = \sum_{k=1}^r p_{ijk} A_k \in E$, the associated (non-negative) structure constants, also being called *intersection numbers*, are given as $p_{ijk} = \frac{n_i}{n_k} \cdot c_{jk}(g_i) \in \mathbb{Z}$, using the *orbit counting numbers* $c_{jk}(g_i) := |\mathcal{O}_j g_i \cap \mathcal{O}_k| \in \mathbb{N}_0$. Thus the (right) regular representation of E, with respect to its Schur basis, is given by the *intersection matrices* $P_j := [p_{ijk}]_{ik} \in \mathbb{Z}^{r \times r}$. In particular, the first row of P_j is given by $p_{1jk} = \delta_{jk}$, that is, consists of the j-th unit vector.

b) Let K be a field. Then, for any E_K-module V, the trace map $\varphi_V \colon E_K \to K \colon A \mapsto \mathrm{Tr}_V(A)$ is called the *character* afforded by V. Letting $\mathrm{Irr}(E_K) := \{\varphi_1, \ldots, \varphi_s\}$ be the set of characters afforded by the irreducible E_K-modules, we obtain the *character table* $\Phi := [\varphi_i(A_j)]_{ij} \in K^{s \times r}$.

Since $\mathbb{C}_H^G$ is a semi-simple $\mathbb{C}[G]$-module, $E_{\mathbb{C}}$ is a (split) semi-simple $\mathbb{C}$-algebra, and we have a natural bijection between the irreducible representations of $E_{\mathbb{C}}$ and the distinct constituents of $\mathbb{C}_H^G$, being called *Fitting correspondence*; in terms of irreducible characters the Fitting correspondent of $\varphi \in \mathrm{Irr}(E_{\mathbb{C}})$ is denoted by χ_φ. Moreover, $E_{\mathbb{C}}$ is commutative if and only if $\mathbb{C}_H^G$ is multiplicity-free.

We have $\varphi(A_1) = m_{\chi_\varphi} = \langle 1_H^G, \chi_\varphi \rangle_G$, the multiplicity of χ_φ as a constituent of 1_H^G. The Fitting correspondent $\varphi_1 \in \mathrm{Irr}(E_{\mathbb{C}})$ of 1_G is given by $\varphi_1(A_j) = n_j$; it is the only irreducible character of $E_{\mathbb{C}}$ whose values on the Schur basis consist of non-negative integers only. We have the following *orthogonality relations* between characters $\varphi, \varphi' \in \mathrm{Irr}(E_{\mathbb{C}})$, where $\overline{\ }$ denotes complex conjugation, and where we have $\overline{\varphi(A_j)} = \varphi(A_{j^*})$:

$$\frac{1}{n} \cdot \sum_{j=1}^{r} \frac{1}{n_j} \cdot \overline{\varphi(A_j)} \cdot \varphi'(A_j) = \delta_{\varphi,\varphi'} \cdot \frac{m_{\chi_\varphi}}{\chi_\varphi(1)}.$$

c) The endomorphism algebra E admits a decomposition theory, similar to the one for group algebras: Let $\mathcal{R}$ be a discrete valuation ring in an algebraic number field $\mathbb{K}$, such that the maximal ideal $\wp \lhd \mathcal{R}$ contains p, and let $\mathbb{F} := \mathcal{R}/\wp$. In practice, in order to keep data consistent, we make the same conventional

choices for $\mathcal{R}$ and $\wp$ as in [7]. Moreover, we assume that $\mathbb{K}$ and $\mathbb{F}$ are large enough so that both $E_{\mathbb{K}}$ and $E_{\mathbb{F}}$ are split.

Then any finitely generated $E_{\mathbb{K}}$-module can be realized by an $E_{\mathcal{R}}$-lattice V, and *$\wp$-modular reduction*, mapping V to $V_{\mathbb{F}} := V \otimes_{\mathcal{R}} \mathbb{F}$, yields a $\mathbb{Z}$-linear decomposition map $D_{\wp}\colon G(E_{\mathbb{K}}) \to G(E_{\mathbb{F}})$ between the associated Grothendieck groups. Its matrix with respect to the $\mathbb{Z}$-bases consisting of the respective irreducible representations is called the associated *decomposition matrix*.

Since $\mathbb{K}$ is a splitting field for $E_{\mathbb{K}}$, we have $\mathrm{Irr}(E_{\mathbb{C}}) = \mathrm{Irr}(E_{\mathbb{K}})$, which is $\mathbb{K}$-linearly independent, so that we may identify $G(E_{\mathbb{K}})$ with $\mathbb{Z}\mathrm{Irr}(E_{\mathbb{C}})$. Since $\mathbb{F}$ is a splitting field for $E_{\mathbb{F}}$, similarly $\mathrm{Irr}(E_{\mathbb{F}})$ is $\mathbb{F}$-linearly independent. Since $\mathrm{Irr}(E_{\mathbb{K}})$ has values in $\mathcal{R}$, for any element of $E_{\mathcal{R}}$, we conclude that $\wp$-modular reduction induces a $\mathbb{Z}$-linear map $D_{\wp}\colon \mathbb{Z}\mathrm{Irr}(E_{\mathbb{C}}) \to \mathbb{F}\mathrm{Irr}(E_{\mathbb{F}})$.

d) Let S be a simple $E_{\mathbb{F}}$-module, with associated projective indecomposable module $P_S \cong e_S E_{\mathbb{F}}$, for some suitable idempotent $e_S \in E_{\mathbb{F}}$. Moreover, for $\varphi \in \mathrm{Irr}(E_{\mathbb{K}})$ let V_φ be an $E_{\mathcal{R}}$-lattice such that $(V_\varphi)_{\mathbb{K}} := V_\varphi \otimes_{\mathcal{R}} \mathbb{K}$ has character φ, and let $e_\varphi \in E_{\mathbb{K}}$ be an idempotent such that $e_\varphi E_{\mathbb{K}} \cong (V_\varphi)_{\mathbb{K}}$.

