Applications: Kuramoto-Sivashinsky and Kot-Schaffer

The Kuramoto-Sivashinski equation

$$u_t = -\nu u_{xxxx} - u_{xx} + 2uu_x$$

on $[0, \infty) \times (-\pi, \pi)$ with periodic boundary conditions: $u(t, -\pi) = u(t, \pi)$.

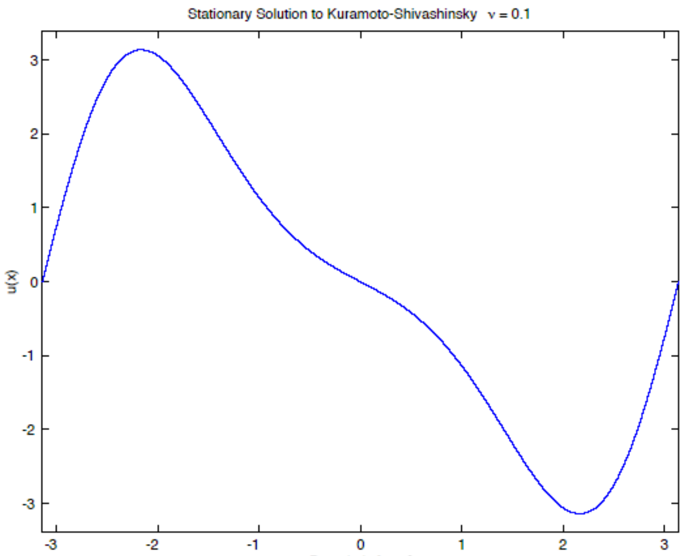
- Jolly, Kevrekidis and Titi, '90: bifurcation diagram for $\nu \in (0.057, \infty)$ for a 12 mode Galerkin approximation;
- here: rigorous proof of existence and localization of several of these equilibria.

Prototype Result

Theorem 1 (Mischaikow,Zgliczyński,01) *Let* $u(x) = \sum_{k=1}^{28} a_k \sin(kx)$, *with the* a_k *as below. Then, for* $\nu = 0.1$ *there exists an equilibrium* $u^*(x)$ *for the KS-equation such that*

$$\|u^* - u\|_{L^2} < 2.8 \cdot 10^{-13}, \qquad \|u^* - u\|_{C^0} < 2.1 \cdot 10^{-13}.$$

$a_1 = 1.07934 \times 10^{-37}$	$a_2 = 1.25665$	$a_3 = -1.92524 \times 10^{-37}$
$a_4 = -0.559867$	$a_5 = 7.81863 \times 10^{-38}$	$a_6 = 0.0881138$
$a_7 = -1.56596 \times 10^{-38}$	$a_8 = -0.0122945$	$a_9 = 2.54974 \times 10^{-39}$
$a_{10} = 0.00143504$	$a_{11} = -3.4963 \times 10^{-40}$	$a_{12} = -0.000156065$
$a_{13} = 4.35072 \times 10^{-41}$	$a_{14} = 1.59816 \times 10^{-05}$	$a_{15} = -5.02979 \times 10^{-42}$
$a_{16} = -1.57158 \times 10^{-06}$	$a_{17} = 5.50953 \times 10^{-43}$	$a_{18} = 1.49677 \times 10^{-07}$
$a_{19} = -5.62586 \times 10^{-44}$	$a_{20} = -1.39049 \times 10^{-08}$	$a_{21} = -8.26547 \times 10^{-45}$
$a_{22} = 1.26591 \times 10^{-09}$	$a_{23} = 1.30584 \times 10^{-43}$	$a_{24} = -1.13347 \times 10^{-10}$
$a_{25} = -9.46577 \times 10^{-43}$	$a_{26} = 1.0008 \times 10^{-11}$	$a_{27} = 1.1614 \times 10^{-40}$
$a_{28} = -8.73294 \times 10^{-13}$		



Line of reasoning

- Consider the evolution equation

  $$\dot{u} = F(u)$$

  on some Hilbert space H.

