Decomposition numbers for generic Iwahori-Hecke algebras of non-crystallographic type

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Abstract

In this note we compute the Φ_e-modular decomposition matrices for the generic Iwahori-Hecke algebras of type $I_2(m)$ for $m \in \mathbb{N}$, $m > 2$, H_3, and H_4, for all $e \in \mathbb{N}$ leading to non-trivial decomposition maps. The results are obtained by a combined use of different ideas from computational representation theory and by application of the computer algebra systems GAP, CHEVIE, VectorEnumerator, and MeatAxe.

1 Introduction

Generic Iwahori-Hecke algebras of the classical series A, B, D, and of the exceptional types G_2, F_4, E_6, E_7, E_8, which together are called the crystallographic types, have gained considerable interest in the representation theory of finite groups of Lie type. Especially their behaviour under certain decomposition maps has become the focus of intensive study. The underlying decomposition theory extends naturally to the closely related non-crystallographic types $I_2(m)$, H_3, H_4, and although there is no interpretation in the framework of the representation theory of groups of Lie type, it seems worth while to have complete results for all types, which also gives us the possibility to check them for similarities and differences between the crystallographic and non-crystallographic cases.
Another motivation to study the non-crystallographic types is to extend results on the automorphisms of generic Iwahori-Hecke algebras of crystallographic type obtained by F. Bleher, M. Geck, W. Kimmerle [2].

In this note we determine the Φ_e-modular decomposition matrices for the generic Iwahori-Hecke algebras of type $I_2(m)$ for $m \in \mathbb{N}$, $m > 2$, H_3, and H_4, for all $e \in \mathbb{N}$ which lead to non-trivial decomposition maps. The general

setup is described in Section 2, the decomposition matrices are determined and depicted in Sections 3, 4, 5, respectively.
Especially, we observe the following facts: All decomposition matrices are of lower unitriangular shape with respect to a suitable ordering of the irreducible representations. All characters are of height 0. Blocks of defect 1 are Brauer tree algebras, where the Brauer tree is a straight line without exceptional vertex. For the blocks of defect 2 all decomposition numbers are 0 or 1.
If e equals the Coxeter number of the corresponding Coxeter group, then there is exactly one block of positive defect, it is of defect 1, and the vertices of the Brauer tree are the successive exterior powers of the reflection representation. This extends [2], Theorem 6.6, to the non-crystallographic types.

The results are obtained by a combined use of different ideas from computational representation theory, which have been applied successfully in the business of determining decomposition numbers, especially for Iwahori-Hecke algebras of exceptional type, see [18]. We make use of the computer algebra systems GAP [21], CHEVIE [9, 10], VectorEnumerator [16], and MeatAxe [20], whose basic features we assume the reader to be acquainted with.

Notation. For $n \in \mathbb{N}$ let $\zeta_n \in \mathbb{C}$ denote the standard n-th primitive root of unity $\zeta_n := \exp(\frac{2\pi i}{n})$. For $d \in \mathbb{N}$ let Φ_d denote the d-th cyclotomic polynomial. For the generic algebras of type H_3, H_4 we use the ordering and labelling of the irreducible characters χ_i as given in CHEVIE. Usually, χ_i is identified by its degree and the smallest symmetric power of the reflection representation χ_i occurs in. These numbers determine χ_i uniquely, with the only exception of the irreducible characters χ_{29}, χ_{30} of the algebra of type H_4, which have labels $(30, 10)$, $(30, 10)'$. Let us agree that $\chi_{29}(T_{s_1}T_{s_2}T_{s_3}T_{s_4}) = \zeta_5 + \zeta_5^4$ holds. To make this note more self-contained, we will incorporate these labels into the decomposition matrices given below. The Schur elements, which in the sequel will be denoted by c_{χ_i}, are also available in CHEVIE. Their factorization into a power product of irreducible polynomials over $\mathbb{Q}(\zeta_5)$ is found using GAP.

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2 Decomposition theory

(2.1) Iwahori-Hecke algebras. We now introduce the algebras we are going to study and state a few basic facts. As general references see [3], Chapter 4, and [5], Chapters 67, 68. Note that we will only deal with the equal parameter case, which is no restriction for the algebras of type H_3, H_4 anyway.

(2.1.1) Definition. Let W be a finite Coxeter group of type Γ with set of standard generators S. Let R be a commutative ring and fix a unit $q \in R$. The Iwahori-Hecke algebra $H_R(\Gamma, q)$ of type Γ over R with parameter q is defined as a finitely presented associative R-algebra with identity T_1 as

$$H_R(\Gamma, q) := \langle T_s; s \in S \quad || \quad \begin{array}{l} T_s^2 = q \cdot T_1 + (q-1) \cdot T_s, \\ \underbrace{T_s T_{s'} T_s \cdots}_{m_{ss'}} = \underbrace{T_{s'} T_s T_{s'} \cdots}_{m_{ss'}}, \; s, s' \in S, s \neq s' \rangle \end{array} ,$$

where $m_{ss'}$ denotes the order of $ss' \in W$. The algebra $H_{\mathbb{Z}[u,u^{-1}]}(\Gamma, u)$, where u is an indeterminate over $\mathbb{Z}$, is called the generic algebra of type Γ.

(2.1.2) $H = H_R(q) = H_R(\Gamma, q)$ is R-free with a basis $\{T_w; w \in W\}$ parametrized by the elements of the Coxeter group W. Letting $T_w \mapsto (-1)^{l(w)} \cdot \mathrm{ind}(T_w) \cdot T_{w^{-1}}^{-1}$ for all $w \in W$ defines an involutory R-algebra automorphism of H, as follows directly from the definition. This automorphism and also its action on the representations of H is called Curtis-Alvis duality. Also we immediately have the existence of two linear representations of H, the sign representation $sgn : T_s \mapsto -1$ for all $s \in S$, and the index representation defined by $ind : T_s \mapsto q$ for all $s \in S$. In fact, H is a symmetric R-algebra with respect to the bilinear form $(T_w, T_{w'}) = \delta_{w^{-1},w'} \cdot ind(T_w)$.
It has been shown by C. Curtis, N. Iwahori, R. Kilmoyer [4], Section 9, that a reflection representation of W has a natural lift, again called a reflection representation, to the generic algebra $H = H_R(u)$, where $\mathbb{Z}[u, u^{-1}] \subseteq R$ is a suitable extension ring. Furthermore, the non-vanishing exterior powers of the module underlying a reflection representation of H become irreducible, pairwise non-isomorphic H-modules.

(2.1.3) Now we consider the generic algebra $H(u)$. Let v be an indeterminate such that $v^2 = u$. If H is of type $I_2(m)$, let $\zeta := \zeta_{2m}$; if H is of type H_3 or H_4, let $\zeta := \zeta_5$. It has been shown by R. Kilmoyer, L. Solomon [15] for type $I_2(m)$, by G. Lusztig [17] for type H_3, and by D. Alvis, G. Lusztig [1] for type H_4 that $K := \mathbb{Q}(\zeta, v)$ is a splitting field for $H_K(u)$. They even have shown that all the irreducible representations of these algebras can be realized over $R := \mathbb{Z}[\zeta, v, v^{-1}]$. Note that since H is defined over $\mathbb{Z}[v, v^{-1}]$, the Galois group $\mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})$ induces an automorphism of the module category of H.

(2.1.4) We will be interested in decomposition maps, as described in (2.2), coming from prime ideals $\mathfrak{p} \lhd R$, such that $R/\mathfrak{p}$ is of characteristic 0. These are of the form $\mathfrak{p} = \langle \Phi \rangle$, where $\Phi \in \mathbb{Z}[\zeta, v]$ is an irreducible polynomial of positive degree. Note that $\mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})$ acts on R and on its set of prime ideals; by the last remark in (2.1.3), without loss of generality we can restrict ourselves to consider only one prime ideal $\mathfrak{p} \lhd R$ out of each Galois orbit.
The other prime ideals are either those of the form $\wp R$, where $\wp \lhd \mathbb{Z}[\zeta]$ is a prime ideal; in this case we let $\wp \cap \mathbb{Z} = \langle l \rangle \lhd \mathbb{Z}$, where $l \in \mathbb{Z}$ is a rational prime. Or they are of the form $\mathfrak{P} = \langle \wp, \tilde{\Phi} \rangle$, where $\wp \lhd \mathbb{Z}$ is a prime ideal and $\tilde{\Phi} \in \mathbb{Z}[\zeta, v]$ such that its natural image $\bar{\tilde{\Phi}} \in (\mathbb{Z}[\zeta]/\wp)[v]$ is irreducible of positive degree. Note that we have $\mathfrak{p} = \langle \Phi \rangle \subseteq \mathfrak{P}$ if and only if $\bar{\tilde{\Phi}}$ divides the natural image $\bar{\Phi} \in (\mathbb{Z}[\zeta]/\wp)[v]$.
By the remarks in (2.2.2) the decomposition map coming from a prime ideal $\mathfrak{P}$, which is of height 2, factors through the decomposition maps coming from the height 1 prime ideals contained in $\mathfrak{P}$. This is one reason to be primarily interested in the height 1 prime ideals. Another reason to stick to prime ideals $\mathfrak{p}$ is that this type of prime ideals has been in the center of research for the crystallographic cases.

