The projective cover of the trivial module in characteristic 11 for the sporadic simple Janko group J_4 revisited

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Abstract

This is a sequel to [9], where we have determined the 11-modular projective indecomposable summands of the permutation character of J_4 on the cosets of an $11'$-subgroup of maximal order, amongst them the projective cover of the trivial module, up to a certain parameter. Here, we fix this parameter, by applying a new condensation method for induced modules which uses enumeration techniques for long orbits.

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1 Introduction

(1.1) The present article is a sequel to [9], which is devoted to answering (to the negative) the question (posed in [8]) whether the projective cover of the trivial module in characteristic 11 for the largest sporadic simple Janko group $G := J_4$ is a permutation module. Actually, this question boils down to determine whether or not the projective permutation character 1_H^G, where $H < G$ is a maximal ($11'$-)subgroup of shape $H \cong 2^{10}\colon L_5(2)$, is projective indecomposable.

According to the ModularAtlasHomepage, virtually nothing is known about the decomposition numbers of G is characteristic 11. Thus the strategy in [9] was to find the decomposition of 1_H^G into projective indecomposable characters. It turned out that there are four distinct indecomposable summands, which are reproduced in Table 1 (on page 8). This can be seen as the first step towards the ambitious goal of determining the 11-modular decomposition matrix of G, which is particularly compelling as G has trivial-intersection Sylow 11-subgroups.

Alone, in [9], we have not been able to fix the parameter $a \in \{0, 1\}$ appearing in Table 1. But this should be done before proceeding to find more decomposition numbers of G. Hence the purpose of the present article is to close this gap, by showing that we actually have $a = 0$, obeying to the conventional choices for decomposition maps made in the ModularAtlas.

To this end, we invoke a maximal subgroup $U < G$ of shape $U \cong U_3(11)\colon 2$, whose 11-modular decomposition matrix is well-known. In order to relate the decomposition matrices of U and G, it turns out that subtle details of the embedding of U into G play a crucial role here. These can be captured by a consideration of the automorphism group of the ordinary character table of G.

Having this in place, letting $\mathcal{O}$ be a G-set affording the permutation character 1_H^G, and letting $\mathbb{F}$ be a field of characteristic 11, we examine how the restriction of the permutation module $\mathbb{F}[\mathcal{O}]$ to U decomposes into projective indecomposable modules. To do so, we are finally led to consider the action of $\mathrm{End}_{\mathbb{F}[G]}(\mathbb{F}[\mathcal{O}])$ on $\mathrm{Hom}_{\mathbb{F}[U]}(\mathbb{F}[\mathcal{O}]|_U, V) \cong \mathrm{Hom}_{\mathbb{F}[G]}(\mathbb{F}[\mathcal{O}], V^G)$, for certain simple $\mathbb{F}[U]$-modules V.

The latter step essentially amounts to computing the 'condensed module' afforded by the module V^G induced from the subgroup U, with respect to the 'condensation subgroup' H. (As a general reference for 'condensation', see for example [10].) A technique to compute condensed modules of induced modules, for subgroups U of smallish index in G, has been developed in [13]. The present approach, combining these ideas with the orbit enumeration techniques available in the GAP package ORB, now allows both subgroups U and H to have large index in G. We expect this to be of independent interest.

(1.2) The present article is organized as follows: In the rest of Section 1 we indicate the computational tools we are using, and we sketch the ideas behind the orbit enumeration techniques available in ORB. In Section 2 we present the piece of theory underlying the new condensation technique advertised above. In Section 3 we collect some character theoretic facts on G and various of its subgroups, and we consider table automorphisms, to clarify where and where not choices can be made. In Section 4 we enumerate $\mathcal{O}$ by U-orbits, by applying ORB. In Section 5, using an idea inspired by [6], we examine certain condensed induced modules, in order to finally determine the missing parameter.

(1.3) Computational tools. To facilitate group theoretic and character theoretic computations we use the computer algebra system GAP [3], and its comprehensive database CTblLib [1] of ordinary and Brauer character tables. In particular, [1] encompasses the data given in the Atlas [2] and the ModularAtlas [4], as well as the additional data collected on the ModularAtlasHomepage [17]. Data concerning explicit permutation representations, ordinary and modular matrix representations, and the embedding of (maximal) subgroups of sporadic simple groups is available in the AtlasOfGroupRepresentations [18], and through the GAP package AtlasRep [19]. To compute with matrix representations over finite fields we use the MeatAxe [14, 15] and its extensions described in [6, 7].

(1.4) Enumerating long orbits. As our computational workhorse to facilitate computations with (large) permutation representations we use the GAP package ORB [11], whose orbit enumeration techniques are comprehensively described in [10, 12]. For convenience, we give a brief sketch of the approach:

Let G be a (large) finite group, and let $\mathcal{O}$ be a (large) transitive G-set, which we assume to be implicitly given, for example as a G-orbit of a vector v_1 in an $F[G]$-module V over a finite field F. Letting $H \leq G$ be a (still large) subgroup, we are interested in finding the H-orbits $\mathcal{O}_j \subseteq \mathcal{O}$, their length n_j, representatives $v_j \in \mathcal{O}_j$, elements $g_j \in G$ such that $v_1 \cdot g_j = v_j$, and the point stabilizers $H_j = \mathrm{Stab}_H(v_j)$. To achieve this, we assume to be able to compute efficiently within H (but not within G), for example by having a (smallish) faithful permutation representation of H at hand.

To find the H-orbits $\mathcal{O}_j$, we choose a (smallish) helper subgroup $K \leq H$, and enumerate the various $\mathcal{O}_j$ by the K-orbits they contain. To do so, we choose a (not too small) helper K-set $\mathcal{Q}$ together with a homomorphism $\pi_K \colon \mathcal{O} \to \mathcal{Q}$ of K-sets, which again we assume to be implicitly given, for example by an $F[K]$-quotient module of V. We assume that K has sufficiently long orbits in $\mathcal{Q}$, and that we are able to classify them, by giving representatives, their point stabilizers in K, as well as complete Schreier trees. Thus for the K-action on $\mathcal{Q}$ we are facing a similar problem as for the H-action on $\mathcal{O}$, apart from the requirement on Schreier trees; so we can just recurse.

For any K-orbit in $\mathcal{Q}$, we choose a representative, called its 'distinguished point'. Then, for any K-orbit $\mathcal{O}' \subseteq \mathcal{O}$, the π_K-preimages of the distinguished point of $\pi_K(\mathcal{O}') \subseteq \mathcal{Q}$ are likewise called the distinguished points of $\mathcal{O}'$. Hence to enumerate an H-orbit $\mathcal{O}_j$ by enumerating the K-orbits it contains, we only have to store the associated distinguished points, and a Schreier tree telling us how to reach them from the orbit representative v_j.

For any $\mathcal{O}_j$ we are content with finding only as many K-orbits contained in it which are needed to cover (more than) half of it; this is equivalent to knowing n_j and $|H_j|$. Then we have a randomized membership test for $\mathcal{O}_j$, and a deterministic test to decide whether the $\mathcal{O}_j$ found are actually pairwise disjoint.

2 Condensing induced modules

(2.1) Endomorphisms of permutation modules. We recall some facts about the structure of endomorphism algebras of permutation modules, thereby fixing the notation used in the sequel; as a general reference, see [5, Ch.II.12].