- Decompose

  $$H = X_m \oplus Y_m,$$

  with $\dim X_m < \infty$.

- Let $P_m : H \to X_m$ and $Q_m : H \to Y_m$ be the associated orthogonal projections.

- With $p = P_m u$ and $q = Q_m u$ we rewrite the system as

$$\begin{aligned}\dot{p} &= P_m F(p,q)\\ \dot{q} &= Q_m F(p,q).\end{aligned}$$

- Consider a restricted domain $W \oplus V \subset X_m \oplus Y_m$.
- Draw conclusions about the dynamics of the differential inclusion

$$\dot{p} \in P_m F(p,V)$$

  by using Conley index arguments.
- Lift the information to the original system (by e.g. using compactness/continuity arguments).

Restricted domain

- Consider a complete orthogonal basis $(\varphi_k)_{k\in\mathbb{N}}$ of H.
- Let $X_m = \mathrm{span}\{\varphi_0,\dots,\varphi_{m-1}\}$ and $A_k : H \to \mathrm{span}\{\varphi_k\}$.
- Let $W \subset X_m$ be compact, $a_k^-, a_k^+ \in \mathbb{R}$, $k = 0, 1, \dots$ and

$$V = \prod_{k=m}^{\infty} [a_k^-, a_k^+].$$

- The bounds W and $\{a_k^\pm\}$ are self-consistent, if
  (i) $a_k^- < 0 < a_k^+$ for $k > M$;
  (ii) $u = \sum_k a_k \varphi_k \in H$ if $a_k \in [a_k^-, a_k^+]$ for all k.
  (iii) F is continuous on $W \oplus V$.

Equivalent countable system

If W and $\{a_k^\pm\}$ are self-consistent bounds, $W \oplus V$ is compact and a function $u : [0,T] \to W \oplus V$,

$$u(t) = \sum_{k=0}^{\infty} u_k(t)\varphi_k,$$

solves $\dot{u} = F(u)$, iff it solves

$$\dot{u}_k = A_k F(u)$$

on $[0,T]$ for all k.

Countable system for Kuramoto-Sivashinsky

- We consider

$$H = \{u \in L^2(-\pi,\pi) \mid u(t,-\pi) = u(t,\pi), u(t,-x) = -u(t,x)\}.$$

- Fourier expansion of $u \in H$:

$$u(t,x) = \sum_{k\in\mathbb{Z}} b_k(t)\exp(ikx),$$

which yields

$$\dot{b}_k = (k^2 - \nu k^4)b_k + ik\sum_{m\in\mathbb{Z}} b_m b_{k-m}, \quad k \in \mathbb{Z}.$$

- Since $u \in H$ is real-valued, $b_k = \overline{b_{-k}}$.

- Since $u \in H$ is odd, $b_k = ia_k$.
- Thus $a_k = -a_{-k}$, $a_0 = 0$ and we arrive at

$$\dot{a}_k = k^2(1-\nu k^2)a_k - k\sum_{n=1}^{k-1} a_n a_{k-n} + 2k\sum_{n=1}^{\infty} a_n a_{n+k},$$

$k = 1, 2, \ldots$.

Isolating blocks

- $\varphi : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m$ continuous flow, generated by $\dot{z} = f(z)$.
- If $N \subset \mathbb{R}^m$ is a compact set such that

$$\mathrm{Inv}(N, \varphi) \subset \mathrm{int}\, N,$$

  then N is an isolating neighborhood.
- If in addition for any $z \in \partial N$ there exists $t_z > 0$ such that

$$\varphi((0, t_z), z) \cap N = \emptyset \quad \text{or} \quad \varphi((-t_z, 0), z) \cap N = \emptyset,$$

  then N is an isolating block.