Any idempotent $e \in E_{\mathbb{F}}$ can be *lifted* to $E_{\mathcal{R}}$, that is, there is an idempotent $\widehat{e} \in E_{\mathcal{R}}$ such that $\widehat{e} \otimes 1_{\mathbb{F}} = e \in E_{\mathbb{F}}$. In particular, there is a projective indecomposable $E_{\mathcal{R}}$-lattice $\widehat{P}_S \cong \widehat{e}_S E_{\mathcal{R}}$ lifting P_S. Thus for the multiplicity of S as a constituent of $V_{\mathbb{F}}$ we have *Brauer reciprocity* $[V_{\mathbb{F}}\colon S] = [(\widehat{P}_S)_{\mathbb{K}}\colon V_{\mathbb{K}}]$, and for the *Cartan numbers* of $E_{\mathbb{F}}$ we have $[P_S\colon S'] = [P_{S'}\colon S]$. In particular, since $E_{\mathbb{F}} \cong \bigoplus_S (P_S)^{\oplus \dim_{\mathbb{F}}(S)}$ as $E_{\mathbb{F}}$-modules, this entails

$$\dim_{\mathbb{F}}(P_S) = \sum_{S'} \dim_{\mathbb{F}}(S') \cdot [P_S\colon S'] = \sum_{S'} \dim_{\mathbb{F}}(S') \cdot [P_{S'}\colon S] = [E_{\mathbb{F}}\colon S].$$

This relates to Fitting correspondence as follows: For an irreducible character χ of G occurring as a constituent of 1_H^G, let V_χ be an $\mathcal{R}[G]$-lattice such that $(V_\chi)_{\mathbb{K}}$ has character χ. Then we have $(\mathbb{K}_H^G)e_\varphi \cong (V_{\chi_\varphi})_{\mathbb{K}}$ as $\mathbb{K}[G]$-modules, and thus

$$[(\mathbb{K}_H^G)\widehat{e}_S\colon (V_{\chi_\varphi})_{\mathbb{K}}] = [\widehat{e}_S E_{\mathbb{K}}\colon (V_\varphi)_{\mathbb{K}}] = [(\widehat{P}_S)_{\mathbb{K}}\colon (V_\varphi)_{\mathbb{K}}] = [(V_\varphi)_{\mathbb{F}}\colon S].$$

(2.2) Enumeration of long orbits. To facilitate computations with (large) permutation representations we use the GAP package ORB [17], where its orbit enumeration techniques are described comprehensively in [16], and an extended worked application is presented in [15]. We give a brief sketch of the approach:

Let G be a (large) finite group, and let $\mathcal{O}$ be a (large) transitive G-set, which we assume to be implicitly given, for example as a G-orbit of a vector v_1 in an $F[G]$-module V over a finite field F. Letting $H \leq G$ be a (still large) subgroup, we are interested in classifying the H-orbits $\mathcal{O}_j$ in $\mathcal{O}$, finding their length n_j, representatives $v_j \in \mathcal{O}_j$, elements $g_j \in G$ such that $v_1 \cdot g_j = v_j$, and the stabilizers $H_j = \mathrm{Stab}_H(v_j)$. To achieve this, we assume to be able to compute efficiently within H (but not within G), for example by having a (smallish) faithful permutation representation of H at hand.

To find the H-orbits in $\mathcal{O}$, we choose a (smallish) helper subgroup $K \leq H$, and enumerate the various H-orbits $\mathcal{O}_j \subseteq \mathcal{O}$ by the K-orbits they contain. To do so, we choose a (not too small) helper K-set $\mathcal{Q}$ together with a homomorphism $\pi_K : \mathcal{O} \to \mathcal{Q}$ of K-sets, which again we assume to be implicitly given, for example by an $F[K]$-quotient module of V.

Moreover, we assume that K has sufficiently long orbits in $\mathcal{Q}$, and that we are able to classify them, by giving representatives, their stabilizers in K, as well as complete Schreier trees. (Thus for the K-action on $\mathcal{Q}$ we are facing a similar problem as for the H-action on $\mathcal{O}$, apart from the requirement on Schreier trees. So we could just recurse. Actually, the full functionality of ORB supports this, but for the present purposes we will get away with a single helper subgroup.)

For any K-orbit in $\mathcal{Q}$, the chosen representative is called its *distinguished point* (although it might be chosen arbitrarily). Then, for any K-orbit $\mathcal{O}' \subseteq \mathcal{O}$, the π_K-preimages of the distinguished point of $\pi_K(\mathcal{O}') \subseteq \mathcal{Q}$ are likewise called the distinguished points of $\mathcal{O}'$. Hence, to enumerate an H-orbit $\mathcal{O}_j$ by enumerating the K-orbits contained in it, we only have to store the associated distinguished points, and a Schreier tree telling us how to reach them from v_j.

For any H-orbit $\mathcal{O}_j$ we are content of finding only as many K-orbits contained in it as are needed to cover more than half of it; this is equivalent to knowing n_j and $|H_j|$. Then we have a randomized membership test for $\mathcal{O}_j$, and a deterministic test to decide whether the H-orbits found are actually pairwise disjoint.

The number of points of $\mathcal{O}_j$ covered by the above enumeration process, divided by the number of distinguished points actually stored is called the *saving factor* achieved. The maximum saving factor possible is $|K|$, which is achieved if and only if K has only regular orbits in the π_K-image of the part of $\mathcal{O}_j$ covered.

(2.3) Computational tools. To facilitate group theoretic and character theoretic computations we use the computer algebra system GAP [4], its comprehensive database CTblLib [1] of ordinary and modular character tables, and its library TomLib [9] of tables of marks. In particular, CTblLib encompasses the data given in the Atlas [3] and in the ModularAtlas [7], as well as the additional data collected on the ModularAtlasHomepage [23].

As far as matrix representations over finite fields are concerned, we use the MeatAxe [21], whose basic ideas go back to [20], where we also use its extensions to compute submodule lattices [10] and direct sum decompositions [11].

Computations with matrix representations over the integers and over the rational numbers are facilitated by the GAP package IntegralMeatAxe [14], which is developed and used heavily in [5], but owes much to [19]. (The IntegralMeatAxe package is as yet unpublished, but I am of course happy to provide the code to everybody interested. Moreover, as an alternative, similar functionality is available in the computer algebra system MAGMA [2].)

Data concerning explicit permutation representations, ordinary and modular matrix representations, and the embedding of (maximal) subgroups of sporadic

Table 1: The permutation character 1_H^G.

χ_i	$\mathbb{Q}(\chi_i)$	m_i	χ_i	$\mathbb{Q}(\chi_i)$	m_i	χ_i	$\mathbb{Q}(\chi_i)$	m_i
1		1	21		2	32		1
8		1	22		1	36	r_5	1
11		1	23	r_3	1	37	r_5	1
14		1	24	r_3	1	38	r_5	1
19	r_{33}	2	29		1	39	r_5	1
20	r_{33}	2	30		1	51		1

simple groups is available in the AtlasOfGroupRepresentations [24], and through the GAP package AtlasRep [25]. For a wealth of group theoretical information used throughout we refer to the Atlas [3], whose notational conventions we follow; in particular we let $r_n := \sqrt{n}$ be the positive square root of $n \in \mathbb{N}$.

3 The principal 11-block of J_4

(3.1) From now on let $G := J_4$.