(2.2) Decomposition maps. We now briefly recall the basic concepts of the decomposition theory of symmetric algebras. As a general reference see [12].

(2.2.1) Let R be an integral domain of characteristic 0, $K := \mathrm{Quot}(R)$, and $\mathfrak{p} \lhd R$ a prime ideal with a perfect quotient field $k := \mathrm{Quot}(R/\mathfrak{p})$. Let $\bar{\ }: R \to k$ denote the natural epimorphism. Let H_R be an R-free R-algebra of finite R-rank, $H_K := H_R \otimes_R K$ and $H_k := H_R \otimes_R k$.
We now assume that each irreducible H_K-module V_K can be realized as a

full R-free H_R-submodule $V_R \subseteq V_K$. Then the H_k-module $V_k := V_R \otimes_R k$ is called a $\mathfrak{p}$-modular reduction of V_K. It can be shown that this defines a $\mathfrak{p}$-modular decomposition map $d_\mathfrak{p} : G_0(H_K) \to G_0(H_k)$ between the Grothendieck groups of the module categories of H_K and H_k.
The elements of the Grothendieck groups are called generalized characters, the basis elements corresponding to the irreducible modules are called irreducible characters. Since $G_0(H_K)$ can be naturally identified with the group of trace functions on the representations of H_K, whose elements usually are also called characters, we will not distinguish between these two notions.
We will be interested in the case where K and k are splitting fields for H_K and H_k, respectively. The decomposition matrices we are going to write down are understood to hold in this situation. For the generic algebras of non-crystallographic type we let R, K be as given in (2.1.3). The computations made in the sequel will show that $k := \mathrm{Quot}(R/\mathfrak{p})$, where $\mathfrak{p} \lhd R$ is as given in (2.1.4), in fact always is a splitting field for H_k.

(2.2.2) In the general situation described in (2.2.1), let $\mathfrak{P} \lhd R$ be a prime ideal containing $\mathfrak{p}$. Let us assume that there is a valuation ring $R' \subseteq \mathrm{Quot}(R/\mathfrak{p})$ with maximal ideal $\mathfrak{P}'$ lying over $(R/\mathfrak{p}, \mathfrak{P}/\mathfrak{p})$, such that $R'/\mathfrak{P}' \cong (R/\mathfrak{p})/(\mathfrak{P}/\mathfrak{p}) \cong R/\mathfrak{P}$. In this situation $d_\mathfrak{P}$ factors through $d_\mathfrak{p}$, see [7], Section 4.2, or [12], Proposition 2.12. Hence this gives us a tool to study $d_\mathfrak{p}$ by studying the auxiliary map $d_\mathfrak{P}$.
This applies to our situation (2.1.3), (2.1.4), where $\mathfrak{p} := \langle \Phi \rangle \lhd R := \mathbb{Z}[\zeta, v, v^{-1}]$ and $\mathfrak{P} := \langle \wp, \tilde{\Phi} \rangle$. Especially, $R/\mathfrak{P}$ is a finite field of characteristic l, hence $d_\mathfrak{P}$-modular reduced representations are computationally much more tractable then $d_\mathfrak{p}$-modular reduced ones.

(2.2.3) Symmetric algebras. From now on we assume that H_R is a symmetric R-algebra such that K is a splitting field for H_K, let $\{T_i\}$, $\{T_i^*\}$ be a pair of R-bases which are dual to each other with respect to the symmetrising form. If χ is an irreducible character of H_K, then the Schur element $c_\chi \in R$ is defined as $c_\chi := \chi(1)^{-1} \cdot \sum_i \chi(T_i)\chi(T_i^*)$. If $c_\chi \neq 0$, then χ corresponds to a projective irreducible representation and the corresponding centrally primitive idempotent $\epsilon_\chi \in H_K$ is given as $\epsilon_\chi := c_\chi^{-1} \cdot \sum_i \chi(T_i)T_i^*$. By Tits' Deformation Theorem, $d_\mathfrak{p}$ induces a bijection between the irreducible characters of H_K and H_k if and only if $c_\chi \in R \setminus \mathfrak{p}$ for all irreducible characters χ of H_K.

(2.2.4) Blocks and defect. From now on we assume that $\mathfrak{p}$ is a principal ideal. Hence $R_{\mathfrak{p}} \subseteq K$ is a discrete valuation ring such that $k \cong R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$. Let $\nu_{\mathfrak{p}}$ denote the corresponding valuation.
The minimal summands in a ring direct decomposition of H_k are called its blocks, and we have a corresponding partition of the irreducible representations of H_K and H_k. The value $\nu_{\mathfrak{p}}(c_\chi) \in \mathbb{N}_0$, where χ is an irreducible representation of H_K, is called the defect of χ. The maximum over the $\nu_{\mathfrak{p}}(c_\chi)$ for all χ belonging to one block is called the defect of the block. By the remarks in (2.2.3), the decomposition matrix of a block of defect 0 is a 1-by-1 unit matrix. Hence in the sequel we will consider only blocks of positive defect.

(2.2.5) Central elements. Let $z \in H_R$ be a central element. By Schur's Lemma z acts as a scalar $x_\chi \in R$ on each irreducible representation χ of H_K. If χ, χ' belong to the same block, then we have $\bar{x}_\chi = \bar{x}_{\chi'}$.
If H is a generic Iwahori-Hecke algebra of type H_3 or H_4, then the basis element $T_{w_0} \in H$ corresponding to the longest element $w_0 \in W$ is a central element. The scalars x_χ can be read off from the character table of H, which can be accessed in CHEVIE.

(2.2.6) Blocks of defect 1. Blocks of defect 1 have been dealt with by M. Geck [7], Theorem 9.6, for generic Iwahori-Hecke algebras of crystallographic type. A careful analysis of the line of reasoning in [7], Section 9, which is a paraphrase of parts of [13], Chapter 11, shows that almost everything works without change for the non-crystallographic cases.
Especially, as we assume that all irreducible representations are realizable over K, we do not have to deal with ramification, hence there is no exceptional vertex. As the ordinary irreducible characters of the Coxeter groups under consideration still are real-valued, we can use contragredient modules as is done in the proof of [7], Proposition 9.5. But as we do not know a priori that $d_{\mathfrak{p}}$ is surjective, we can only conclude that the decomposition matrix is described by a Brauer graph which is a connected regular graph of valence 2, hence it is a straight line or a circle.
Assume the latter case occurs. As we have $\nu_{\mathfrak{p}}(c_i + c_j) \geq 2$ by [7], Proposition 9.3, where c_i, c_j denote the Schur elements for two adjacent characters in the Brauer graph, the circle must be of even length. But the projectively indecomposable characters, see (2.3), corresponding to a circular Brauer graph of even length are $\mathbb{Z}$-linearly dependent, contradicting the remarks in (2.3.2).

Hence the Brauer graph is a straight line without exceptional vertex.

(2.3) Projective characters. One of our main tools to compute decomposition numbers are projective characters.

(2.3.1) There is a natural bijection between the irreducible H_k-modules and the projectively indecomposable H_k-modules. Then Brauer reciprocity gives us a homomorphism

$$e_{\mathfrak{p}} : G_0(H_k) \longrightarrow G_0(H_K) : \varphi \longmapsto \sum_{\chi} d_{\chi\varphi} \cdot \dim_k(\mathrm{End}_{H_k}(\varphi)) \cdot \chi,$$

where χ runs over the irreducible representations of H_K and $d_{\chi\varphi}$ denotes the corresponding decomposition number.
The subgroup $e_{\mathfrak{p}}(G_0(H_k)) \leq G_0(H_K)$ is called the group of generalized projective characters. Let $G_0^+(H_k)$ be the submonoid of $G_0(H_k)$ generated by the irreducible H_k-modules. Then the elements of the submonoid $e_{\mathfrak{p}}(G_0^+(H_k))$ of $G_0(H_K)$ are called projective characters.