Let G be a finite group, let $H \leq G$ be a subgroup, let $\mathcal{O}$ be a transitive G-set with associated point stabilizer H, that is there is $v_1 \in \mathcal{O}$ such that $\mathrm{Stab}_G(v_1) = H$, and let $n := |\mathcal{O}| = [G \colon H]$.

If R is a principal ideal domain, let $R[\mathcal{O}]$ be the associated permutation $R[G]$-lattice. For subgroups $L \leq M \leq G$ let $\mathrm{Fix}_{R[\mathcal{O}]}(M) := \{v \in R[\mathcal{O}]; vg = v \text{ for all } g \in M\} \leq R[\mathcal{O}]$ be the R-sublattice of M-fixed points, and let

$$\mathrm{Tr}_L^M \colon \mathrm{Fix}_{R[\mathcal{O}]}(L) \to \mathrm{Fix}_{R[\mathcal{O}]}(M) \colon x \mapsto x \cdot \sum_{g \in L \backslash M} xg$$

be the associated trace operator, where g runs through a set of representatives of the cosets of L in M. For $L = \{1\}$ we just write $\mathrm{Tr}^M := \mathrm{Tr}_{\{1\}}^M$.

Let $E_R := \mathrm{End}_{R[G]}(R[\mathcal{O}])$ be the R-algebra of $R[G]$-endomorphisms of $R[\mathcal{O}]$. Then E_R is R-free of rank $r = |H\backslash G/H|$, that is the number of double cosets of H in G. In other words, we have $r = \langle 1_H^G, 1_H^G\rangle_G$, where 1_H^G is the permutation character afforded by $\mathcal{O}$, and $\langle\cdot,\cdot\rangle_G$ denotes the usual scalar product on the complex class functions on G. More precisely:

Let $\{v_1,\ldots,v_r\}$ be a set of representatives of the H-orbits $\mathcal{O}_i := (v_i)^H \subseteq \mathcal{O}$, where v_1 is as specified above, let $g_i \in G$ such that $v_1 g_i = v_i$, let $H_i := \mathrm{Stab}_H(v_i) = H^{g_i} \cap H$, and let $n_i := |\mathcal{O}_i| = [H\colon H_i]$. Then E_R has a distinguished R-basis $\{A_1,\ldots,A_r\}$, being called its Schur basis, where A_i is given by

$$A_i\colon v_1 \mapsto \mathcal{O}_i^+ := \sum_{v\in\mathcal{O}_i} v = v_i \cdot \mathrm{Tr}_{H_i}^H,$$

and extension to all of $\mathcal{O}$ by G-transitivity.

(2.2) Restriction to subgroups. Keeping the above notation, let $U \leq G$ be a subgroup. Then we may consider $\mathcal{O}$ as an intransitive U-set:

Let $\{\omega_1,\ldots,\omega_s\}$ be a set of representatives of the U-orbits $\Omega_j := (\omega_j)^U \subseteq \mathcal{O}$, where $\omega_1 := v_1$ and $s = |H\backslash G/U|$, let $\gamma_j \in G$ such that $v_1\gamma_j = \omega_j$, and let $U_j := \mathrm{Stab}_U(\omega_j) = H^{\gamma_j} \cap U$; then we have $|\Omega_j| = [U\colon U_j]$.

We get a direct sum decomposition $R[\mathcal{O}]|_U = \bigoplus_{j=1}^s R[\Omega_j]$ into transitive permutation $R[U]$-lattices, and a corresponding decomposition

$$\mathcal{E}_R := \mathrm{End}_{R[U]}(R[\mathcal{O}]|_U) = \bigoplus_{j=1}^s \bigoplus_{k=1}^s \mathrm{Hom}_{R[U]}(R[\Omega_j], R[\Omega_k]).$$

We abbreviate $\mathcal{E}_{jk,R} := \mathrm{Hom}_{R[U]}(R[\Omega_j], R[\Omega_k])$. Then $\mathcal{E}_{jk,R}$ has a distinguished R-basis $\{\mathcal{A}_{jk,1},\ldots,\mathcal{A}_{jk,t}\}$, again called its Schur basis, given as follows:

Let $\{u_{jk,1},\ldots,u_{jk,t}\} \subseteq U$ be a set of representatives of the double cosets $U_k\backslash U/U_j$, where $t := |U_k\backslash U/U_j|$, and let $\omega_{jkl} := \omega_k \cdot u_{jkl} \in \Omega_k$, for $l \in \{1,\ldots,t\}$. Then the U_j-orbits in Ω_k are given as $\Omega_{jkl} := (\omega_{jkl})^{U_j}$, where

$$\mathrm{Stab}_U(\omega_{jkl}) = H^{\gamma_k u_{jkl}} \cap U = H^{\gamma_k u_{jkl}} \cap U^{u_{jkl}} = (H^{\gamma_k} \cap U)^{u_{jkl}} = U_k^{u_{jkl}},$$

implying that $|\Omega_{jkl}| = \frac{|U_j|}{|U_k^{u_{jkl}} \cap U_j|}$. Then $\mathcal{A}_{jkl} \in \mathcal{E}_{jk,R}$ is given by

$$\mathcal{A}_{jkl}\colon \omega_j \mapsto \Omega_{jkl}^+ = \omega_{jkl} \cdot \mathrm{Tr}_{U_k^{u_{jkl}} \cap U_j}^{U_j} = \omega_k \cdot u_{jkl} \cdot \mathrm{Tr}_{U_k^{u_{jkl}} \cap U_j}^{U_j},$$

and extension to all of Ω_j by U_j-transitivity. The action of $u_{jkl} \cdot \mathrm{Tr}_{U_k^{u_{jkl}} \cap U_j}^{U_j}$ only depends on the parameters j,k,l, but not on the particular choice of the u_{jkl}.

(2.3) Embedding endomorphisms. Next, we describe the embedding of E_R into $\mathcal{E}_R$, in terms of their Schur bases:

For $i \in \{1, \ldots, r\}$ and $j \in \{1, \ldots, s\}$ we have

$$\omega_j \cdot A_i = v_1\gamma_j \cdot A_i = v_1 \cdot A_i \cdot \gamma_j = v_i \cdot \mathrm{Tr}_{H_i}^H \cdot \gamma_j = \mathcal{O}_i^+ \cdot \gamma_j.$$

Hence, for $k \in \{1, \ldots, s\}$ and $l \in \{1, \ldots, t\}$, where $t = |U_k \backslash U/U_j|$, let the U**-orbit counting numbers** be defined as

$$c_{jkl}(i) := |\mathcal{O}_i\gamma_j \cap \Omega_{jkl}| \quad \text{and} \quad c_{jk}(i) := |\mathcal{O}_i\gamma_j \cap \Omega_k| = \sum_{l=1}^{t} c_{jkl}(i).$$