Then G has the principal block B_0 as its only 11-block of positive defect. The defect groups of B_0, that is, the Sylow 11-subgroups of G, are extraspecial of shape 11_+^{1+2}, and have the rare property of being trivial-intersection subgroups. There are $k_0 := 49$ irreducible ordinary characters and $l_0 := 40$ irreducible modular characters belonging to B_0.

Sadly enough, this is virtually all what is known about the decomposition numbers of B_0, according to the ModularAtlasHomepage, where B_0 is a prominent gap, in particular in view of the trivial-intersection property of its defect groups.

(3.2) Using the conjugacy classes of maximal subgroups of G, as reproduced in the Atlas, it turns out that the unique class of subgroups of $11'$-subgroups of maximal order is given by the maximal subgroups of G of shape $2^{10}\colon L_5(2)$. Let $H < G$ be a representative of this class.

We have $|H| = 10.239.344.640$ and $[G\colon H] = 8.474.719.242$. The decomposition of the permutation character 1_H^G into the irreducible ordinary characters χ_i of G is given in Table 1, where the χ_i are ordered as in the Atlas, we indicate generators of their (quadratic) character fields, and $m_i := \langle 1_H^G, \chi_i\rangle_G$. In particular, we have $r := \langle 1_H^G, 1_H^G\rangle_G = 27$, and all constituents of 1_H^G belong to B_0.

All constituents except $\chi_{19/20}$ are 11-rational characters, being fixed by the Frobenius automorphism. Amongst them, $\chi_{23/24}$, $\chi_{36/37}$ and $\chi_{38/39}$ are pairs of Galois conjugate characters. The constituents $\chi_{19/20}$ are non-11-rational, Galois conjugate characters, restricting to the same character on 11-regular classes.

(3.3) The projective indecomposable character $\Psi_{\mathbb{F}_G}$ is a summand of 1_H^G. Thus, writing $\Psi_{\mathbb{F}_G} = \sum_{i\in\mathcal{I}} d_i\chi_i$, where $d_i \in \mathbb{N}_0$ and $\mathcal{I}$ is the index set occurring in the first column of Table 1, we have $0 \le d_i \le m_i$; in particular $d_1 = m_1 = 1$.

Moreover, we have $\Psi_{\mathbb{F}_G}(g) = 0$ whenever $g \in G$ is 11-singular. Enforcing these conditions, it turns out that from the $2^{15} \cdot 3^3 = 884736$ candidates for $\Psi_{\mathbb{F}_G}$ allowed by the above inequalities, there are just 75 admissible candidates left.

The above conditions say that $\Psi_{\mathbb{F}_G}$ is a generalized projective character. Thus, by [18, Cor.2.17, La.2.21], we conclude that $\frac{1}{|C_G(g)|_{11}} \cdot \Psi_{\mathbb{F}_G}(g)$ is an algebraic integer, for any $g \in G$, in particular entailing that $11^3 \mid \Psi_{\mathbb{F}_G}(1)$. Moreover, it turns out that they already imply $d_{19} = d_{20}$, and neither complex conjugation nor the Frobenius automorphism yield further conditions. (Note that $\Psi_{\mathbb{F}_G}$ is not necessarily a rational character, although the trivial character 1_G is.)

Unfortunately, we have not been able to find further purely character-theoretic conditions to narrow down the set of admissible candidates for $\Psi_{\mathbb{F}_G}$. In particular, neither restriction to maximal subgroups of G and decomposition into their projective indecomposable characters (providing lower bounds on the d_i), nor induction of the projective cover of the trivial module of maximal subgroups (providing upper bounds on the d_i) did yield any improvement. At this point we have decided to revert to explicit computations, in particular applying ORB.

4 The permutation action

(4.1) To do explicit computations, we pick the 112-dimensional (absolutely) irreducible representation of G over $\mathbb{F}_2$ from the AtlasRep database. The latter is given in terms of (two) standard generators, in the sense of [22]. Words in the standard generators providing generators of a maximal subgroup $2^{10}\colon L_5(2) \cong H < G$ are also available in the AtlasRep database.

Let $V \cong \mathbb{F}_2^{112}$ be the module underlying the above representation of G. Using the MeatAxe, it turns out that H possesses a 1-dimensional fixed space in V. Hence letting $v_1 \in V$ be the unique non-zero H-fixed vector, the G-action on the orbit $\mathcal{O} := (v_1)^G \subseteq V$ is equivalent to its action on the cosets of H. Hence this provides an implicit realization of the latter action.

To store a vector in V, including header information, we need $(\lceil \frac{112}{8} \rceil + 4)$ Byte $=$ 18 Byte. Thus to store $\mathcal{O}$ completely we would need at least

$$([G\colon H] \cdot 18) \text{ Byte} = 152.544.946.356 \text{ Byte} \sim 150 \text{ GB}$$

of memory space, plus some more header information. Although this would in principle be feasible nowadays, in view of the computations we are going to make within $\mathcal{O}$, we apply ORB, trying to achieve a saving factor of ~ 150, say. We set up the required framework:

(4.2) **i)** A faithful permutation representation of H is found as follows: Using the MeatAxe, we determine the submodule lattice of the restriction V_H of V to

H. It turns out that it possesses a unique 16-dimensional $\mathbb{F}_2[H]$-submodule U. Moreover, there is a faithful H-orbit of vectors in U of length 310. Computing the H-action on the latter yields an explicit permutation representation of H.

ii) Before actually choosing a helper subgroup, we already look for a helper H-set, being an epimorphic image of $\mathcal{O}$ as H-sets:

Since V is a self-contragredient $\mathbb{F}_2[G]$-module, V_H has a unique 16-dimensional quotient $\mathbb{F}_2[H]$-module W, being the dual of the submodule U considered above, and thus in particular being a faithful $\mathbb{F}_2[H]$-module as well. Having cardinality $2^{16} = 65536$, it is small enough to enumerate all its elements explicitly.

Using the MeatAxe, we compute the natural quotient map $V_H \to W$ of $\mathbb{F}_2[H]$-modules, which gives rise to a homomorphism $\pi_H \colon \mathcal{O} \to W$ of H-sets.

iii) As helper subgroup we now choose $K := N_H(31) \cong 31 \colon 5$, the normalizer in H of a Sylow 31-subgroup, having order $|K| = 155$. Words in the chosen generators of H providing generators of K are found with the help of GAP, employing the permutation representation of H constructed above.

Returning to the matrix representation of K on V, and going over to the quotient W, it turns out that the K-orbits of vectors in W have lengths $[1^2, 31^{14}, 155^{420}]$, where the exponents denote multiplicities. Hence the expected value of the length of the K-orbit of a randomly chosen vector in W is ~ 154, so that we indeed expect a suitable saving factor as envisaged above.