(2.3.2) If $e_{\mathfrak{p}}$ is an injective map, then $e_{\mathfrak{p}}(G_0(H_k))$ has a basis consisting of the projectively indecomposable characters $\{e_{\mathfrak{p}}(\varphi)\}$, where φ runs through the irreducible H_k-modules. In this case the problem of finding decomposition numbers is equivalent to determining projectively indecomposable characters.
If H is an Iwahori-Hecke algebra then $e_{\mathfrak{p}}$ is injective. Indeed, it has been shown in [10], Section 3.2, that the statements concerning conjugacy classes and class polynomials proved for the crystallographic types in [11] hold without change for the non-crystallographic types. It has then been shown by M. Geck and R. Rouquier [12], Lemma 3.1 and Theorem 5.2, that this implies the injectivity of $e_{\mathfrak{p}}$.

(2.4) Computational methods. We now introduce the main computational concepts that will be applied in the explicit determination of decomposition numbers.

(2.4.1) First of all we need a method to generate projective characters.
Let H be a finite dimensional algebra over a field and $H' \leq H$ be a unitary subalgebra such that H is a free right H'-module and a free left H'-module.

In this case it follows from [6], Lemma I.4.6, Theorem I.4.8, that the restriction of a projective H-module is a projective H'-module, the induction of a projective H'-module is a projective H-module, and each projective H-module is a direct summand of an induced projective H'-module.
If H is an Iwahori-Hecke algebra and H' is a parabolic subalgebra, then the assumptions made above are fulfilled. By Tits' Deformation Theorem, the induction map $G_0(H'_K) \to G_0(H_K)$ equals the one for the corresponding Coxeter groups $G_0(W') \to G_0(W)$. Hence induction can be carried out on the level of ordinary characters of the groups W', W.

(2.4.2) Given a set of projective characters, we then have to find a basis for the subgroup of $G_0(H_K)$ it generates, which again consists of projective characters. This can be done algorithmically by the FBA algorithm due to R. Parker, see [14], Section 5. We use a sligthly modified version, which is described in [18], Section 4.5.

(2.4.3) We next need a criterion which allows us to prove that a given projective character is projectively indecomposable or to find candidates for the projectively indecomposable summands it contains.
For a character $\vartheta \in G_0(H_K)$ let $\vartheta = \sum_\chi \vartheta_\chi \cdot \chi$, $\vartheta_\chi \in \mathbb{Z}$, be its decomposition into the basis of $G_0(H_K)$ consisting of the irreducible characters. If ϑ is a generalized projective character, then it has been conjectured by the author and subsequently been proved by M. Geck and R. Rouquier [12], Proposition 4.4, that $\nu_\mathfrak{p}(\sum_\chi \vartheta_\chi c_\chi^{-1}) \geq 0$ holds.
If ψ is a projective character, then a summand ϑ contained in ψ necessarily fulfills $0 \leq \vartheta_\chi \leq \psi_\chi$ for all irreducible characters χ and the valuative condition given above. Hence we find a set of candidates for projectively indecomposable summands by looking through the finite set of all characters ϑ fulfilling the first condition, and testing its elements for the second condition.

(2.4.4) Let again H be an Iwahori-Hecke algebra and H' a parabolic subalgebra. We finally need a method to explicitly induce an H'-module up to H. As H is given as a finitely presented algebra and the generators for H' are explicitly given even as a subset of the generators for H we can use the VectorEnumerator to perform this task, provided we have found a finite module presentation for the H'-module we are going to induce up. For more details on how to find module presentations and on induction of modules

using the VectorEnumerator, see [19].
We will only use explicit induction for modules over some finite field F' which are defined over the prime field $F \leq F'$, hence the computations can entirely be carried out over F, see [19]. Note that it is easy using the MeatAxe over F to find the F'-constituents of a module which is defined over F.

3 The algebras of type $I_2(m)$, $m > 2$

(3.1) We are going to consider the generic Iwahori-Hecke algebra of type $I_2(m)$ with parameter $u = v^2$ over the field $\mathbb{Q}(\zeta_{2m}, v)$. Its standard generators will be denoted by T_s and T_t. The Poincaré polynomial $P_m(u) := P_{H(I_2(m))}(u) \in \mathbb{Z}[u]$ for the generic algebra of type $I_2(m)$ with parameter u is given by $P_m(u) = \Phi_2(u) \cdot \prod_{d|m,d>1} \Phi_d(u)$.
The irreducible representations of the algebras of type $I_2(m)$ have been determined by R. Kilmoyer, L. Solomon [15], where also the corresponding Schur elements can be found.
If $m = 2k + 1$ is odd, then $H(I_2(m))$ has exactly two linear representations, the index and the sign representation. The Schur elements are given as $c_{ind} = P_m(u)$ and $c_{sgn} = u^{-m} P_m(u)$. Furthermore, there are exactly $\frac{m-1}{2}$ pairwise non-isomorphic 2-dimensional irreducible representations X_j, where $j \in \mathbb{N}$, $1 \leq j \leq \frac{m-1}{2}$. They are given by

$$X_j : T_s \mapsto \begin{bmatrix} -1 & v(\zeta_{2m}^j + \zeta_{2m}^{-j}) \\ 0 & v^2 \end{bmatrix}, \; T_t \mapsto \begin{bmatrix} v^2 & 0 \\ v(\zeta_{2m}^j + \zeta_{2m}^{-j}) & -1 \end{bmatrix}.$$

The Schur element for X_j is given as

$$c_{X_j} = m \cdot u^{-1} \cdot \frac{(u - \zeta_m^j)(u - \zeta_m^{-j})}{(1 - \zeta_m^j)(1 - \zeta_m^{-j})}.$$

If $m = 2k$ is even, then $H(I_2(m))$ has exactly four linear representations, the index and the sign representation, and the two 'mixed' types λ and λ^*. And there are exactly $\frac{m}{2} - 1$ pairwise non-isomorphic 2-dimensional irreducible representations X_j. They are given by the same formulas as above, except that now $1 \leq j \leq \frac{m}{2} - 1$. For the Schur elements the formulas given above also hold. For the additional linear representations we find $c_\lambda = c_{\lambda^*} = \frac{m}{2} \cdot u^{-1} \cdot \Phi_2(u)^2$.
As can be seen from the Schur elements given above, to get a non-trivial decomposition map we have only to consider the cases $\Phi = \Phi_e$ where e

divides m. We then have a standard choice for R by imposing the condition $v - \zeta_{2e} \in \mathfrak{p} \lhd R$.

(3.2) Φ_e-modular decomposition numbers for $e > 2$. For $e > 2$ we see from the Schur elements that in both cases m even or odd, the only non-projective irreducible representations are *ind*, *sgn*, and $X_{m/e}$. Hence they form one block, which is of defect 1. $X_{m/e}$ is left fixed by Curtis-Alvis duality whereas *ind* and *sgn* are exchanged. Since the Φ_e-modular reductions of *ind* and *sgn* are non-isomorphic, the decomposition matrix of this block is as follows.

ind	1	.
$X_{m/e}$	1	1
sgn	.	1

Note that for $e = m$, which is the Coxeter number of the corresponding Coxeter group, $X_{m/e} = X_1$ is one of the reflection representations and *sgn* is obtained as the exterior square of X_1.

(3.3) Φ_e-modular decomposition numbers for $e = 2$. For $e = 2$, we find that exactly the linear representations are the non-projective ones. Furthermore, all their Φ_2-modular reductions are equal, hence for m odd they form a block of defect 1, for m even of defect 2, having the following decomposition matrix.

ind	1
sgn	1

ind	1
sgn	1
λ	1
λ^*	1

We add as a remark that this result can be obtained without use of the Schur elements by considering the eigenspaces of the action of T_s and T_t and the trace of the action of T_sT_t on the module X_j.

4 The algebra of type H_3

(4.1) The Poincaré polynomial $P_{H_3}(u) \in \mathbb{Z}[u]$ for the generic algebra of type H_3 with parameter u is given by $P_{H_3} = \Phi_2^3 \cdot \Phi_3 \cdot \Phi_5 \cdot \Phi_6 \cdot \Phi_{10}$.
It turns out that all of the c_{χ_i} divide $P_{H_3}(u)$ in $\mathbb{Q}(\zeta, u, u^{-1})$. Hence we only have to consider $\Phi = \Phi_e$, where $e \in \{2, 3, 5, 6, 10\}$. Let $\Phi'_{2e}(v) \in \mathbb{Q}(\zeta_5)[v]$ be the minimum polynomial of ζ_{2e} over $\mathbb{Q}(\zeta_5)$. Again we will use a standard choice for R, given by $\Phi'_{2e}(v) \in \mathfrak{p} \lhd R$.