Recall that $U_j = H^{\gamma_j} \cap U$, so that $\mathcal{O}_i\gamma_j$ is U_j-stable, thus so is $\mathcal{O}_i\gamma_j \cap \Omega_{jkl}$, implying that Ω_{jkl} is either disjoint from $\mathcal{O}_i\gamma_j$, or contained in it, so that either $c_{jkl}(i) = 0$ or $c_{jkl}(i) = |\Omega_{jkl}|$. Thus we have

$$A_i = \sum_{j=1}^{s}\sum_{k=1}^{s}\sum_{l=1}^{|U_k\backslash U/U_j|} \frac{c_{jkl}(i)}{|\Omega_{jkl}|} \cdot \mathcal{A}_{jkl} \in \mathcal{E}_R,$$

where the coefficients are in $\{0, 1\}$, saying that A_i splits into a sum of certain pairwise distinct Schur basis elements of $\mathcal{E}_R$. This is illustrated by the following generic examples:

i) Let $U = G$; thus $\mathcal{E}_R = E_R$. We have $s = 1$ and $\Omega_1 = \mathcal{O}$, so that $\Omega_{1,1,l} = \mathcal{O}_l$ for $l \in \{1, \ldots, r\}$, where $|U_1\backslash U/U_1| = |H\backslash G/H| = r$. This yields $c_{1,1,l}(i) = \delta_{i,l} \cdot |\mathcal{O}_l|$ for $i \in \{1, \ldots, r\}$, and the above triple sum boils down to the tautology $A_i = A_i$.

ii) Let $U = \{1\}$; thus $\mathcal{E}_R = \mathrm{End}_R(R[\mathcal{O}]) \cong R^{n\times n}$. We have $s = n$, and $\Omega_k = \{v_1\gamma_k\}$ is a singleton set, and $|U_k\backslash U/U_j| = 1$ for $j, k \in \{1, \ldots, n\}$. This yields $c_{j,k,1}(i) = 1$ if $v_1\gamma_k \in \mathcal{O}_i\gamma_j$, and $c_{j,k,1}(i) = 0$ otherwise, for $i \in \{1, \ldots, r\}$. Hence, identifying $\mathcal{A}_{j,k,1}$ with $E_{jk} \in R^{n\times n}$, having entry 1 at position $[j, k]$, and zero entries otherwise, we recover the natural representation of E_R on $R[\mathcal{O}]$.

iii) Let $U = H$. We have $s = r$, and $\Omega_j = \mathcal{O}_j$ for $j \in \{1, \ldots, r\}$. Thus we get the (H-)orbit counting numbers $c_{jk}(i) = |\mathcal{O}_i g_j \cap \mathcal{O}_k|$. These are related to the regular representation of E_R with respect to the Schur basis as follows:

For $i, j \in \{1, \ldots, r\}$ we write $A_jA_i = \sum_{k=1}^{r} p_{jk}(i) \cdot A_k$, the associated structure constants $p_{jk}(i) \in \mathbb{N}_0$ being called intersection numbers. Then we have

$$v_1 \cdot A_jA_i = v_1 g_j \cdot \mathrm{Tr}_{H_j}^H \cdot A_i = v_1 \cdot A_i \cdot g_j \cdot \mathrm{Tr}_{H_j}^H = \mathcal{O}_i^+ \cdot g_j \cdot \mathrm{Tr}_{H_j}^H.$$

From $v_1 \cdot A_k = \mathcal{O}_k^+$, and $|\mathcal{O}_i g_j h \cap \mathcal{O}_k| = |\mathcal{O}_i g_j \cap \mathcal{O}_k| = c_{jk}(i)$, for $h \in H$, we get

$$p_{jk}(i) = \sum_{h\in H_j\backslash H} \frac{|\mathcal{O}_i g_j h \cap \mathcal{O}_k|}{|\mathcal{O}_k|} = \frac{n_j}{n_k} \cdot c_{jk}(i).$$

(2.4) Condensing $\mathcal{E}$. a) Let V be an $R[U]$-lattice. Then $\mathcal{E}_R$ acts naturally (from the left) on $\mathcal{H}(V) := \mathrm{Hom}_{R[U]}(R[\mathcal{O}]|_U, V)$, by

$$\alpha \colon \mathcal{H}(V) \to \mathcal{H}(V) \colon \varphi \mapsto \alpha \cdot \varphi, \quad \text{for } \alpha \in \mathcal{E}_R.$$

Note that we have $\mathcal{H}(V) \cong \mathrm{Hom}_{R[G]}(R[\mathcal{O}], V^G) = \mathrm{Hom}_{R[G]}(R_H^G, V^G)$, where R_H denotes the trivial $R[H]$-module, and superscripts denote induction.

The direct sum decomposition $R[\mathcal{O}]|_U = \bigoplus_{j=1}^s R[\Omega_j]$ entails a decomposition

$$\mathcal{H}(V) = \bigoplus_{j=1}^s \mathrm{Hom}_{R[U]}(R[\Omega_j], V) \cong \bigoplus_{j=1}^s \mathrm{Fix}_V(U_j);$$

we write $\varphi = \sum_{j=1}^s \varphi_j$. The latter isomorphism of R-lattices is given componentwise by $\mathrm{Hom}_{R[U]}(R[\Omega_j], V) \to \mathrm{Fix}_V(U_j) \colon \varphi_j \mapsto \omega_j \varphi_j$.

For the Schur basis element $\mathcal{A}_{jkl} \in \mathcal{E}_{jk,R}$, letting $\varphi_i \in \mathrm{Hom}_{R[U]}(R[\Omega_i], V)$, we have $\mathcal{A}_{jkl} \cdot \varphi_i = 0$ whenever $i \neq k$. If $i = k$, then we get

$$\omega_j \cdot \mathcal{A}_{jkl}\varphi_k = \omega_k u_{jkl} \cdot \mathrm{Tr}_{U_k^{u_{jkl}} \cap U_j}^{U_j} \cdot \varphi_k = \omega_k \varphi_k \cdot u_{jkl} \cdot \mathrm{Tr}_{U_k^{u_{jkl}} \cap U_j}^{U_j} \in \mathrm{Fix}_V(U_j),$$

where indeed $\omega_k \varphi_k \in \mathrm{Fix}_V(U_k)$, hence $\omega_k \varphi_k \cdot u_{jkl} \in \mathrm{Fix}_V(U_k^{u_{jkl}})$. Thus, in terms of fixed spaces, $\mathcal{A}_{jkl}$ annihilates $\mathrm{Fix}_V(U_i)$ for $i \neq k$, and for $i = k$ we get

$$\mathcal{A}_{jkl} \colon \mathrm{Fix}_V(U_k) \to \mathrm{Fix}_V(U_j) \colon v \mapsto v \cdot u_{jkl} \cdot \mathrm{Tr}_{U_k^{u_{jkl}} \cap U_j}^{U_j}.$$

b) In view of the application envisaged here, let $R[\mathcal{O}]$ be a projective $R[G]$-module, which is equivalent to $|H|$ being a unit in R. Then let

$$e_H := \frac{1}{|H|} \cdot \mathrm{Tr}^H \in R[H]$$

be the associated 'fixed-point' idempotent, that is the primitive idempotent of $R[H]$ associated with the trivial representation of H; recall that e_H projects any $R[H]$-lattice onto its R-sublattice of H-fixed points. Moreover, we have $R[\mathcal{O}] \cong R_H^G \cong e_H R[G]$, so that as R-lattices we get

$$\mathcal{H}(V) = \mathrm{Hom}_{R[U]}(R[\mathcal{O}]|_U, V) \cong \mathrm{Hom}_{R[G]}(e_H R[G], V^G) \cong V^G \cdot e_H.$$