(4.3) We are now prepared to run ORB, in order to find the decomposition $\mathcal{O} = \coprod_{j=1}^{27} \mathcal{O}_j$ into H-orbits; recall that there are $r = 27$ orbits indeed:

We let $\mathcal{O}_1 := (v_1)^H = \{v_1\}$. Then we randomly choose elements $g \in G$, and check whether $v_1 g \in \mathcal{O}$ belongs to one of the H-orbits already found. If not, then we have found a new H-orbit, $\mathcal{O}_j$ say, and let $g_j := g$ and $v_j := v_1 g$. We also consider $v_1 g_j^{-1} \in \mathcal{O}$ in order to detect non-self-paired H-orbits.

Letting this run for a certain while (actually some 2 hours on a single 3 GHz CPU), we have been able to find 24 of the 27 orbits, making up all of $\mathcal{O}$ up to 27001 vectors. Thus only a fraction of $\sim 3.2 \cdot 10^{-6}$ of $\mathcal{O}$ is missing at this stage, making it highly improbable to conclude simply by random search.

Hence we set out to find the missing (small) three orbits by identifying the associated (large) point stabilizers: Using the library TomLib we find all subgroup orders of $L_5(2)$, and allowing for factors 2^i, where $i \in \{0, \ldots, 10\}$, we get a set of numbers encompassing the subgroup orders of H. Checking all 3-tuples thereof whose indices in H add up to 27001 leaves the following candidates for the missing three orbit lengths:

$$[31, 930, 26040], \quad [465, 496, 26040], \quad [31, 7440, 19530].$$

(4.4) We proceed to decide which case occurs: Recalling that $H \cong O_2(H) \colon L$, where $L \cong L_5(2)$, is a split extension, we have an embedding of L into H. Since

Table 2: The H-orbits in $\mathcal{O}$.

j	j^*	n_j	$\lvert H_j\rvert$	H_j	
1		1	10239344640	$2^{10}.L_5(2)$	1
2		31	330301440	$2^{10}.2^4.L_4(2)$	1
3		930	11010048	$2^9.2^6.(L_3(2) \times 2)$	1
4		17360	589824	$2^7.2^8.3^2.2$	1
5		26040	393216	$2^{1+12}.(D_8 \times S_3)$	1
6		27776	368640	$2^9.S_6$	1
7	8	416640	24576	$2^5.2^6.D_{12}$	1
8	7	416640	24576	$2^5.2^6.D_{12}$	1
9		624960	16384	$(4 \times 2^4).2^4.2^4$	1
10		333120	3072	$2^3.2^3.2^3.S_3$	155
11		4999680	2048	$(D_8 \times 2^4).2^4$	1
12	13	6666240	1536	$2^6.3.D_8$	1
13	12	6666240	1536	$2^6.3.D_8$	155
14		9999360	1024	$(D_8 \times D_8).2.2^3$	155
15		13332480	768	$2^7.S_3$	155
16		53329920	192	$2^2.2^2.D_{12}$	155
17		66060288	155	31.5	102
18	19	79994880	128	$2^3.2^4$	155
19	18	79994880	128	$2^3.2^4$	155
20		159989760	64	$2^3.2^3$	155
21		159989760	64	$2^3.2^3$	155
22		319979520	32	$D_8 \times 2^2$	155
23		341311488	30	$D_{10} \times 3$	152
24		1279918080	8	2^3	155
25		1279918080	8	D_8	152
26		2047868928	5	5	150
27		2559836160	4	4	152

representatives of the conjugacy classes of subgroups of L can be straightforwardly determined, this allows us to find representatives S of the conjugacy classes of subgroups of H of a fixed index.

In turn, using the MeatAxe, we compute the fixed space of S on V, and check whether it contains a vector v having an H-orbit of desired length and belonging to $\mathcal{O}$. The latter property is verified by picking random elements $g \in G$, and checking whether $vg \in V$ is contained in one of the known H-orbits, $\mathcal{O}_j$ say. If so, then we may choose v as a new H-orbit representative, and since we have enumerated $\mathcal{O}_j$ already, we are readily able to find $h \in H$ such that $v_j h = vg$; thus we have $v_1 \cdot g_j h g^{-1} = v$.

i) We first consider subgroups $S < H$ of index 31. For these we have $O_2(H) < S$ and $S/O_2(H) \cong 2^4{:}\, L_4(2)$. There are two conjugacy classes. For one of them

we are successful, excluding the second of the above cases.

ii) Next, we consider subgroups $S < H$ of index 930. Letting $S^* := S \cap O_2(H)$, we have either $S^* = O_2(H)$ and $S/S^* \cong 2^6 \colon L_3(2)$, or $|S^*| = 2^9$ and $S/S^* \cong 2^6 \colon (L_3(2) \times 2)$. For one of the conjugacy classes of subgroups of the second shape we are successful, which brings us down to the first of the above cases.

iii) Finally, we consider subgroups $S < H$ of index 26040, for which we have $2^7 \mid |S^*|$. For one of the conjugacy classes of subgroups such that $|S^*| = 2^7$ we are successful indeed. It turns out that in this case $S/S^* \cong 2^6 \colon (D_8 \times S_3)$.

The results on the H-orbits in $\mathcal{O}$ are collected in Table 2: In the second column we indicate the non-self-paired orbits. In the fifth column we indicate the shape of H_j, where groups having the same shape are actually isomorphic. (This is clear for $H_{7/8}$, $H_{12/13}$ and $H_{18/19}$ coming from paired orbits anyway.) But, letting $H_j^* := H_j \cap O_2(H)$, it turns out that $H_{7/8}^*$ and $H_{12/13}^*$ are different.

In the last column we also give the saving factors achieved for the various H-orbits, which amounts to an average saving factor of ~ 152, indicating that our choice of K and $\mathcal{Q} = W$ was not too bad. Still, we observe that the shorter H-orbits tend to have a saving factor of 1, amounting to no saving at all. This could be due to our fairly ambitious choice of a helper H-set, rather than just a helper K-set, so that the quotient map might very well send a full H-orbit to the zero vector in W; but we have not analyzed this thoroughly.