(4.2) Φ_{10}-modular decomposition numbers. It turns out that the non-projective irreducible characters are $\chi_1, \chi_2, \chi_5, \chi_7$. All of them are of defect 1, hence they form a union of blocks of defect 1. Now we consider the Φ_{10}-modular reductions of the class functions corresponding to these characters. GAP shows that they span a space of rank 3. Hence these characters form exactly one block. Furthermore it turns out that $c_{\chi_1} + c_{\chi_5}$ is divisible by Φ_{10}^2, but $c_{\chi_1} + c_{\chi_7}$ is not. This determines the decomposition matrix.

1	$(1,15)$	1	.	.
5	$(3,6)$	1	1	.
7	$(3,1)$	.	1	1
2	$(1,0)$	.	.	1

Note that $e = 10$ equals the Coxeter number of the corresponding Coxeter group. Furthermore, χ_7 is the reflection representation, χ_5 is its exterior square and $sgn = \chi_1$ is its exterior cube. Finally, we have $ind = \chi_2$.

(4.3) Φ_6-modular decomposition numbers. It turns out that the non-projective irreducible characters are $\chi_1, \chi_2, \chi_3, \chi_4$. An argument similar to the one given in (4.2) shows that the decomposition matrix is as follows.

1	$(1,15)$	1	.	.
3	$(5,5)$	1	1	.
4	$(5,2)$	.	1	1
2	$(1,0)$	.	.	1

(4.4) Φ_5-modular decomposition numbers. It turns out that the non-projective irreducible characters are $\chi_1, \chi_2, \chi_6, \chi_8, \chi_9, \chi_{10}$. All of them are of defect 1. But in this case, we find the following Φ_5-modular reductions $\bar{x}_{\chi_i}$ of the scalars x_{χ_i}, see (2.2.5): $\bar{x}_{\chi_1} = \bar{x}_{\chi_8} = \bar{x}_{\chi_{10}} = -1$, $\bar{x}_{\chi_2} = \bar{x}_{\chi_6} = \bar{x}_{\chi_9} = 1$. This shows that there are exactly two blocks whose decomposition matrices are as follows.

		Λ_1	Λ_2
1	$(1,15)$	1	.
10	$(4,4)$	1	1
8	$(3,3)$	.	1

		Λ_3	Λ_4
2	$(1,0)$	1	.
9	$(4,3)$	1	1
6	$(3,8)$	.	1

(4.5) Φ_3-modular decomposition numbers. It turns out that the non-projective irreducible characters are $\chi_1, \chi_2, \chi_3, \chi_4, \chi_9, \chi_{10}$. Similarly to (4.4) we find the decomposition matrix using the following Φ_3-modular reductions: $\bar{x}_{\chi_1} = \bar{x}_{\chi_3} = \bar{x}_{\chi_{10}} = -1$, $\bar{x}_{\chi_2} = \bar{x}_{\chi_4} = \bar{x}_{\chi_9} = 1$.

		Λ_1	Λ_2
1	$(1,15)$	1	.
3	$(5,5)$	1	1
10	$(4,4)$	.	1

		Λ_3	Λ_4
2	$(1,0)$	1	.
4	$(5,2)$	1	1
9	$(4,3)$	.	1

(4.6) Φ_2-modular decomposition numbers. The projective irreducible characters are χ_9, χ_{10}. All the other irreducible characters are of defect 3. Now we induce the projectively indecomposable characters, see (3.3), from the parabolic subalgebra of type $I_2(5)$ up to H_3 and restrict them to the defect 3 characters. This gives the following projective characters, where Λ_1 corresponds to the projectively indecomposable character of $I_2(5)$ belonging to the block of defect 1.

		Λ_1	Λ_2	Λ_3
1	$(1,15)$	1	.	.
2	$(1,0)$	1	.	.
3	$(5,5)$	1	1	1
4	$(5,2)$	1	1	1
5	$(3,6)$	1	1	.
6	$(3,8)$	1	.	1
7	$(3,1)$	1	1	.
8	$(3,3)$	1	.	1

We look through the subsums of Λ_1, Λ_2, Λ_3, see (2.4.3), and find that Λ_2 and Λ_3 are indecomposable, whereas Λ_1 has at most one of Λ_2 or Λ_3 as a summand. The field automorphism of $\mathbb{Q}(\zeta_5)$ defined by $\zeta_5 \mapsto \zeta_5^2$ interchanges χ_5 and χ_6 leaving χ_1 fixed. Since R is fixed under this field automorphism, it follows that the Φ_2-modular reduction $\bar{\chi}_1$ of χ_1 is a constituent of $\bar{\chi}_5$ if and only if it is one of $\bar{\chi}_6$. Hence Λ_1 is also indecomposable.

5 The algebra of type H_4

(5.1) The Poincaré polynomial $P_{H_4}(u) \in \mathbb{Z}[u]$ for the generic algebra of type H_4 with parameter u is given by

$$P_{H_4} = \Phi_2^4 \cdot \Phi_3^2 \cdot \Phi_4^2 \cdot \Phi_5^2 \cdot \Phi_6^2 \cdot \Phi_{10}^2 \cdot \Phi_{12} \cdot \Phi_{15} \cdot \Phi_{20} \cdot \Phi_{30}.$$

Again all of the c_{χ_i} divide $P_{H_4}(u)$ in $\mathbb{Q}(\zeta, u, u^{-1})$. Hence we only have to consider $\Phi = \Phi_e$, where $e \in \{2, 3, 4, 5, 6, 10, 12, 15, 20, 30\}$. We will use the standard choice for R as was described in (4.1).
In the sequel we will look at the induced index representation of H_3. Using CHEVIE we find the character of the induced representation to be given as $\chi_1 + \chi_3 + \chi_5 + \chi_{11} + \chi_{13} + \chi_{18} + \chi_{20} + \chi_{27} + \chi_{31}$.

(5.2) Φ_{30}-modular decomposition numbers. It turns out that the non-projective irreducible characters are $\chi_1, \chi_2, \chi_3, \chi_4, \chi_7$, all of which are of defect 1. The rank argument as was described in (4.2) shows that these all belong to the same block. Now the defect 1 component of the induced index representation of H_3 is given by $\chi_1 + \chi_3$, see (5.1). Since Φ_{30} does not divide P_{H_3}, the induced character is projective. This determines the decomposition matrix.

1	$(1,0)$	1	.	.	.
3	$(4,1)$	1	1	.	.
7	$(6,12)$	.	1	1	.
4	$(4,31)$	.	.	1	1
2	$(1,60)$	.	.	.	1

Note that $e = 30$ equals the Coxeter number of the corresponding Coxeter group. Furthermore, χ_3 is the reflection representation, χ_7 is its exterior square, χ_4 is its exterior cube, and $sgn = \chi_2$ is its exterior fourth power. Finally, we have $ind = \chi_1$.

(5.3) Φ_{20}-modular decomposition numbers. It turns out that the non-projective irreducible characters are $\chi_1, \chi_2, \chi_{11}, \chi_{12}, \chi_{16}$. A similar argument as was used in (5.2), the defect 1 block component of the induced index representation now being given by $\chi_1 + \chi_{11}$, shows that the decomposition matrix is as follows.

1	$(1,0)$	1	.	.	.
11	$(9,2)$	1	1	.	.
16	$(16,11)$	.	1	1	.
12	$(9,22)$	.	.	1	1
2	$(1,60)$	.	.	.	1

(5.4) Φ_{15}-modular decomposition numbers. It turns out that the non-projective irreducible characters are χ_1, χ_2, χ_5, χ_6, χ_{18}, χ_{19}, χ_{20}, χ_{21}, χ_{24}, χ_{29}, again all of defect 1. Computing the scalars $\bar{x}_{\chi_i}$, see (2.2.5), gives $\bar{x}_{\chi_1} = \bar{x}_{\chi_2} = \bar{x}_{\chi_{18}} = \bar{x}_{\chi_{19}} = \bar{x}_{\chi_{29}} = 1$, $\bar{x}_{\chi_5} = \bar{x}_{\chi_6} = \bar{x}_{\chi_{20}} = \bar{x}_{\chi_{21}} = \bar{x}_{\chi_{24}} = -1$. The rank argument, see (4.2), shows that this is the block distribution of these characters. Now we again use the induced index representation, see (5.1), which gives us $(\chi_1 + \chi_{18}) + (\chi_5 + \chi_{20})$ as a projective character.