This shows that $\mathcal{H}(V)$ can be seen as the 'condensed module' of the induced module V^G, with respect to the 'condensation subgroup' H. A technique to compute condensed modules of shape $V^G \cdot e_H$ for subgroups U of smallish index in G has been invented in [13]; the present approach now allows for both subgroups U and H to have large index. In the spirit of 'condensation techniques', recalling that $|\Omega_{jkl}| = \frac{|U_j|}{|U_k^{u_{jkl}} \cap U_j|}$, on $\mathrm{Fix}_V(U_k^{u_{jkl}})$ we get

$$\mathrm{Tr}_{U_k^{u_{jkl}} \cap U_j}^{U_j} = e_{U_k^{u_{jkl}} \cap U_j} \cdot \mathrm{Tr}_{U_k^{u_{jkl}} \cap U_j}^{U_j} = \frac{|\Omega_{jkl}|}{|U_j|} \cdot \mathrm{Tr}^{U_j} = |\Omega_{jkl}| \cdot e_{U_j}.$$

Thus, in terms of fixed-point idempotents, the action of $\mathcal{A}_{jkl}$ can be written as

$$\mathcal{A}_{jkl} \colon \mathrm{Fix}_V(U_k) \to \mathrm{Fix}_V(U_j) \colon v \mapsto |\Omega_{jkl}| \cdot v \cdot e_{U_k} u_{jkl} e_{U_j}.$$

Note that the latter 'condensation formula' actually holds more generally if $R[\mathcal{O}]|_U$ is a projective $R[U]$-module, that is all $R[\Omega_j]$ are projective $R[U]$-modules, which is equivalent to $\prod_{j=1}^s |U_j|$ being a unit in R.

(2.5) Condensing E. Finally, combining the above observations, still assuming that $R[\mathcal{O}]$ is a projective $R[G]$-module, we derive a 'condensation formula' for the action of the Schur basis elements of E_R on $\mathcal{H}(V) \cong \bigoplus_{j=1}^{s} \mathrm{Fix}_V(U_j)$:

Fixing R-bases for the fixed spaces $\mathrm{Fix}_V(U_j)$, the action of $\mathcal{E}_R$ is given by block matrices, where the blocks in position $[j,k]$ have size $\mathrm{rk}_R(\mathrm{Fix}_V(U_j)) \times \mathrm{rk}_R(\mathrm{Fix}_V(U_k))$, for $j,k \in \{1,\ldots,s\}$. Then the matrix representing the Schur basis element $\mathcal{A}_{jkl} \in \mathcal{E}_R$, where $l \in \{1,\ldots,t\}$ and $t := |U_k \backslash U / U_j|$, has its only non-zero block in position $[j,k]$, where the latter block represents the R-linear map $\mathrm{Fix}_V(U_k) \to \mathrm{Fix}_V(U_j)$ induced by the action of $|\Omega_{jkl}| \cdot e_{U_k} u_{jkl} e_{U_j}$.

Let $A_i \in E_R$ be a Schur basis element, where $i \in \{1,\ldots,r\}$. Then A_i is represented by a block matrix as above, whose block in position $[j,k]$ represents the R-linear map $\mathrm{Fix}_V(U_k) \to \mathrm{Fix}_V(U_j)$ induced by the action of

$$\sum_{l=1}^{|U_k \backslash U/U_j|} c_{jkl}(i) \cdot e_{U_k} u_{jkl} e_{U_j} = \sum_{l=1}^{|U_k \backslash U/U_j|} |\mathcal{O}_i \gamma_j \cap \Omega_{jkl}| \cdot e_{U_k} u_{jkl} e_{U_j}.$$

Since the elements $u_{jkl} \in U$ may be chosen arbitrarily as representatives of the double cosets $U_k \backslash U / U_j$, for any $v \in \Omega_k$ we let $u_k(v) \in U$ be any element such that $\omega_k \cdot u_k(v) = v$. Then the block in position $[j,k]$ represents the map

$$\sum_{v \in \mathcal{O}_i \cap (\Omega_k \gamma_j^{-1})} e_{U_k} \cdot u_k(v\gamma_j) \cdot e_{U_j}.$$

3 Characters of J_4 and its subgroups

(3.1) From now on let $G := J_4$ and $p := 11$. Let $\mathrm{Irr}(G)$ be the set of irreducible (ordinary) characters of G. We order the conjugacy classes of G and the $\mathrm{Irr}(G)$ is specified in [2], thus we may identify $\mathrm{Irr}(G)$ with the character table of G.

The principal p-block B_0 is the only one of positive defect. There are $k_0 := 49$ irreducible characters and $l_0 := 40$ irreducible Brauer characters belonging to B_0. According to [17], this is essentially all what until recently has been known about the decomposition numbers of B_0.

Now, in [9] we have been able to determine four of the projective indecomposable characters of B_0, amongst them the one belonging to the projective cover of the trivial module. The decomposition of newly found projective indecomposable characters Ψ_α, where $\alpha \in \{1,\ldots,4\}$, into irreducible characters is reproduced in Table 1, where $a \in \{0,1\}$. We also indicate the ordinal numbers of the irreducible characters occurring, their degree and their character field, where $r_n := \sqrt{n}$ denotes the positive square root of $n \in \mathbb{N}$.

Actually, $\chi_{19/20}$, $\chi_{23/24}$, $\chi_{36/37}$, and $\chi_{38/39}$ form four pairs of mutually algebraically conjugate characters. Since the quadratic fields $\mathbb{Q}(r_3)$, $\mathbb{Q}(r_5)$, and $\mathbb{Q}(r_{33})$ are disjoint, there are Galois automorphisms of $\mathbb{Q}(r_3, r_5, r_{11})$ inducing each of the involutions $(\chi_{19} \leftrightarrow \chi_{20})$, $(\chi_{23} \leftrightarrow \chi_{24})$, and $(\chi_{36} \leftrightarrow \chi_{37})(\chi_{38} \leftrightarrow \chi_{39})$.

Table 1: Projective indecomposable characters of G, taken from [9].

χ	$\chi(1)$	$\mathbb{Q}(\chi)$	Ψ_1	Ψ_2	$\Psi_{3,a}$	$\Psi_{4,a}$
1	1		1	.	.	.
8	889111		.	1	.	.
11	1776888		.	.	1	.
14	4290927		1	.	.	.
19	35411145	$\mathbb{Q}(r_{33})$	1	.	1	.
20	35411145	$\mathbb{Q}(r_{33})$	1	.	1	.
21	95288172		1	.	1	.
22	230279749		1	.	.	.
23	259775040	$\mathbb{Q}(r_3)$	.	.	.	1
24	259775040	$\mathbb{Q}(r_3)$	.	1	.	.
29	460559498		.	.	.	1
30	493456605		.	.	.	1
32	786127419		.	1	.	.
36	885257856	$\mathbb{Q}(r_5)$	1	.	.	.
37	885257856	$\mathbb{Q}(r_5)$	.	.	.	1
38	1016407168	$\mathbb{Q}(r_5)$	.	.	a	$1-a$
39	1016407168	$\mathbb{Q}(r_5)$	.	.	$1-a$	a
51	1842237992		.	.	.	1

Thus we may choose the rows of the decomposition matrix belonging to $\chi_{23/24}$ and $\chi_{36/37}$ as is shown in Table 1, while the non-p-rational characters $\chi_{19/20}$ have the same restriction to the p-regular conjugacy classes of G anyway. But then, as we will see below, there is no further choice possible for $\chi_{38/39}$, leaving two possible cases parameterized by $a \in \{0, 1\}$.