5 The endomorphism ring

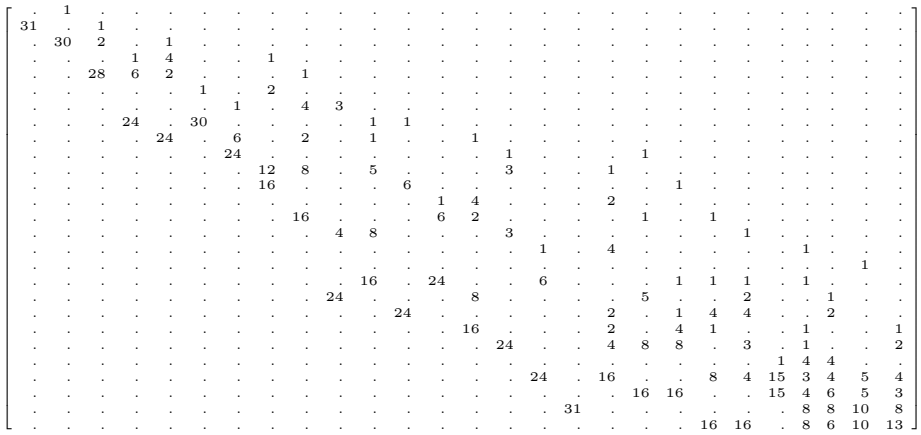
(5.1) We consider the regular representation of the endomorphism algebra E of $\mathbb{Z}_H^G$, which is $\mathbb{Z}$-free of rank $r = 27$. Computing the intersection matrices P_j boils down to determining the orbit counting numbers $c_{jk}(g_i) = |\mathcal{O}_j g_i \cap \mathcal{O}_k|$:

If n_j is small enough so that $\mathcal{O}_j$ can be enumerated explicitly, applying $g_i \in G$ and using the randomized orbit membership test in ORB, we determine lower bounds for the numbers $c_{jk}(g_i)$. We are done once the figures we have found, running through all k, sum up to n_j. For example, P_2 is given in Table 3.

The orbits $\mathcal{O}_j$ being ordered by increasing length, we successively compute $P_2, P_3, \ldots$, until the $\mathbb{Q}$-algebra generated by the matrices found so far has $\mathbb{Q}$-dimension 27, and hence equals $E_{\mathbb{Q}}$. Since E is non-commutative, we need at least two generators, where it turns out that P_3 belongs to the $\mathbb{Q}$-algebra generated by P_2, but that $\{P_2, P_4\}$ suffice to generate $E_{\mathbb{Q}}$.

The P_j are associated with the regular representation of E, with respect to its Schur basis. Hence the $\mathbb{Q}$-dimension of a candidate subalgebra as above can be determined by 'spinning up' the first unit vector by applying the 'standard basis algorithm', in the sense of [20], using the generators in question. Moreover, since the first row of P_j equals the j-th unit vector, decomposing it into the 'standard basis' found above yields the complete intersection matrix P_j. In practice, all of this is done using the IntegralMeatAxe.

Table 3: The intersection matrix P_2.



(5.2) We proceed to determine the character table of $E_{\mathbb{C}}$ and the Fitting correspondence: To do so, we decompose the regular representation of $E_{\mathbb{Q}}$, which is semi-simple, but not split. The ordinary constituents of $\mathbb{Q}_H^G$, as compared to those of $\mathbb{C}_H^G$ given in Table 1, tell us that $E_{\mathbb{Q}}$ has 14 irreducible representations, of degree $[1^9, 2^4, 4] = [1^9, (1+1)^3, 2, (2+2)]$, where exponents denote their multiplicity, and brackets denote their splitting over $\mathbb{C}$.

Using the IntegralMeatAxe and GAP, we compute the characteristic polynomials of P_2 and P_4, their factorization over $\mathbb{Q}$, and for the irreducible divisors f and g occurring, respectively, we determine the $E_{\mathbb{Q}}$-submodules

$$E_{f,g} := \ker(f^{m_f}(P_2)) \cap \ker(g^{m_g}(P_4)) \leq E_{\mathbb{Q}},$$

and their $\mathbb{Q}$-dimension $d_{f,g}$, where m_f and m_g are the associated multiplicities in the respective characteristic polynomials. The result is given Table 4, where (to save space) we abbreviate $f_4 := X^4 - 16X^3 - 75X^2 + 1706X - 2768 \in \mathbb{Q}[X]$ and $g_4 := X^4 - 1288X^3 + 405424X^2 - 2113152X - 2701694976 \in \mathbb{Q}[X]$.

Since we obtain 14 pairwise distinct submodules, we conclude that these coincide with the homogeneous components of $E_{\mathbb{Q}}$.

i) The cases such that $d_{f,g} = 1$ correspond to the rational constituents of 1_H^G with multiplicity 1, where the very first case corresponds to the trivial character 1_G. The action of the P_j on the various $E_{f,g}$ directly yields the associated characters of $E_{\mathbb{C}}$. The orthogonality relations yield the degree of their Fitting correspondents, determining the Fitting correspondence in these cases.

ii) We consider the cases such that $d_{f,g} = 2$, and compute the splitting fields of the irreducible divisors of degree 2 occurring. From this we conclude that these cases correspond to the non-rational constituents of 1_H^G with multiplicity 1:

Table 4: Simultaneous generalized eigenspaces of P_2 and P_4.

f | char.pol.(P_2)	m_f	g | char.pol.(P_4)	m_g	$d_{f,g}$	spl.	χ_i	m_i
$X-31$	1	$X-17360$	1	1		χ_1	1
$X-16$	1	$X-1640$	1	1		χ_8	1
$X-9$	1	$X+20$	1	1		χ_{22}	1
$X-5$	1	$X+120$	1	1		χ_{32}	1
$X-1$	3	$X-284$	1	1		χ_{29}	1
		$X-196$	1	1		χ_{30}	1
		$X-20$	1	1		χ_{51}	1
$X+2$	1	$X-112$	1	1		χ_{14}	1
$X+12$	1	$X-1192$	1	1		χ_{11}	1
$X^2-6X-39$	1	$X^2+100X-1388$	1	2	r_3	$\chi_{23/24}$	1
X^2-45	1	$X^2+130X+3820$	1	2	r_5	$\chi_{38/39}$	1
$X^2+12X+31$	1	$X^2+110X-620$	1	2	r_5	$\chi_{36/37}$	1
$X^2+3X-64$	2	$X^2-424X-11520$	2	4		χ_{21}	2
f_4	2	g_4	2	8	r_{33}	$\chi_{19/20}$	2

In the first case we get simultaneous splitting field $\mathbb{Q}(r_3)$, over which $E_{f,g}$ splits into two 1-dimensional submodules. Thus this case corresponds to $\chi_{23/24}$, and as above we determine the associated characters of $E_{\mathbb{C}}$.

Similarly, the second and third cases yield simultaneous splitting field $\mathbb{Q}(r_5)$, over which both submodules split. Thus these cases correspond to $\chi_{36/37;38/39}$, and as above we determine the associated characters of $E_{\mathbb{C}}$. The orthogonality relations yield the degree of the associated Fitting correspondents, showing that these cases correspond to $\chi_{38/39}$ and $\chi_{36/37}$, respectively.

iii) Hence we conclude that the cases $d_{f,g} > 2$ correspond to the remaining constituents of 1_H^G, that is, $\chi_{19/20;21}$, each of which occurs with multiplicity 2.