1	$(1,0)$	1	.	.	.
18	$(16,3)$	1	1	.	.
29	$(30,10)$	.	1	1	.
19	$(16,21)$	.	.	1	1
2	$(1,60)$	.	.	.	1

5	$(4,7)$	1	.	.	.
20	$(16,6)$	1	1	.	.
24	$(24,7)$	.	1	1	.
21	$(16,18)$	.	.	1	1
6	$(4,37)$	.	.	.	1

(5.5) Φ_{12}-modular decomposition numbers. It turns out that the non-projective irreducible characters are $\chi_1, \chi_2, \chi_{27}, \chi_{28}, \chi_{34}$. A similar argument as was used in (5.2), giving $\chi_1 + \chi_{27}$ as a projective character, shows that the decomposition matrix is as follows.

1	$(1,0)$	1	.	.	.
27	$(25,4)$	1	1	.	.
34	$(48,9)$	.	1	1	.
28	$(25,16)$	.	.	1	1
2	$(1,60)$	.	.	.	1

(5.6) Φ_{10}-modular decomposition numbers. It turns out that the non-projective irreducible characters are χ_1, χ_2, χ_5, χ_6, χ_{13}, χ_{14}, χ_{26}, χ_{30}, χ_{31}, χ_{32}, χ_{33}, which are of defect 2, and $\chi_3, \chi_4, \chi_{11}, \chi_{12}, \chi_{15}$, which are of defect 1. Now we use the formula, see (2.2.3), giving the centrally primitive idempotents $e_{\chi_i} \in H_K$ in terms of a pair of mutually dual bases. We find that even $\sum_{i \in \{3,4,11,12,15\}} e_{\chi_i} \in H_R$ holds. Hence both the defect 1 and the defect 2 characters form a union of blocks.

(5.6.1) The defect 1 block. The rank argument, see (4.2), shows that the defect 1 characters form one block. We now induce the projective character $\chi_2 + \chi_7$, see (4.2), from H_3 up to H_4. Its component in the defect 1 block equals $2 \cdot (\chi_3 + \chi_{11})$. This determines the decomposition matrix.

3	$(4,1)$	1	.	.	.
11	$(9,2)$	1	1	.	.
15	$(10,12)$	.	1	1	.
12	$(9,22)$	.	.	1	1
4	$(4,31)$	.	.	.	1

(5.6.2) The defect 2 block. Now we concentrate on the defect 2 characters. We first induce up all the Φ_{10}-modular projectively indecomposable characters of H_3, see (4.2), and restrict them to their defect 2 block components. This gives us a set of projective characters belonging to the defect 2 blocks. Applying the idea described in (2.4.2) we get a basis consisting of the characters coming from $\Lambda_3, \Lambda_1, \chi_8, \chi_6, \chi_4, \chi_3, \chi_9$. It is shown below in this order. Using the idea described in (2.4.3) we find that Λ_7^1 is projectively indecomposable. Furthermore, we find that all possible summands of the Λ_i^1 are again in the subgroup generated by $\Lambda^1 := \{\Lambda_i^1\}_{i=1,\ldots,7}$. Finally, we observe that the matrix in shown below is of lower unitriangular shape.

		Λ_1^1	Λ_2^1	Λ_3^1	Λ_4^1	Λ_5^1	Λ_6^1	Λ_7^1
1	$(1,0)$	1	.	.	.	.	.	.
2	$(1,60)$	.	1	.	.	.	.	.
5	$(4,7)$	1	.	1	.	.	.	.
6	$(4,37)$	.	1	.	1	.	.	.
13	$(9,6)$	1	.	1	.	1	.	.
14	$(9,26)$	.	1	.	1	.	1	.
26	$(24,6)$	1	1	.	.	1	1	1
30	$(30,10)'$	1	1	1	1	1	1	1
31	$(36,5)$	3	.	2	.	2	1	1
32	$(36,15)$	.	3	.	2	1	2	1
33	$(40,8)$	1	1	1	1	2	2	1

In application of (2.2.2) we let $l := 11$, which is fully decomposed in $\mathbb{Z}[\zeta]$, let $\wp \lhd \mathbb{Z}[\zeta]$ be a prime ideal over l, $\tilde{\Phi} := v^2 - \zeta_{10} \in R$, and $\mathfrak{P} := \langle \wp, \tilde{\Phi} \rangle \lhd R$. Hence we have $R/\mathfrak{P} \cong GF(11^2)$. Now we use the VectorEnumerator to compute the $d_{\mathfrak{P}}$-modular reduction of the induced index representation of H_3 explicitly, see (2.4.4). Applying the MeatAxe to the induced module, which is of dimension 120, shows that it has the following constituents over $GF(11^2)$: $1a^2, 4a^2, 4b^2, 5a, 9a^2, 22a, 25a, 32a$. The defect 2 block component of the corresponding character is given as $\chi_1 + \chi_5 + \chi_{13} + \chi_{31}$, see (5.1).

There is exactly one linear constituent, and it occurs with multiplicity 2. A search as is described in (2.4.3) shows that there is exactly one possible summand of Λ_1^1 which has an entry 1 in the row corresponding to χ_1, and the sum of the entries at χ_5, χ_{13}, χ_{31} is less or equal to 1. Hence the projectively indecomposable character corresponding to the Φ_{10}-modular reduction of χ_1 is uniquely determined. It is shown below as Λ_1^2. Curtis-Alvis duality then gives the projectively indecomposable character Λ_2^2.
Now that the constituent $1a$ is accounted for as the $d_{\mathfrak{P}}$-modular reduction of χ_1, it follows that the Φ_{10}-modular reduction of χ_5 occurs with multiplicity at most 2 in the induced module. The same line of reasoning as above now gives the projectively indecomposable characters Λ_3^2 and Λ_4^2.
Now combining this with the result in (5.6.1), all the constituents $1a$, $4a$, $4b$, $5a$ are accounted for as $d_{\mathfrak{P}}$-modular reductions of irreducible Φ_{10}-modular characters. The consideration of χ_{13} and the same type of argument shows that Λ_5^2 and Λ_6^2 are projectively indecomposable.

		Λ_1^2	Λ_2^2	Λ_3^2	Λ_4^2	Λ_5^2	Λ_6^2	Λ_7^2
1	$(1,0)$	1	.	.	.	.	.	.
2	$(1,60)$	.	1	.	.	.	.	.
5	$(4,7)$	.	.	1	.	.	.	.
6	$(4,37)$	.	.	.	1	.	.	.
13	$(9,6)$	.	.	.	.	1	.	.
14	$(9,26)$	.	.	.	.	.	1	.
26	$(24,6)$	1	1	.	.	.	.	1
30	$(30,10)'$	.	.	1	1	.	.	1
31	$(36,5)$	1	.	1	.	1	.	1
32	$(36,15)$	.	1	.	1	.	1	1
33	$(40,8)$	.	.	.	.	1	1	1

(5.7) Φ_6-modular decomposition numbers. It turns out that the non-projective irreducible characters are χ_1, χ_2, χ_3, χ_4, χ_5, χ_6, χ_{22}, χ_{25}, χ_{26}, χ_{27}, χ_{28}, which are all of defect 2. First we induce up all the projectively indecomposable characters of H_3, see (4.3), and we get a basis consisting of characters coming from $\Lambda_3, \Lambda_1, \chi_7, \chi_5, \chi_8, \chi_6, \chi_9$. It is shown below. We find that Λ_3^1, Λ_4^1, Λ_5^1, Λ_6^1, Λ_7^1 are projectively indecomposable. Unfortunately, Λ_1^1 and Λ_2^1 may even have summands which are not in the subgroup generated by Λ^1.

		Λ_1^1	Λ_2^1	Λ_3^1	Λ_4^1	Λ_5^1	Λ_6^1	Λ_7^1
1	$(1,0)$	1	.	.	.	.	.	.
2	$(1,60)$	.	1	.	.	.	.	.
3	$(4,1)$	1	.	1	.	.	.	.
4	$(4,31)$	.	1	.	1	.	.	.
5	$(4,7)$	1	.	.	.	1	.	.
6	$(4,37)$	.	1	.	.	.	1	.
22	$(18,10)$	1	1	.	.	.	.	1
25	$(24,12)$	1	1	.	.	1	1	1
26	$(24,6)$	1	1	1	1	.	.	1
27	$(25,4)$	3	.	1	.	1	.	1
28	$(25,16)$	.	3	.	1	.	1	1

We again compute the constituents of the induced index representation, as was described in (5.6.2), now for the case $l := 7$, which is fully ramified in $\mathbb{Z}[\zeta]$. We let $\wp := \langle 7 \rangle \lhd \mathbb{Z}[\zeta]$, $\tilde{\Phi} := v - \omega_{12} \in R$, where $\omega_{12} \in \mathbb{Z}[\zeta]$ is a preimage of a primitive 12-th root of unity in $\mathbb{Z}[\zeta]/\wp \cong GF(7^4)$. Hence we have $R/\mathfrak{P} \cong GF(7^4)$. We find: $1a^2, 8a^2, 16a, 18a, 32a, 36a$. The defect 2 block component of the character of the induced index representation of H_3 now equals $\chi_1 + \chi_3 + \chi_5 + \chi_{27}$, see (5.1). The same line of reasoning as in (5.6.2) shows that Λ_1^2 shown below is projectively indecomposable. Curtis-Alvis duality gives Λ_2^2. Finally we have $\Lambda_1^1 - \Lambda_1^2 = \Lambda_3^1 + \Lambda_5^1$. This completes the decomposition matrix.