(3.2) In order to get a comprehensive overview about the possible choices on the character theoretic side, and what has to be decided explicitly in the end, we use the following terminology: Let $\mathcal{A}(G) := \mathrm{Aut}(\mathrm{Irr}(G))$ be the group of table automorphisms of $\mathrm{Irr}(G)$, that is the permutations of the conjugacy classes of G compatible with power maps and inducing permutations of the rows of $\mathrm{Irr}(G)$.

According to [1], the group $\mathcal{A}(G)$ has order 432, is generated by

$$\begin{aligned}\mathcal{A}(G) \;=\; &\langle (12,13)(24,25)(26,27)(32,33)(39,40)(48,49)(55,56),\ (43,44,45),\\ &(30,31)(53,54),\ (37,38),\ (46,47)(61,62),\ (50,51,52),\ (57,58,59)\rangle.\end{aligned}$$

Its action on $\mathrm{Irr}(G)$ is given as

$$\begin{aligned}\mathcal{A}(G) \;\to\; &\langle (2,3)(4,5)(6,7)(9,10)(12,13)(15,16)(17,18),\ (19,20)(33,34),\\ &(23,24),\ (36,37)(38,39),\ (46,47,48),\ (53,54,55),\ (56,57,58)\rangle.\end{aligned}$$

In particular, the latter contains the action of the Galois automorphisms mentioned above, where on the irreducible characters considered here we indeed see the involution $(19, 20)$. (On all of $\mathrm{Irr}(B_0)$ we see $(19, 20)(33, 34)$ instead, which has to be taken into account as soon as as projective characters also having constituents $\chi_{33/34}$ are considered.)

(3.3) In the sequel, for various subgroups $M < G$, we will compute the set of 'possible class fusions' from the conjugacy classes of M to those of G, that is the maps compatible with power maps and restrictions of irreducible characters. This set is acted on naturally by $\mathcal{A}(M)\times\mathcal{A}(G)$ via $[\alpha, \beta]\colon f \mapsto \alpha^{-1}\cdot f\cdot\beta$. Possible class fusions are considered equivalent if they belong to the same $(\mathcal{A}(M)\times\mathcal{A}(G))$-orbit. Hence 'choosing' a class fusion amounts to picking an orbit representative, f say, and keeping it fixed. But this restricts the table automorphisms remaining admissible in subsequent 'choices up to equivalence' to $\mathrm{Stab}_{\mathcal{A}(M)\times\mathcal{A}(G)}(f) \leq \mathcal{A}(M) \times \mathcal{A}(G)$, and its projections $\mathcal{A}_f(M) \leq \mathcal{A}(M)$ and $\mathcal{A}_f(G) \leq \mathcal{A}(G)$ to the first and second direct factors, respectively.

Now we bring Brauer characters into play: Let $\mathrm{IBr}_p(G)$ be the (as yet unknown) p-modular Brauer character table of G, whose columns are identified with the p-regular conjugacy classes of G. This amounts to saying that the class fusion from p-regular to all conjugacy classes of G has been chosen, so that we have to go down to the admissible subgroup $\mathcal{A}_p(G) \leq \mathcal{A}(G)$ inducing permutations of the rows of $\mathrm{IBr}_p(G)$.

Similarly, let $\mathrm{IPr}_p(G)$ be the (as yet unknown) table of p-modular projective indecomposable characters of G. Since $\mathrm{IPr}_p(G)$ is the dual basis of $\mathrm{IBr}_p(G)$, extended by zeroes on the p-singular conjugacy classes of G, with respect to the usual scalar product on the complex class functions on G, the group $\mathcal{A}_p(G)$ coincides with the subgroup of $\mathcal{A}(G)$ inducing permutations of the rows of $\mathrm{IPr}_p(G)$. If only an $\mathcal{A}_p(G)$-stable subset $\Psi \subseteq \mathrm{IPr}_p(G)$ is known, then the subgroup $\mathcal{A}_\Psi(G) \leq \mathcal{A}(G)$ inducing permutations of Ψ contains $\mathcal{A}_p(G)$, thus can serve as an upper approximation of the latter.

(3.4) Now, let $H < G$ be a (maximal) subgroup of shape $2^{10}\colon L_5(2)$. Then H is an $11'$-subgroup of order $|H| = 10\,239\,344\,640$. It turns out that $\mathcal{A}(H)$ has order 24, and that there are six possible class fusions from H to G, consisting of a single $(\mathcal{A}(H) \times \mathcal{A}(G))$-orbit. As a representative f we choose the class fusion stored in [1]. We get $\mathcal{A}_f(G) = \mathcal{A}(G)$, saying that upon choosing the class fusion from H to G all table automorphisms of G remain admissible.

Moreover, let 1_H^G be the permutation character afforded by a transitive G-set with associated point stabilizer H. Then 1_H^G is $\mathcal{A}(G)$-invariant, and it is a projective character of G, which by [9] splits into four projective indecomposable characters as $1_H^G = \Psi_1 + \Psi_2 + \Psi_{3,a} + \Psi_{4,a}$. Hence the set $\Psi := \{\Psi_1, \Psi_2, \Psi_{3,a}, \Psi_{4,a}\}$ is $\mathcal{A}_p(G)$-stable. From the description of the action of $\mathcal{A}(G)$ on $\mathrm{Irr}(G)$ we infer that $\mathcal{A}_\Psi(G)$ fixes Ψ element-wise, and thus so does $\mathcal{A}_p(G)$. Hence we get $\mathcal{A}_\Psi(G) = \mathrm{Stab}_{\mathcal{A}(G)}(\chi_{23}, \chi_{36})$, for both $a \in \{0, 1\}$, that is

$$\begin{aligned}\mathcal{A}_\Psi(G) \;=\; \langle &(12,13)(24,25)(26,27)(32,33)(39,40)(48,49)(55,56),\\ &(43,44,45),\ (46,47)(61,62),\ (50,51,52),\ (57,58,59)\rangle.\end{aligned}$$

Thus we have $[\mathcal{A}(G)\colon \mathcal{A}_\Psi(G)] = 4$, reflecting the couple of choices made between two alternatives each, and $\mathcal{A}_\Psi(G)$ acts on $\mathrm{Irr}(G)$ as

$$\begin{aligned}\mathcal{A}_\Psi(G) \;\to\; &\langle (2,3)(4,5)(6,7)(9,10)(12,13)(15,16)(17,18),\\ &\;(19,20)(33,34),\,(46,47,48),\,(53,54,55),\,(56,57,58)\rangle.\end{aligned}$$

We conclude that $\mathcal{A}_\Psi(G)$ acts on the constituents of the permutation character 1_H^G as a subgroup of $\langle (19,20)\rangle$, and so does $\mathcal{A}_p(G)$. In particular, the cases $a \in \{0,1\}$ are genuinely different, so that we have to decide which one holds.

(3.5) In order to do so, let $U < G$ be a (maximal) subgroup of shape $U_3(11)\colon 2$, having order $|U| = 141\,831\,360$, and let $U' \cong U_3(11)$ be its derived subgroup of index 2. Moreover, let $S < U'$ be a Sylow 11-subgroup, hence S is extra-special of shape 11_+^{1+2}, and is a Sylow 11-subgroup of G as well.