Since χ_{21} is rational, while $\chi_{19/20}$ is not, we infer that the case $d_{f,g} = 4$ corresponds to χ_{21}; thus the trace map afforded by the action of the P_j on $E_{f,g}$ is twice the associated character of $E_{\mathbb{C}}$.

Hence the case $d_{f,g} = 8$ corresponds to $\chi_{19/20}$. Over $\mathbb{Q}(r_{33})$, both f_4 and g_4 split into two irreducible factors of degree 2, and $E_{f,g}$ splits into a direct sum of two 4-dimensional submodules. The trace maps afforded by the action of the P_j on the latter are twice the associated characters of $E_{\mathbb{C}}$.

(5.3) Collecting the traces of the intersection matrices $P_1, \ldots, P_{27}$ on the various generalized eigenspaces yields the character table $\Phi = [\varphi_1, \ldots, \varphi_{18}] \in \mathbb{C}^{18\times 27}$ of $E_{\mathbb{C}}$, see Table 5. We also indicate the pairing of H-orbits, Galois conjugate characters, the Fitting correspondence, and the degree of the Fitting

Table 5: The character table of $E_{\mathbb{C}}$.

φ	φ^*	χ_φ	$\chi_\varphi(1)$	1	2	3	4	5	6	$8^*=7$	$7^*=8$	9	10
1		1	1	1	31	930	17360	26040	27776	416640	416640	624960	3333120
2		8	889111	1	16	225	1640	2670	1856	13920	13920	21240	41280
3		11	1776888	1	−12	113	1192	−1222	1632	−10608	−10608	6792	36912
4		14	4290927	1	−2	−27	112	168	672	−1008	−1008	−588	1792
5	6	19	354111145	2	$8+r_{33}$	$141-10r_{33}$	$644-40r_{33}$	$519+31r_{33}$	$720+8r_{33}$	$264+152r_{33}$	$264+152r_{33}$	$4422-318r_{33}$	$2976-544r_{33}$
6	5	20	354111145	2	$8-r_{33}$	$141+10r_{33}$	$644+40r_{33}$	$519-31r_{33}$	$720-8r_{33}$	$264-152r_{33}$	$264-152r_{33}$	$4422+318r_{33}$	$2976+544r_{33}$
7		21	95288172	2	−3	75	424	−570	720	0	0	−264	7552
8		22	230279749	1	9	50	−20	80	−120	−480	−480	−720	−320
9	10	23	259775040	1	$3-4r_3$	$26-24r_3$	$-50-36r_3$	$224-8r_3$	$-36+88r_3$	$-564+160r_3$	$-564+160r_3$	$-108-16r_3$	$-872+880r_3$
10	9	24	259775040	1	$3+4r_3$	$26+24r_3$	$-50+36r_3$	$224+8r_3$	$-36-88r_3$	$-564-160r_3$	$-564-160r_3$	$-108+16r_3$	$-872-880r_3$
11		29	46055498	1		1	−30	284	0	−256	0	0	−864	1152
12		30	493456605	1		1	−30	196	0	−168	0	0	−336	448
13		32	786127419	1		5	−6	−120	−168	96	192	192	384	−256
14	15	36	885257856	1	$-6+r_5$	$10-12r_5$	$-55-27r_5$	$40+76r_5$	$10+50r_5$	$90-170r_5$	$90-170r_5$	$110-70r_5$	$-640+520r_5$
15	14	37	885257856	1	$-6-r_5$	$10+12r_5$	$-55+27r_5$	$40-76r_5$	$10-50r_5$	$90+170r_5$	$90+170r_5$	$110+70r_5$	$-640-520r_5$
16	17	38/39	1016407168	1	3	14	$-65-9r_5$	$-28-48r_5$	$6+6r_5$	$42+6r_5$	$42+6r_5$	$-666+66r_5$	$904-48r_5$
17	16	39/38	1016407168	1	−3	14	$-65+9r_5$	$-28+48r_5$	$6-6r_5$	$42-6r_5$	$42-6r_5$	$-666-66r_5$	$904+48r_5$
18		51	1842237992	1	1	−30	20	0	8	0	0	720	−960

φ	11	$13^*=12$	$12^*=13$	14	15	16	17	$19^*=18$	$18^*=19$
1	4999680	6666240	6666240	9999360	13332480	53329920	66060288	79994880	79994880
2	77760	49920	49920	72000	96000	61440	−73728	230400	230400
3	21312	28416	28416	−23424	−50496	29184	12288	−137856	−137856
4	−19488	−1344	−1344	23856	66304	146944	−172032	−45696	−45696
5	$-3456-1328r_{33}$	$-6720-704r_{33}$	$-6720-704r_{33}$	$1752+1256r_{33}$	$-384-192r_{33}$	$14208+1792r_{33}$	$26112+4608r_{33}$	$-18816-640r_{33}$	$-18816-640r_{33}$
6	$-3456+1328r_{33}$	$-6720+704r_{33}$	$-6720+704r_{33}$	$1752-1256r_{33}$	$-384+192r_{33}$	$14208-1792r_{33}$	$26112-4608r_{33}$	$-18816+640r_{33}$	$-18816+640r_{33}$
7	−7680	−2496	−2496	−5376	−3200	7168	−7680	10752	10752
8	3920	−4160	−4160	−8000	9280	1920	−3072	−640	−640
9	$-2644+1024r_3$	$1312-64r_3$	$1312-64r_3$	$736-1376r_3$	$136+2224r_3$	$4704+3392r_3$	$9600-8960r_3$	$224+64r_3$	$224+64r_3$
10	$-2644-1024r_3$	$1312+64r_3$	$1312+64r_3$	$736+1376r_3$	$136-2224r_3$	$4704-3392r_3$	$9600-8960r_3$	$224-64r_3$	$224-64r_3$
11	2400	−1536	−1536	−1536	−1920	768	0	3072	3072
12	−2352	2688	2688	2688	5824	−6272	0	−5376	−5376
13	1728	−768	−768	2304	256	−6144	−6144	−6144	−6144
14	$-690+146r_5$	$320+112r_5$	$320+112r_5$	$-2040-280r_5$	$1040-8r_5$	$-3760-1440r_5$	$-3296-1440r_5$	$3240+328r_5$	$3240+328r_5$
15	$-690-146r_5$	$320-112r_5$	$320-112r_5$	$-2040+280r_5$	$1040+8r_5$	$-3760+1440r_5$	$-3296+1440r_5$	$3240-328r_5$	$3240-328r_5$
16	$878-318r_5$	$592+480r_5$	$592+480r_5$	$1864-696r_5$	$-1304+528r_5$	$1776-384r_5$	$2976+1632r_5$	$-424+2040r_5$	$-424+2040r_5$
17	$878+318r_5$	$592-480r_5$	$592-480r_5$	$1864+696r_5$	$-1304-528r_5$	$1776+384r_5$	$2976-1632r_5$	$-424-2040r_5$	$-424+2040r_5$
18	−240	−480	−480	−480	−1920	2880	0	960	960