		Λ_1^2	Λ_2^2	Λ_3^2	Λ_4^2	Λ_5^2	Λ_6^2	Λ_7^2
1	$(1,0)$	1	.	.	.	.	.	.
2	$(1,60)$	.	1	.	.	.	.	.
3	$(4,1)$	.	.	1	.	.	.	.
4	$(4,31)$	.	.	.	1	.	.	.
5	$(4,7)$	.	.	.	.	1	.	.
6	$(4,37)$	.	.	.	.	.	1	.
22	$(18,10)$	1	1	.	.	.	.	1
25	$(24,12)$	.	.	.	.	1	1	1
26	$(24,6)$	.	.	1	1	.	.	1
27	$(25,4)$	1	.	1	.	1	.	1
28	$(25,16)$	.	1	.	1	.	1	1

(5.8) Φ_5-modular decomposition numbers. It turns out that the non-projective irreducible characters are χ_1, χ_2, χ_3, χ_4, χ_8, χ_9, χ_{11}, χ_{12}, χ_{17}, χ_{18}, χ_{19}, χ_{20}, χ_{21}, χ_{24}, χ_{26}, χ_{31}, χ_{32}, χ_{34}, which are of defect 2, and χ_5, χ_6, χ_{10}, χ_{13}, χ_{14}, χ_{22}, which are of defect 1. Considering centrally primitive idempotents, see (2.2.3), and the scalars $\bar{x}_{\chi_i}$, see (2.2.5), shows that the sets of defect 2 characters χ_1, χ_2, χ_8, χ_9, χ_{11}, χ_{12}, χ_{18}, χ_{19}, χ_{26} and χ_3, χ_4, χ_{17}, χ_{20}, χ_{21}, χ_{24}, χ_{31}, χ_{32}, χ_{34} and the sets of defect 1 characters χ_5, χ_6, χ_{10} and χ_{13}, χ_{14}, χ_{22} each form a union of blocks.

(5.8.1) The defect 1 blocks. The decomposition matrices are as follows.

5	$(4,7)$	1	.
10	$(8,13)$	1	1
6	$(4,37)$	.	1

13	$(9,6)$	1	.
22	$(18,10)$	1	1
14	$(9,22)$	.	1

(5.8.2) The defect 2 blocks. First we induce all of the projectively indecomposable characters of H_3 and restrict them to their components belonging to the first defect 2 block. We find a basis coming from the following characters: $\Lambda_3, \Lambda_1, \Lambda_2, \chi_4, \chi_3$. It is shown below. An application of (2.4.3) shows that Λ_3^1, Λ_4^1, Λ_5^1 are projectively indecomposable characters. Furthermore, if Λ_1^1 were not indecomposable, its decomposition were $\Lambda_1^1 = (\Lambda_1^1 - \Lambda_4^1) + \Lambda_4^1$. An analogous situation holds for Λ_2^1 by Curtis-Alvis duality.

		Λ_1^1	Λ_2^1	Λ_3^1	Λ_4^1	Λ_5^1
1	$(1,0)$	1	.	.	.	.
2	$(1,60)$	.	1	.	.	.
8	$(6,20)$	.	.	1	.	.
9	$(8,12)$	1	1	1	.	.
11	$(9,2)$	1	.	.	1	.
12	$(9,22)$	.	1	.	.	1
18	$(16,3)$	2	.	1	1	.
19	$(16,21)$	.	2	1	.	1
26	$(24,6)$	1	1	1	1	1

Now we again induce the index representation of H_3 using the VectorEnumerator, this time letting $l := 11$, $\tilde{P} := v - \zeta_{10}$, and $R/\mathfrak{P} \cong GF(11)$. The MeatAxe finds the following constituents: $1a^2$, $4a^2$, $4b$, $6a$, $9a^2$, $9b$, $16a^2$, $16b$, $25a$. The component of the character of the induced representation belonging to the block under consideration is by (5.1) given by $\chi_1 + \chi_{11} + \chi_{18}$.

Now the $d_{\mathfrak{P}}$-modular reduction has exactly one linear constituent, and it occurs with multiplicity 2. Hence it follows that $\Lambda^2_1 := \Lambda^1_1 - \Lambda^1_4$ is a projective character, and the decomposition matrix is as follows.

		Λ^2_1	Λ^2_2	Λ^2_3	Λ^2_4	Λ^2_5
1	$(1,0)$	1	.	.	.	.
2	$(1,60)$	.	1	.	.	.
8	$(6,20)$	.	.	1	.	.
9	$(8,12)$	1	1	1	.	.
11	$(9,2)$	.	.	.	1	.
12	$(9,22)$	.	.	.	.	1
18	$(16,3)$	1	.	1	1	.
19	$(16,21)$	.	1	1	.	1
26	$(24,6)$	.	.	1	1	1

(5.8.3) Now we turn our attention to the second defect 2 block. Similarly to the case dealt with in (5.8.2) we find a basis coming from $\chi_7, \chi_5, \chi_4, \Lambda_4, \chi_3$, which is shown below. Furthermore, we find that Λ^1_5 is projectively indecomposable or it decomposes as $\Lambda^1_5 = \Lambda' + \Lambda''$, where Λ', Λ'' are also given below.

		Λ^1_1	Λ^1_2	Λ^1_3	Λ^1_4	Λ^1_5	Λ'	Λ''
3	$(4,1)$	1	.	.	.	.	.	.
4	$(4,31)$	.	1	.	.	.	.	.
17	$(16,13)$	.	.	1	1	1	1	.
20	$(16,6)$	1	.	1	.	.	.	.
21	$(16,18)$	.	1	.	2	1	.	1
24	$(24,7)$	1	1	1	1	1	1	.
31	$(36,5)$	2	.	2	1	1	1	.
32	$(36,15)$	.	2	1	3	2	1	1
34	$(48,9)$	1	1	2	3	2	1	1
	$(\Lambda_2)_{H_3}\uparrow^{H_4}$	.	.	2	1	-2		

Now the component of the induced projectively indecomposable character $(\Lambda_2)_{H_3}\uparrow^{H_4}$ belonging to the block under consideration decomposes into Λ^1 as also shown above. Hence Λ^1_5 is decomposable and Λ', Λ'' are projectively indecomposable characters. Clearly, the Curtis-Alvis dual counterpart Λ''' of Λ'' is also projectively indecomposable. We add these characters to the list Λ^1, and an application of (2.4.2) gives us the basis Λ^2 shown below, where $\Lambda^2_1 := \Lambda^1_1$, $\Lambda^2_2 := \Lambda^1_2$, $\Lambda^2_3 := \Lambda'$, $\Lambda^2_4 := \Lambda'''$, $\Lambda^2_5 := \Lambda''$.

		Λ_1^2	Λ_2^2	Λ_3^2	Λ_4^2	Λ_5^2
3	$(4,1)$	1	.	.	.	.
4	$(4,31)$	.	1	.	.	.
17	$(16,13)$	.	.	1	.	.
20	$(16,6)$	1	.	.	1	.
21	$(16,18)$	.	1	.	.	1
24	$(24,7)$	1	1	1	.	.
31	$(36,5)$	2	.	1	1	.
32	$(36,15)$	.	2	1	.	1
34	$(48,9)$	1	1	1	1	1

As was already seen above, Λ_3^2, Λ_4^2, Λ_5^2 are projectively indecomposable characters. Furthermore we find that Λ_1^2 is projectively indecomposable or it decomposes as $\Lambda_1^2 = (\Lambda_1^2 - \Lambda_4^2) + \Lambda_4^2$. An analogous situation holds for Λ_2^2 by Curtis-Alvis duality. We finally use again the $d_{\mathfrak{P}}$-modular reduction of the induced module already considered in (5.8.2). Since the occurrence of the constituent $1a$ is already accounted for, it follows that the Φ_5-modular reduction of χ_3 occurs at most with multiplicity 2 in the Φ_5-modular reduction of the induced module. Hence it follows that Λ_1^2 and Λ_2^2 are decomposable and the decomposition matrix is as follows.