Let $\mathrm{IBr}_p(U)$ be as specified in [4]. In particular, let S_8 be the (absolutely irreducible) adjoint module of U' of degree 8, and let $S_8^\pm$ be its extensions to U, where the Brauer character of $S_8^\pm$ on the conjugacy class of involutions not contained in U' has value ± 2. Then S_8^+ and S_8^- have Brauer characters φ_3 and φ_4, respectively. Moreover, let $\Phi_8^\pm$ be the projective indecomposable characters of U associated with $S_8^\pm$.

It turns out that the group $\mathcal{A}(U)$ has order 96, but the admissible subgroup $\mathcal{A}_p(U) \leq \mathcal{A}(U)$ has order 2, whose non-trivial element is the transposition interchanging the conjugacy classes of elements of order 44 not belonging to U'. It turns out that there are 24 possible class fusions from U to G, which fall into two $(\mathcal{A}_p(U) \times \mathcal{A}(G))$-orbits of length 12; the latter are even $\mathcal{A}(U)$-invariant. Orbit representatives are given (in terms of conjugacy class numbers) as

$$\begin{gathered}[1,2,4,5,6,8,8,10,14,17,17,19,20,20,21,22,22,\\ 30,31,34,50,51,52,50,51,52,53,54,53,54,60,\\ 3,5,11,15,18,18,21,30+y,31-y,35,37,38,60,60],\end{gathered}$$

where $y \in \{0,1\}$, the case $y = 0$ being the one stored in [1]. The conjugacy classes of U whose fusion to G depends on y are those containing elements of order 20 not belonging to U'.

The picture changes when we restrict to $\mathcal{A}_\Psi(G)$: Both of the above orbits split into four $(\mathcal{A}_p(U) \times \mathcal{A}_\Psi(G))$-orbits of length three. Thus now there are eight orbits, representatives of which are given by the maps

$$\begin{gathered}[1,2,4,5,6,8,8,10,14,17,17,19,20,20,21,22,22,\\ 30+x,31-x,34,50,51,52,50,51,52,53+x,54-x,53+x,54-x,60,\\ 3,5,11,15,18,18,21,30+y,31-y,35,37+z,38-z,60,60],\end{gathered}$$

where $x, y, z \in \{0,1\}$; the case $x = y = z = 0$ being the one stored in [1]. The conjugacy classes of U whose fusion to G depends on the parameters x or y consist of elements of order 20 and 40, where the conjugacy classes $\{53,54\}$ of G square to the conjugacy classes $\{30,31\}$; the conjugacy classes of U whose fusion to G depends on the parameter z consist of elements of order 24.

(3.6) We can do slightly better, as far as the fusion from conjugacy classes of U consisting of elements of order 20 to G is concerned:

To this end, let $N := N_G(S) < G$, which is a (maximal) subgroup of shape $S\colon T \cong 11_+^{1+2}\colon (5 \times 2.\mathcal{S}_4)$, having order $|N| = 319\,440$, where $T \cong 5 \times 2.\mathcal{S}_4$ is unique up to N-conjugacy. Hence we may assume that $T \cap U \cong 5 \times \mathrm{QD}_{16}$, thus

$$N_U(S) = N \cap U = S\colon (T \cap U) \cong 11_+^{1+2}\colon (5 \times \mathrm{QD}_{16}).$$

We compare the embeddings $T \cap U < U$ and $T \cap U < T$: It turns out that there are eight possible class fusions from $T \cap U$ to T, and two possible class fusions from T to G. Then composition yields two possible class fusions from $T \cap U$ to G which factor through T. Similarly, there are four possible class fusions from $T \cap U$ to U. Then composing either of the 24 possible class fusions from U to G with the latter, and checking whether a possible class fusion factoring through T is obtained, leaves 12 possible class fusions from U to G.

It turns out that these fall into four $(\mathcal{A}_p(U) \times \mathcal{A}_\Psi(G))$-orbits, which are given by the parameters $x = y$, leaving the following four maps from the above list:

$$\begin{gathered}[1, 2, 4, 5, 6, 8, 8, 10, 14, 17, 17, 19, 20, 20, 21, 22, 22,\\ 30 + y, 31 - y, 34, 50, 51, 52, 50, 51, 52, 53 + y, 54 - y, 53 + y, 54 - y, 60,\\ 3, 5, 11, 15, 18, 18, 21, 30 + y, 31 - y, 35, 37 + z, 38 - z, 60, 60].\end{gathered}$$

(3.7) Having this in place, restricting Ψ_α to U, for $\alpha \in \{1, \ldots, 4\}$ and both cases $a \in \{0, 1\}$, and using the various class fusions for $y, z \in \{0, 1\}$, we may write $\Psi_\alpha|_U$ uniquely as an integral linear combination of projective indecomposable characters of U. By construction, the multiplicities occurring are $(\mathcal{A}_p(U) \times \mathcal{A}_\Psi(G))$-invariant. It turns out that in all cases these multiplicities are non-negative, so that this does not yield further immediate restrictions.

But, amongst others, the multiplicity $[\Psi_\alpha|_U\colon \Phi_8^\pm]$ of $\Phi_8^\pm$ in a direct sum decomposition of $\Psi_\alpha|_U$ subtly depends on the parameters a, y, z. We get the following pattern, where we also indicate $[1_H^G|_U\colon \Phi_8^\pm]$:

	Φ_8^+	Φ_8^-
$\Psi_1\|_U$	$77 - y$	$71 + y$
$\Psi_2\|_U$	$67 + z$	$52 - z$
$\Psi_{3,a}\|_U$	80	$56 + (-1)^{y+a}$
$\Psi_{4,a}\|_U$	$299 + y - z$	$260 - y + z - (-1)^{y+a}$
$1_H^G\|_U$	523	439

In order to determine the multiplicities $[\Psi_\alpha|_U\colon \Phi_8^\pm]$ explicitly, let $\mathbb{F} = \mathbb{F}_{11}$, and let P_α be the projective indecomposable $\mathbb{F}[G]$-module affording Ψ_α, where since $[1_H^G\colon \Psi_\alpha] = 1$ implies that Ψ_α is indeed realizable over $\mathbb{F}$. We have

$$[\Psi_\alpha|_U\colon \Phi_8^\pm] = \dim_{\mathbb{F}}(\mathrm{Hom}_{\mathbb{F}[U]}(P_\alpha|_U, S_8^\pm)).$$

Using the notation introduced in Section 2, in particular letting $\mathcal{O}$ be a G-set affording the permutation character 1_H^G, we have $\mathbb{F}[\mathcal{O}] \cong \bigoplus_{i=1}^4 P_\alpha$ as $\mathbb{F}[G]$-modules, where $P_\alpha \cong \mathbb{F}[\mathcal{O}] \cdot e_\alpha$ for a set $\{e_1, \ldots, e_4\} \subseteq E_\mathbb{F}$ of pairwise orthogonal primitive idempotents. The natural action of $E_\mathbb{F} \subseteq \mathcal{E}_\mathbb{F}$ on

$$\mathcal{H}(S_8^\pm) := \mathrm{Hom}_{\mathbb{F}[U]}(\mathbb{F}[\mathcal{O}]|_U, S_8^\pm) \cong \bigoplus_{j=1}^{s} \mathrm{Fix}_{S_8^\pm}(U_j)$$

yields $\mathrm{Hom}_{\mathbb{F}[U]}(P_\alpha|_U, S_8^\pm) \cong e_\alpha \cdot \mathcal{H}(S_8^\pm)$. Thus we have to determine the action of $E_\mathbb{F}$ on $\mathcal{H}(S_8^\pm)$, and the $\mathbb{F}$-dimension $d_\alpha := \dim_\mathbb{F}(e_\alpha \cdot \mathcal{H}(S_8^\pm))$, for $\alpha \in \{1, \ldots, 4\}$. At this stage, we switch to explicit computations:

4 Enumerating $\mathcal{O}$ again

(4.1) We pick the 112-dimensional absolutely irreducible representation of G over $\mathbb{F}_2$ from [19], and let $V \cong \mathbb{F}_2^{112}$ be the underlying module. The representation is given in terms of (two) standard generators, in the sense of [16]. Words in the generators providing (non-standard) generators of maximal subgroups $2^{10}\colon L_5(2) \cong H < G$ and $U_3(11)\colon 2 \cong U < G$ are available in [19] as well.

It turns out that H possesses a 1-dimensional fixed space in V. Hence letting $v_1 \in V$ be the unique non-zero H-fixed vector, we let $\mathcal{O} := (v_1)^G \subseteq V$, providing an implicit realization of $\mathcal{O}$. In [9], using ORB, we have already enumerated $\mathcal{O}$ by H-orbits, of which there are $r := \langle 1_H^G, 1_H^G \rangle_G = 27$. For the H-orbits $\mathcal{O}_i \subseteq \mathcal{O}$ we have in particular determined their lengths n_i, which are reproduced in Table 2, as well as elements $g_i \in G$ yielding orbit representatives $v_i := v_1 g_i \in \mathcal{O}_i$.

We are now going to enumerate $\mathcal{O}$ again, this time by U-orbits. It turns out that there are $s := \langle 1_H^G, 1_U^G \rangle_G = 131$ such U-orbits; note that s is independent of the class fusions chosen. To do so, we set up a new framework to apply ORB, adjusted to our present needs. In particular, comparing with [9], we have to pick another helper subgroup, since the one chosen there is a subgroup of H, but is not conjugate to a subgroup of U.

(4.2) By a random search we replace the non-standard generators of U we have so far by standard ones. A faithful permutation representation of U on 1332 points, in terms of standard generators, is available in [19]. Actually, the associated point stabilizers are conjugate to $11_+^{1+2}\colon (5 \times \mathrm{QD}_{16}) \cong N_U(S) < U$, which was already encountered in (3.6).

We choose a (single) helper subgroup $K < U$: Let $z \in U'$ be an involution, which is unique up to U'-conjugacy; specifically, we choose z as the square of the second standard generator of U, which has order 4. Then we let

$$K := C_U(z) \cong (\mathrm{SL}_2(11) \times V_4)\colon 2,$$

being computed in the permutation representation of U; we have $|K| = 10560$.

Table 2: H-orbit lengths in $\mathcal{O}$.

i	n_i	i	n_i	i	n_i
1	1	10	333120	19	79994880
2	31	11	4999680	20	159989760
3	930	12	6666240	21	159989760
4	17360	13	6666240	22	319979520
5	26040	14	9999360	23	341311488
6	27776	15	13332480	24	1279918080
7	416640	16	53329920	25	1279918080
8	416640	17	66060288	26	2047868928
9	624960	18	79994880	27	2559836160

Keeping K fixed, we choose (two) helper K-sets: It turns out that the restriction of V to K decomposes as $V|_K \cong V_{80} \oplus V_{32}$, subscripts denoting $\mathbb{F}_2$-dimension, where K acts faithfully on both summands. Moreover, V_{32} has a unique $\mathbb{F}_2[K]$-quotient V_{20} of $\mathbb{F}_2$-dimension 20, on which K acts non-faithfully by its quotient $(L_2(11) \times V_4): 2$. Actually, V_{20} is uniserial with a unique (absolutely irreducible) constituent of $\mathbb{F}_2$-dimension 10, on which K acts as $L_2(11): 2$. As helper K-sets we now choose the natural epimorphisms $V|_K \to V_{32} \to V_{20}$ of $\mathbb{F}[K]$-modules.

(4.3) We are now prepared to run ORB, in order to find the decomposition $\mathcal{O} = \coprod_{j=1}^{131} \Omega_j$ into U-orbits: We randomly choose elements $g \in G$, and check whether $v_1 g \in \mathcal{O}$ belongs to one of the U-orbits already found. If not, then we have found a previously unseen U-orbit, Ω_j say. In this case, we store $\gamma_j := g$ and the orbit representative $\omega_j := v_1\gamma_j \in \Omega_j$, we enumerate half of Ω_j, and using the faithful permutation representation of U we determine $U_j := \mathrm{Stab}_U(\omega_j)$. The lengths of the U-orbits in $\mathcal{O}$ are summarized in Table 3. Recalling that $|U| = 141\,831\,360$, we infer that the point stabilizers U_j have order at most 360.

To detect all U-orbits in $\mathcal{O}$ we need approximately half an hour on a single 3 GHz CPU. Setting up ORB anew, and using the γ_j instead of a random search, the orbit enumeration database is rebuilt in about ten minutes of CPU time. The statistics provided by ORB shows that the 'saving factor', that is the quotient between the number of points in a U-orbit actually stored in the orbit enumeration database, and the length of the piece of the U-orbit enumerated, varies between 8022 (for one of the shorter U-orbits) and 10497 (for one of the regular U-orbits). Thus we achieve an average saving factor of $\sim 10304 \sim 0,97 \cdot |K|$. The total memory usage of the orbit enumeration database amounts to manageable ~ 115 MB, and the infrastructure needs additional ~ 195 MB.

Table 3: U-orbit lengths in $\mathcal{O}$.

393976	2216115	4×11819280	22×35457840
738705	3×2954820	14183136	2×47277120
984940	3939760	12×17728920	28×70915680
2×1181928	5909640	3×23638560	36×141831360
1477410	9×8864460	2×28366272	

5 Condensing the induced module $(S_8^{\pm})^G$

(5.1) Finding idempotents. We proceed to determine pairwise orthogonal primitive idempotents $\{e_1, \ldots, e_4\} \subseteq E_{\mathbb{F}}$, and their action on $\mathcal{H}(S_8^{\pm})$. The idea pursued here is inspired by [6].

In [9] we have computed the 11-modular character table of $E_{\mathbb{F}}$; it is reproduced in Table 4. Here, notation is chosen such that the irreducible character φ_α of $E_{\mathbb{F}}$ corresponds to the projective indecomposable character Ψ_α of G. Since all irreducible characters are linear, the character values coincide with the eigenvalues of the action of the Schur basis elements in the various irreducible representations. We consider the action of A_2, where we observe that the character values $[\varphi_1(A_2), \ldots, \varphi_4(A_2)] = [9, 5, 10, 1]$ are pairwise different.