φ	20	21	22	23	24	25	26	27
1	159989760	159989760	319979520	341311488	1279918080	1279918080	2047868928	2559836160
2	276480	138240	460800	12288	0	46080	−1179648	−645120
3	−341760	186624	439296	−288768	172032	766464	−147456	−580608
4	26880	−32256	−182784	−258048	−258048	451584	344064	−43008
5	$-1248+4832r_{33}$	$32928+3680r_{33}$	$-43392-2944r_{33}$	$86784+5376r_{33}$	$-60672+46336r_{33}$	$-108288+36608r_{33}$	$247296-19968r_{33}$	$-150528-76800r_{33}$
6	$-1248-4832r_{33}$	$32928-3680r_{33}$	$-43392+2944r_{33}$	$86784-5376r_{33}$	$-60672-46336r_{33}$	$-108288-36608r_{33}$	$247296+19968r_{33}$	$-150528+76800r_{33}$
7	12480	−29376	−39168	−39936	218112	−91392	−192000	153600
8	−4800	−18880	25600	17920	17920	17920	−27648	−20480
9	$5856-7616r_3$	$10880-3744r_3$	$-2048-12256r_3$	$1408+4992r_3$	$-32192-12288r_3$	$17920+13376r_3$	$-78720-11520r_3$	$63232+13632r_3$
10	$5856+7616r_3$	$10880+3744r_3$	$-2048+12256r_3$	$1408-4992r_3$	$-32192+12288r_3$	$17920-13376r_3$	$-78720+11520r_3$	$63232-13632r_3$
11	7680	21504	−19968	−12288	−12288	12288	0	0
12	−13440	−8064	5376	21504	21504	−21504	0	0
13	−2304	11520	−12288	12288	0	12288	−30720	30720
14	$-4720+1696r_5$	$8000+320r_5$	$-1320-2136r_5$	$-3520+256r_5$	$6160+5008r_5$	$320-6336r_5$	$12576+5344r_5$	$-15520-2208r_5$
15	$-4720-1696r_5$	$8000-320r_5$	$-1320+2136r_5$	$-3520-256r_5$	$6160-5008r_5$	$320+6336r_5$	$12576-5344r_5$	$-15520+2208r_5$
16	$2976-1200r_5$	$-6640+1008r_5$	$1192+2760r_5$	$-3392-3456r_5$	$-5840+2448r_5$	$-6272+768r_5$	$24480+8928r_5$	$-13280-6624r_5$
17	$2976+1200r_5$	$-6640-1008r_5$	$1192-2760r_5$	$-3392+3456r_5$	$-5840-2448r_5$	$-6272-768r_5$	$24480-8928r_5$	$-13280+6624r_5$
18	2400	−9120	9600	−3840	−3840	3840	0	0

correspondents. The rows are ordered such that the Fitting correspondents appear as in Table 1, except that we only know that $\varphi_{16/17}$ correspond to $\chi_{38/39}$.

Indeed, the character fields of the φ_j coincide with the character fields of their respective Fitting correspondents. Since the quadratic fields $\mathbb{Q}(r_3)$, $\mathbb{Q}(r_5)$ and $\mathbb{Q}(r_{33})$ are disjoint, it follows that there are Galois automorphisms inducing the involutions $(\chi_{19} \longleftrightarrow \chi_{20})$ and $(\chi_{23} \longleftrightarrow \chi_{24})$ and $(\chi_{36} \longleftrightarrow \chi_{37})(\chi_{38} \longleftrightarrow \chi_{39})$. A consideration of the ordinary character table of G shows that actually there is no table automorphism interchanging $\chi_{36/37}$ and fixing $\chi_{38/39}$. This says that by considering ordinary character theory alone the Fitting correspondence is only determined up to the ambiguity above.

We define $\varphi_{14/15/16,17}$ by letting $\varphi_{14}(P_2) = -6 + r_5$ and $\varphi_{15}(P_2) = -6 - r_5$, and $\varphi_{16}(P_2) = 3r_5$ and $\varphi_{17}(P_2) = -3r_5$, being the roots of $X^2 + 12X + 31$ and $X^2 - 45$, respectively, see Table 4. Then, choosing $\chi_{\varphi_{14}} := \chi_{36}$ and $\chi_{\varphi_{15}} := \chi_{37}$, it remains to decide by further inspection whether $\chi_{\varphi_{16}} = \chi_{38}$ or $\chi_{\varphi_{16}} = \chi_{39}$.

6 Decomposition numbers

(6.1) Let now $\mathbb{F} := \mathbb{F}_{11}$. Having the regular representation of $E_{\mathbb{F}}$ at hand, we apply the MeatAxe to compute the simple $E_{\mathbb{F}}$-modules S and their multiplicities $[E_{\mathbb{F}} \colon S]$ in the regular $E_{\mathbb{F}}$-module. We get

$$(1a)^{10}, \quad (1b)^8, \quad (1c)^6, \quad (1d)^3,$$

saying that all constituents are 1-dimensional, and where exponents denote their multiplicity. In particular, $\mathbb{F}$ already is a splitting field for $E_{\mathbb{F}}$.

Let P_i, where $i \in \{a, b, c, d\}$, be the projective indecomposable $E_{\mathbb{F}}$-module associated with the simple module $S_i := (1i)$, that is, such that $P_i/\mathrm{rad}(P_i) \cong S_i$. Since H is an $11'$-group, we have $11 \nmid \prod_{j=1}^{27} n_j$, implying that $E_{\mathbb{F}}$ is a symmetric algebra, thus having a symmetric Cartan matrix and $\dim_{\mathbb{F}}(P_i) = [E_{\mathbb{F}} \colon S_i]$.

Using the MeatAxe again to compute the indecomposable direct summands of the regular module, we find the following Cartan matrix of $E_{\mathbb{F}}$, giving the multiplicities $[P_i \colon S_j]$, both rows and columns being parameterized by $\{a, b, c, d\}$:

$$\begin{bmatrix} 7 & 3 & . & . \\ 3 & 5 & . & . \\ . & . & 6 & . \\ . & . & . & 3 \end{bmatrix}.$$

This shows that $E_{\mathbb{F}}$ has three blocks, given by $\{S_a, S_b\} \dot\cup \{S_c\} \dot\cup \{S_d\}$.