		Λ_1^3	Λ_2^3	Λ_3^3	Λ_4^3	Λ_5^3
3	$(4,1)$	1	.	.	.	.
4	$(4,31)$	.	1	.	.	.
17	$(16,13)$	.	.	1	.	.
20	$(16,6)$	.	.	.	1	.
21	$(16,18)$	.	.	.	.	1
24	$(24,7)$	1	1	1	.	.
31	$(36,5)$	1	.	1	1	.
32	$(36,15)$	.	1	1	.	1
34	$(48,9)$	.	.	1	1	1

(5.9) Φ_4-modular decomposition numbers. It turns out that the non-projective irreducible characters are χ_1, χ_2, χ_{10}, χ_{11}, χ_{12}, χ_{13}, χ_{14}, χ_{23}, χ_{24}, χ_{27}, χ_{28}. All of them are of defect 2. Now Φ_4 does not divide the Poincaré polynomial P_{H_3}, hence the irreducible characters are projectively indecomposable. Inducing these up to H_4, the induced characters coming from χ_2, χ_1, χ_9, χ_4, χ_5, χ_8, χ_6 give the basis depicted below, which is by (2.4.3) shown to consist of projectively indecomposable characters.

1	$(1,0)$	1	.	.	.	.	.	.
2	$(1,60)$	.	1	.	.	.	.	.
10	$(8,13)$	.	.	1	.	.	.	.
11	$(9,2)$	1	.	.	1	.	.	.
12	$(9,22)$	.	1	.	.	1	.	.
13	$(9,6)$	1	.	.	.	.	1	.
14	$(9,26)$	.	1	.	.	.	.	1
23	$(24,11)$	.	.	1	.	.	1	1
24	$(24,7)$	.	.	1	1	1	.	.
27	$(25,4)$	1	.	1	1	.	1	.
28	$(25,16)$	.	1	1	.	1	.	1

(5.10) Φ_3-modular decomposition numbers. It turns out that the non-projective irreducible characters are χ_1, χ_2, χ_3, χ_4, χ_5, χ_6, χ_9, χ_{10}, χ_{15}, χ_{16}, χ_{17}, χ_{18}, χ_{19}, χ_{20}, χ_{21}, χ_{27}, χ_{28}, χ_{33}. All of them are of defect 2. Using the scalar technique, see (2.2.5), we find that these characters can be divided into the subsets χ_1, χ_2, χ_9, χ_{15}, χ_{18}, χ_{19}, χ_{27}, χ_{28}, χ_{33}, and χ_3, χ_4, χ_5, χ_6, χ_{10}, χ_{16}, χ_{17}, χ_{20}, χ_{21}, each of which forms a union of blocks.

(5.10.1) The first block. We induce up the projectively indecomposable characters of H_3 and find a basis consisting of the characters coming from $\chi_7, \chi_5, \chi_8, \chi_6, \Lambda_2$, which is shown below. Again these are projectively indecomposable.

3	$(4,1)$	1	.	.	.	.
4	$(4,31)$	.	1	.	.	.
5	$(4,7)$	.	.	1	.	.
6	$(4,37)$	.	.	.	1	.
10	$(8,13)$	.	.	.	.	1
16	$(16,11)$	1	1	.	.	1
17	$(16,13)$	.	.	1	1	1
20	$(16,6)$	1	.	1	.	1
21	$(16,18)$	.	1	.	1	1

(5.10.2) The second block. The same technique gives us the basis shown below, which consists of the characters coming from $\Lambda_3, \Lambda_1, \Lambda_2, \chi_7, \chi_5$.

		Λ_1^1	Λ_2^1	Λ_3^1	Λ_4^1	Λ_5^1
1	$(1,0)$	1	.	.	.	.
2	$(1,60)$	.	1	.	.	.
9	$(8,12)$	.	.	1	.	.
15	$(10,12)$	1	1	1	.	.
18	$(16,3)$	2	.	.	1	.
19	$(16,21)$	.	2	2	.	1
27	$(25,4)$	3	.	1	1	.
28	$(25,16)$	.	3	3	.	1
33	$(40,8)$	2	2	3	1	1
	$\Lambda_4 \uparrow^{H_4}$	.	.	1	2	-2

By (2.4.3) we find that Λ_5^1, Λ_4^1 are projectively indecomposable. Furthermore, the component of the induced character $\Lambda_4 \uparrow^{H_4}$ belonging to the block under consideration decomposes into Λ^1 as is also shown above. Hence, using (2.4.3), $\Lambda_3^2 := \Lambda_3^1 - 2 \cdot \Lambda_5^1$ is a projectively indecomposable character. Finally, the only possible decompositions of Λ_1^1 into projectively indecomposable ones are $\Lambda_1^1 = (\Lambda_1^1 - c \cdot \Lambda_4^1) + c \cdot \Lambda_4^1$ where $c \in \{0, 1, 2\}$.
Now we again induce the index representation of H_3 using the VectorEnumerator, with $l := 7$ and $\tilde{P} := v - \omega_6$, where $\omega_6 \in \mathbb{Z}[\zeta]$ is a preimage of a primitive 6-th root of unity in $\mathbb{Z}[\zeta]/\wp \cong GF(7^4)$. The MeatAxe finds the following constituents: $1a^2, 8a^2, 8b, 8c, 16a^2, 18a, 36a$. The character of the component of the induced module belonging to the present block is by (5.1) given as $\chi_1 + \chi_{18} + \chi_{27}$. As its $d_{\mathfrak{P}}$-modular reduction has exactly one linear constituent, which occurs with multiplicity 2, it follows that $\Lambda_1^2 = \Lambda_1^1 - 2 \cdot \Lambda_4^1$ and, by Curtis-Alvis duality, $\Lambda_2^2 = \Lambda_2^1 - 2 \cdot \Lambda_5^1$ are projective characters. The decomposition matrix is shown below.

		Λ_1^2	Λ_2^2	Λ_3^2	Λ_4^2	Λ_5^2
1	$(1,0)$	1	.	.	.	.
2	$(1,60)$	.	1	.	.	.
9	$(8,12)$	.	.	1	.	.
15	$(10,12)$	1	1	1	.	.
18	$(16,3)$	.	.	.	1	.
19	$(16,21)$	.	.	.	.	1
27	$(25,4)$	1	.	1	1	.
28	$(25,16)$	.	1	1	.	1
33	$(40,8)$	.	.	1	1	1

(5.11) Φ_2-modular decomposition numbers. It turns out that the non-projective irreducible characters are χ_1, χ_2, χ_3, χ_4, χ_5, χ_6, χ_7, χ_8, χ_{11}, χ_{12}, χ_{13}, χ_{14}, χ_{15}, χ_{22}, χ_{27}, χ_{28}, χ_{29}, χ_{30}, χ_{31}, χ_{32}, which are of defect 4, and χ_{18}, χ_{19}, χ_{20}, χ_{21}, which are of defect 1. The consideration of centrally primitive idempotents, see (2.2.3), shows that the characters of defect 4 form a union of blocks.

(5.11.1) The defect 1 blocks. The same argument shows that the sets χ_{18}, χ_{21}, and χ_{19}, χ_{20} form blocks, whose decomposition matrices are as follows.

18	$(16,3)$	1
21	$(16,18)$	1

19	$(16,21)$	1
20	$(16,6)$	1

(5.11.2) The defect 4 block. Inducing up the projectively indecomposable characters of H_3 gives a basis consisting of the characters coming from $\Lambda_1, \Lambda_2, \Lambda_3, \chi_9$, which is shown below. By (2.4.3) we find that Λ_4^1 is projectively indecomposable.