Let $\mu := \prod_{\alpha=1}^{4}(X - \varphi_\alpha(A_2))^{h_\alpha} \in \mathbb{F}[X]$ be the minimum polynomial of the action of A_2 in the (faithful) regular representation of $E_{\mathbb{F}}$. The multiplicities $h_\alpha \in \mathbb{N}$ are not needed explicitly in the sequel, but they are actually easily determined: The intersection matrices of $E_{\mathbb{Z}}$ have been determined in [9], so that the minimum polynomial of their 11-modular reduction is straightforwardly computed, yielding $[h_1, \ldots, h_4] = [5, 3, 5, 4]$.

Let $\mu_\alpha := (X - \varphi_\alpha(A_2))^{h_\alpha}$, and let $\mu'_\alpha := \frac{\mu}{\mu_\alpha}$ be the associated co-factor. Then, μ_α and μ'_α being coprime, there are $f_\alpha, f'_\alpha \in \mathbb{F}[X]$ such that $f_\alpha \mu_\alpha + f'_\alpha \mu'_\alpha = 1 \in \mathbb{F}[X]$. Hence we have $E_{\mathbb{F}} \cong \bigoplus_{\alpha=1}^{4} \ker(\mu_\alpha(A_2))$ as $\mathbb{F}[A_2]$-modules. Moreover,

$$e_\alpha := f'_\alpha(A_2)\mu'_\alpha(A_2) = 1 - f_\alpha(A_2)\mu_\alpha(A_2) \in \mathbb{F}[A_2]$$

annihilates $\ker(\mu'_\alpha(A_2)) = \bigoplus_{\beta \neq \alpha} \ker(\mu_\beta(A_2))$, while it acts as the identity on $\ker(\mu_\alpha(A_2))$. Thus $\{e_1, \ldots, e_4\} \subseteq E_{\mathbb{F}}$ is a set of pairwise orthogonal, hence primitive idempotents. Moreover, e_α acts on any $E_{\mathbb{F}}$-module as a projection onto the generalized eigenspace of A_2 with respect to the eigenvalue $\varphi_\alpha(A_2)$. In particular, it does so on the simple $E_{\mathbb{F}}$-modules, so that e_α is associated with the irreducible character φ_α indeed.

(5.2) Thus we are left with determining the action of A_2 on $\mathcal{H}(S_8^{\pm})$, which is given in terms of fixed spaces by the 'condensation formula' in (2.5). In order to apply it, we first observe that the point stabilizers U_j are small enough such

Table 4: The character table of $E_{\mathbb{F}}$.

φ_α	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	1	9	6	2	3	1	4	4	6	10	4	9	9	8
2	1	5	5	1	8	8	5	5	10	8	1	2	2	5
3	1	10	3	4	10	4	7	7	5	7	5	3	3	6
4	1	1	3	9	0	8	0	0	5	8	2	4	4	4

φ_α	15	16	17	18	19	20	21	22	23	24	25	26	27
1	7	6	8	9	9	7	7	3	1	1	1	6	2
2	3	5	5	5	5	6	3	10	1	0	1	3	8
3	5	1	1	7	7	10	9	0	4	3	6	10	5
4	5	9	0	3	3	2	10	8	10	10	1	0	0

---

that the action of the fixed-point idempotent e_{U_j} on $S_8^{\pm}$ is straightforwardly computed by running through all the elements of U_j explicitly.

Starting with $v_2 \in \mathcal{O}_2$, the H-orbit $\mathcal{O}_2$, having length $n_2 = 31$, is easily enumerated explicitly. Then, running through the points $v \in \mathcal{O}_2$, we apply the elements $\gamma_j \in G$ in turn, for $j \in \{1, \ldots, s\}$, and check using ORB to which U-orbit the point $w := v\gamma_j \in \mathcal{O}$ belongs. If $w \in \Omega_k$, say, then we use the functionality ORB readily offers to find an element $u_k(w) \in U$ such that $\omega_k \cdot u_k(w) = w$.

Having this in place, we apply the 'condensation formula' to compute the action of A_2 on $\mathcal{H}(S_8^{\pm}) \cong \bigoplus_{j=1}^{s} \mathrm{Fix}_{S_8^{\pm}}(U_j)$. This straightforwardly yields the $\mathbb{F}$-dimension $d_\alpha^{\pm}$ of the generalized eigenspace of the action of A_2 with respect to the eigenvalue $\varphi_\alpha(A_2)$, and for comparison the multiplicity $h_\alpha^{\pm}$ of the irreducible factor $X - \varphi_\alpha(A_2)$ in the minimum polynomial of this action:

	h_α	d_α^+	h_α^+	d_α^-	h_α^-
$\varphi_1(A_2)$	5	76	4	72	2
$\varphi_2(A_2)$	3	67	3	52	2
$\varphi_3(A_2)$	5	80	4	55	2
$\varphi_4(A_2)$	4	300	4	260	4
$\mathcal{H}(S_8^{\pm})$		523		439	

(5.3) We just remark that the same approach works for all the shorter H-orbits in $\mathcal{O}$, in particular including $\mathcal{O}_4$ of length $n_4 = 17360$.

Now, in [9] we have shown that $E_{\mathbb{Q}} = \mathbb{Q}[A_2, A_4]$, by writing all the Schur basis elements of $E_{\mathbb{Q}}$ explicitly as words in the generating set $\{A_2, A_4\}$. Letting $\mathbb{Z}_{(11)} \subseteq \mathbb{Q}$ be the ring of 11-adic integers in $\mathbb{Q}$, it turns out that the latter words actually belong to $\mathbb{Z}_{(11)}[A_2, A_4]$. This implies that the Schur basis elements of $E_{\mathbb{F}}$ are given by the very words, now considered via 11-modular reduction as belonging to $\mathbb{F}[A_2, A_4]$. Thus we have $E_{\mathbb{F}} = \mathbb{F}[A_2, A_4]$.

As already follows from comparing $h_\alpha^\pm$ with h_α, the algebra $E_\mathbb{F}$ acts non-faithfully on $\mathcal{H}(S_8^\pm)$, where it turns out that on $\mathcal{H}(S_8^+)$ and $\mathcal{H}(S_8^-)$ it acts by an algebra of $\mathbb{F}$-dimension 24 and 15, respectively, while $\dim_\mathbb{F}(E_\mathbb{F}) = 27$.

(5.4) Conclusion. We have found the multiplicities $[\Psi_\alpha|_U : \Phi^\pm] = d_\alpha^\pm$ as given in (5.2). Comparing with the possible parameter choices left in (3.7), we get

$$y = 1 \quad \text{and} \quad z = 0 \quad \text{and} \quad a = 0.$$

Thus the projective indecomposable summands Ψ_α of the permutation character 1_H^G are, up to admissible table automorphisms, as shown in Table 1, upon specifying $a := 0$. Moreover, the class fusion from U to G is given, again up to admissible table automorphisms, as follows, where it turns out that it differs from the one stored in [1] precisely in the (eight) positions printed in bold face:

$$\begin{gathered}[1, 2, 4, 5, 6, 8, 8, 10, 14, 17, 17, 19, 20, 20, 21, 22, 22,\\ \mathbf{31}, \mathbf{30}, 34, 50, 51, 52, 50, 51, 52, \mathbf{54}, \mathbf{53}, \mathbf{54}, \mathbf{53}, 60,\\ 3, 5, 11, 15, 18, 18, 21, \mathbf{31}, \mathbf{30}, 35, 37, 38, 60, 60].\end{gathered}$$

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