Let $e_i \in E_{\mathbb{F}}$ be a primitive idempotent associated with S_i, where we may assume that the e_i are pairwise orthogonal. Then we have $P_i \cong e_i E_{\mathbb{F}}$ as $E_{\mathbb{F}}$-modules, and $E_{\mathbb{F}} = (e_a E_{\mathbb{F}} \oplus e_b E_{\mathbb{F}}) \oplus e_c E_{\mathbb{F}} \oplus e_d E_{\mathbb{F}}$, brackets indicating blocks.

Hence, applying the primitive idempotents $e_i \in E_{\mathbb{F}}$ to the permutation module $\mathbb{F}_H^G$ we get the direct sum decomposition $\mathbb{F}_H^G = \mathbb{F}_H^G e_a \oplus \mathbb{F}_H^G e_b \oplus \mathbb{F}_H^G e_c \oplus \mathbb{F}_H^G e_d$

Table 6: The character table of $E_{\mathbb{F}}$.

$(\varphi_j)_{\mathbb{F}}$	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	1	9	6	2	3	1	4	4	6	10	4	9	9	8
3	1	10	3	4	10	4	7	7	5	7	5	3	3	6
9	1	1	3	9	0	8	0	0	5	8	2	4	4	4
2	1	5	5	1	8	8	5	5	10	8	1	2	2	5

$(\varphi_j)_{\mathbb{F}}$	15	16	17	18	19	20	21	22	23	24	25	26	27
1	7	6	8	9	9	7	7	3	1	1	1	6	2
3	5	1	1	7	7	10	9	0	4	3	6	10	5
9	5	9	0	3	3	2	10	8	10	10	1	0	0
2	3	5	5	5	5	6	3	10	1	0	1	3	8

into indecomposable, pairwise non-isomorphic $\mathbb{F}[G]$-modules. In particular, this says that $P_{\mathbb{F}_G}$ is *not* a permutation module.

(6.2) Although this already answers the question from the beginning, we can do better, and try and determine the projective indecomposable characters of G being contained in 1_H^G, in particular encompassing $\Psi_{\mathbb{F}_G}$. To this end, we determine the 11-modular decomposition matrix of E:

Using GAP, we compute the 11-modular reduction $\Phi_{\mathbb{F}} := [(\varphi_1)_{\mathbb{F}}, \ldots, (\varphi_{18})_{\mathbb{F}}] \in \mathbb{F}^{18\times 27}$ of the character table of $E_{\mathbb{C}}$. It turns out that $\{(\varphi_1)_{\mathbb{F}}, (\varphi_3)_{\mathbb{F}}, (\varphi_9)_{\mathbb{F}}, (\varphi_2)_{\mathbb{F}}\}$ are pairwise distinct and $\mathbb{F}$-linearly independent, all having degree 1, see Table 6. (Hence $\{\varphi_1, \varphi_3, \varphi_9, \varphi_2\}$ is a 'Basic Set' in the sense of [6, Def.3.1.1].)

Then the relations between the rows of $\Phi_{\mathbb{F}}$, together with the fact the character degrees involved are at most 2, yield the complete 11-modular decomposition matrix of E. In particular, this shows that the blocks of characters consist of

$$\{\varphi_1, \varphi_3, \varphi_4, \varphi_5, \varphi_6, \varphi_7, \varphi_8, \varphi_{14}, \varphi_{17}\}, \{\varphi_9, \varphi_{11}, \varphi_{12}, \varphi_{15}, \varphi_{16}, \varphi_{18}\}, \{\varphi_2, \varphi_{10}, \varphi_{13}\}.$$

The blocks of the 11-modular decomposition matrix of E are given in Table 7, where we also repeat the Fitting correspondence from Table 5 and the multiplicities m_i from Table 1.

Comparing the dimension of the projective indecomposable $E_{\mathbb{F}}$-modules and the multiplicity of the simple $E_{\mathbb{F}}$-modules as constituents of the regular module yields the following correspondence:

$$S_a \longleftrightarrow (\varphi_1)_{\mathbb{F}}, \quad S_b \longleftrightarrow (\varphi_3)_{\mathbb{F}}, \quad S_c \longleftrightarrow (\varphi_9)_{\mathbb{F}}, \quad S_d \longleftrightarrow (\varphi_2)_{\mathbb{F}}.$$

(6.3) Thus, by Fitting correspondence, we have determined four columns of the 11-modular decomposition matrix of G, up to the ambiguity for the Fitting

Table 7: The 11-modular decomposition matrix of E.

φ_j	χ_i	m_i	$(\varphi_1)_{\mathbb{F}}$	$(\varphi_3)_{\mathbb{F}}$
1	1	1	1	.
3	11	1	.	1
4	14	1	1	.
5	19	2	1	1
6	20	2	1	1
7	21	2	1	1
8	22	1	1	.
14	36	1	1	.
17	38/39	1	.	1

φ_j	χ_i	m_i	$(\varphi_9)_{\mathbb{F}}$
9	23	1	1
11	29	1	1
12	30	1	1
15	37	1	1
16	39/38	1	1
18	51	1	1

φ_j	χ_i	m_i	$(\varphi_2)_{\mathbb{F}}$
2	8	1	1
10	24	1	1
13	32	1	1

correspondents of $\varphi_{16/17}$, having fixed those of $\varphi_{5/6}$, $\varphi_{9/10}$ and $\varphi_{14/15}$; see Table 8, where $a \in \{0, 1\}$. (Actually, it is possible to decide which of the two cases left actually occurs, again by a computational attack similar to the one described here; details about this will appear elsewhere [13].)

In view of the remarks in (3.3), we observe that the projective indecomposable character $\Psi_{\mathbb{F}_G}$ associated with the trivial character, which corresponds to the projective indecomposable $E_{\mathbb{F}}$-module P_a, is a non-rational character indeed.

Finally, the results of this article constitute the first steps towards the ambitious goal of finding the complete 11-modular decomposition matrix of $G = J_4$. Richard Parker would have been keen to pursue this!

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Table 8: The permutation character on the cosets of H in G again.

χ_i	χ_i^*	$\Psi_{\mathbb{F}_G}$			
1		1	.	.	.
8		.	1	.	.
11		.	.	1	.
14		1	.	.	.
19	20	1	.	1	.
20	19	1	.	1	.
21		1	.	1	.
22		1	.	.	.
23	24	.	.	.	1
24	23	.	1	.	.
29		.	.	.	1
30		.	.	.	1
32		.	1	.	.
36	37	1	.	.	.
37	36	.	.	.	1
38	39	.	.	a	$1-a$
39	38	.	.	$1-a$	a
51		.	.	.	1

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