		Λ_1^1	Λ_2^1	Λ_3^1	Λ_4^1
1	$(1,0)$	1	.	.	.
2	$(1,60)$	1	.	.	.
3	$(4,1)$	2	1	.	.
4	$(4,31)$	2	1	.	.
5	$(4,7)$	2	.	1	.
6	$(4,37)$	2	.	1	.
7	$(6,12)$	2	2	.	.
8	$(6,20)$	2	.	2	.
11	$(9,2)$	3	2	1	.
12	$(9,22)$	3	2	1	.
13	$(9,6)$	3	1	2	.
14	$(9,26)$	3	1	2	.
15	$(10,12)$	2	2	2	.
22	$(18,10)$	2	2	2	1
27	$(25,4)$	5	3	3	1
28	$(25,16)$	5	3	3	1
29	$(30,10)$	6	4	4	1
30	$(30,10)'$	6	4	4	1
31	$(36,5)$	8	5	5	1
32	$(36,15)$	8	5	5	1

(5.11.3) In the case of the decomposition map d_{Φ_2}, we can improve the algorithm described in (2.4.3) by taking into account that the Φ_2-modular decomposition numbers of characters which are conjugate to each other by Curtis-Alvis duality are equal. The orbits of length two are $\chi_1 \leftrightarrow \chi_2$, $\chi_3 \leftrightarrow \chi_4$, $\chi_5 \leftrightarrow \chi_6$, $\chi_{11} \leftrightarrow \chi_{12}$, $\chi_{13} \leftrightarrow \chi_{14}$, $\chi_{27} \leftrightarrow \chi_{28}$, $\chi_{31} \leftrightarrow \chi_{32}$. Furthermore, the field automorphism of $\mathbb{Q}(\zeta_5)$ defined by $\zeta_5 \mapsto \zeta_5^2$ has the following orbits of length two on the irreducible characters of defect 4: $\chi_3 \leftrightarrow \chi_5$, $\chi_4 \leftrightarrow \chi_6$, $\chi_7 \leftrightarrow \chi_8$, $\chi_{11} \leftrightarrow \chi_{13}$, $\chi_{12} \leftrightarrow \chi_{14}$, $\chi_{29} \leftrightarrow \chi_{30}$. The search for the projectively indecomposable character corresponding to the Φ_2-modular reduction $\bar{\chi}_1$ can even be further improved by using the fact that its decomposition into irrreducible characters gives multiplicities which are constant on the above orbits.

(5.11.4) Letting $l := 7$, $\tilde{\Phi} := v - \omega_4 \in R$, where $\omega_4 \in \mathbb{Z}[\zeta]$ is a preimage of a primitive 4-th root of unity in $\mathbb{Z}[\zeta]/\wp \cong GF(7^4)$, we again induce the index representation of H_3. The MeatAxe finds the following constituents: $1a^2, 8a^4, 10a^2, 17a^2, 32a$. The character of the component of the induced module belonging to the present block is by (5.1) given as $\chi_1+\chi_3+\chi_5+\chi_{11}+\chi_{13}+\chi_{27}+\chi_{31}$. Since the multiplicity of the linear constituent $1a$ equals 2, the projectively indecomposable character corresponding to $\bar{\chi}_1$ is an element of the set of all subsums of Λ_1^1 such that the criteria given in (5.11.3) hold, the multiplicity of χ_1 equals 1, and the sum of the multiplicities of χ_3, χ_5, χ_{11}, χ_{13}, χ_{27}, and χ_{31} is less than 2. This set consists of the subsums Λ_1^2, Λ_1' shown below.

		Λ_1^2	Λ_1'	Λ_2^2	Λ_3^2	Ψ_1	Ψ_2	Ψ_3	Λ_4^2	Λ_4'
1	$(1,0)$	1	1	.	.	.	.	.	.	.
2	$(1,60)$	1	1	.	.	.	.	.	.	.
3	$(4,1)$	.	.	1	.	.	.	.	.	.
4	$(4,31)$	.	.	1	.	.	.	.	.	.
5	$(4,7)$	.	.	.	1	.	.	.	.	.
6	$(4,37)$	.	.	.	1	.	.	.	.	.
7	$(6,12)$	1	.	.	.	1	2	.	1	1
8	$(6,20)$	1	.	.	.	1	.	2	.	.
11	$(9,2)$	.	.	.	1	1	2	.	1	.
12	$(9,22)$	.	.	.	1	1	2	.	1	.
13	$(9,6)$	.	.	1	.	1	.	2	.	1
14	$(9,26)$	.	.	1	.	1	.	2	.	1
15	$(10,12)$	.	2	.	.	2	2	2	1	1
22	$(18,10)$	1	1	.	.	1	2	2	.	.
27	$(25,4)$	.	.	1	1	1	2	2	.	.
28	$(25,16)$	.	.	1	1	1	2	2	.	.
29	$(30,10)$	.	2	.	2	2	4	2	1	.
30	$(30,10)'$	.	2	2	.	2	2	4	.	1
31	$(36,5)$	1	1	1	1	3	4	4	1	1
32	$(36,15)$	1	1	1	1	3	4	4	1	1

Now we induce the reflection representation χ_7 of H_3 up to H_4, using the VectorEnumerator, with $l := 11$, $\tilde{\Phi} := v^2 + 1 \in R$, hence $R/\mathfrak{P} \cong GF(11^2)$. The MeatAxe finds the following constituents of the induced module M, which is of dimension 360: $1a^3, 4a^6, 4b^6, 5a^5, 5b^3, 16a, 17a^5, 24a, 24b, 32a, 40a, 48a$. The character of the induced representation is given as $\chi + \chi'$, where $\chi := \chi_3 + \chi_7 + \chi_{11} + \chi_{27} + \chi_{29} + \chi_{30} + 2 \cdot \chi_{31}$ and $\chi' := \chi_{18} + \chi_{20} + \chi_{16} + \chi_{24} + \chi_{26} + \chi_{33} + \chi_{34}$ are the components of the character belonging to the block under consideration respectively not belonging to it.
The multiplicity of the linear constituent of M shows that Λ_1^2 is the projectively indecomposable characacter corresponding to $\bar{\chi}_1$. The list of constituents of M also shows that the Φ_2-modular reduction $\bar{\chi}_3$ of χ_3 is irreducible. By the field automorphism the reduction $\bar{\chi}_5$ of χ_5 is also irreducible. Now we compute all subsums of Λ_2^1 such that the criteria described in (5.11.3) hold, the multiplicity of χ_3 equals 1, and the sum of the multiplicities of χ_7, χ_{11}, χ_{27}, χ_{29}, χ_{30}, plus twice the multiplicity of χ_{31} is less than 6. It turns out, that this subsum, Λ_2^2 above, is uniquely determined,

which hence is the projectively indecomposable character corresponding to $\bar{\chi}_3$. Applying the field automorphism gives the projectively indecomposable character Λ_3^2 corresponding to $\bar{\chi}_5$.
Furthermore, we have the following projective characters

$$\Psi_1 := \Lambda_1^1 - \Lambda_1^2 - 2 \cdot \Lambda_2^2 - 2 \cdot \Lambda_3^2, \Psi_2 := \Lambda_2^1 - \Lambda_2^2, \Psi_3 := \Lambda_3^1 - \Lambda_3^2.$$

Now the list of constituents of M shows that the Φ_2-modular reduction $\bar{\chi}_7$ of χ_7 decomposes into irreducibles as $\bar{\chi}_7 = \bar{\chi}_1 + \varphi$. To find the projectively indecomposable character corresponding to φ, we search for subsums of Ψ_1 such that again the criteria of (5.11.3) hold, the multiplicity of χ_7 equals 1, and the sum of the multiplicities of χ_{11}, χ_{27}, χ_{29}, χ_{30}, plus twice the multiplicity of χ_{31} is less than 5. We find two candidates, Λ_4^2, Λ_4' above. But the projectively indecomposable character corresponding to φ is also a subsum of Ψ_2, which excludes Λ_4'. Finally, the field automorphism gives the projectively indecomposable character Λ_5^2.
Finally, we let $\Lambda_6^2 := \Lambda_4^1$, which gives us the $\mathbb{Z}$-linear independent set shown below. We find that all the induced projectively indecomposable characters of H_3 decompose into this set with nonnegative integral coefficients, hence this is indeed the decomposition matrix.

		Λ_1^2	Λ_2^2	Λ_3^2	Λ_4^2	Λ_5^2	Λ_6^2
1	$(1,0)$	1	.	.	.	.	.
2	$(1,60)$	1	.	.	.	.	.
3	$(4,1)$	.	1	.	.	.	.
4	$(4,31)$	.	1	.	.	.	.
5	$(4,7)$	.	.	1	.	.	.
6	$(4,37)$	.	.	1	.	.	.
7	$(6,12)$	1	.	.	1	.	.
8	$(6,20)$	1	.	.	.	1	.
11	$(9,2)$	.	.	1	1	.	.
12	$(9,22)$	.	.	1	1	.	.
13	$(9,6)$	.	1	.	.	1	.
14	$(9,26)$	.	1	.	.	1	.
15	$(10,12)$	.	.	.	1	1	.
22	$(18,10)$	1	.	.	.	.	1
27	$(25,4)$	.	1	1	.	.	1
28	$(25,16)$	.	1	1	.	.	1
29	$(30,10)$	.	.	2	1	.	1
30	$(30,10)'$	.	2	.	.	1	1
31	$(36,5)$	1	1	1	1	1	1
32	$(36,15)$	1	1	1	1	1	1

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