Local sections

- Isolating blocks can be constructed via local sections:
- $\Xi \subset \mathbb{R}^m$ is a local section for φ, if

$$\varphi : (-\varepsilon, \varepsilon) \times \Xi \to \varphi((-\varepsilon, \varepsilon), \Xi)$$

  is a homeomorphism and $\varphi((-\varepsilon, \varepsilon), \Xi)$ is open.
- **Example**: hypersurface Ξ which is transversal to the flow, i.e. for each $z \in \Xi$,

$$n(z) \cdot f(z) \neq 0,$$

  where $n(z)$ is a normal vector at $z \in \Xi$.

Isolating blocks for linear systems

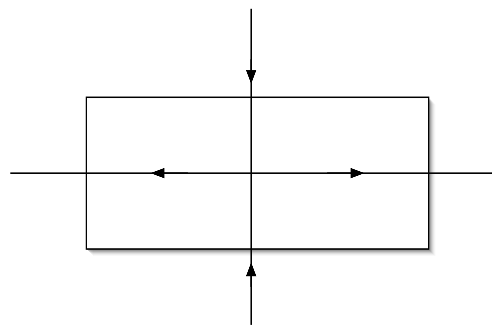
Consider

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

with $\lambda_1, \lambda_2 \neq 0$. Then

$$[a_1^-, a_1^+] \times [a_2^-, a_2^+]$$

with $a_i^- < 0 < a_i^+$ is an isolating block.



Robust isolating blocks

Consider the nonlinear, perturbed system

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} f_1(z) \\ f_2(z) \end{bmatrix} + \begin{bmatrix} \varepsilon_1(z) \\ \varepsilon_2(z) \end{bmatrix} \tag{1}$$

where $|f_i|(z) = \mathcal{O}(\|z\|^2)$ and $\max_{z\in N} |\varepsilon_i(z)| \le c_i$.

If

$$\lambda_i a_i^{\pm} + f_i(z) + \varepsilon_i(z) \tag{2}$$

has the same sign as $\lambda_i a_i^{\pm}$ on the sets $\{z \in N, z_i = a_i^{\pm}\}$, then N is an isolating block for (1).

(2) $\rightsquigarrow$ system of inequalities

Conley index

- Let N be an isolating block for φ. Let L be the closed subset of ∂N such that for all $z \in L$

$$\varphi((0,\varepsilon),z) \cap N = \emptyset$$

for a sufficiently small $\varepsilon > 0$. The Conley index of $\mathrm{Inv}(N,\varphi)$ is

$$CH_*(\mathrm{Inv}(N,\varphi)) = H_*(N,L).$$

- McCord, 88: If the Conley index has the form

$$CH_j(\mathrm{Inv}(N,\varphi)) \cong \begin{cases} \mathbb{Z} & \text{if } j = q \\ 0 & \text{otherwise,} \end{cases}$$

for some q, then N contains a fixed point.

Lifting to higher order modes

- **Idea**: construct isolating block N for the m-mode system

$$\dot{p} = P_m F(p,q) \tag{3}$$

  such that it is robust for all $q \in V$.

- **Definition**: The compact sets $N \subset W$ and the bounds $\{a_k^\pm\}$ are topologically self-consistent, if W and $\{a_k^\pm\}$ are self-consistent

  (i) for $u \in W \oplus V$ and $k > m$

$$A_k F(u) < 0 \quad \text{if } A_k u = a_k^+,$$
$$A_k F(u) > 0 \quad \text{if } A_k u = a_k^-,$$

  (ii) and N is an isolating block for (3) for all $q \in V$.

Lifting to higher order modes

Let $N \subset W$ and $\{a_k^\pm\}$ be topologically self-consistent. Consider

$$\hat{N} = N \times \prod_{k=m+1}^{r} [a_k^-, a_k^+].$$

Then $\hat{N}$ is an isolating block for the system

$$\dot{a}_k = A_k F\left(\sum_{i=1}^{r} a_i \varphi_i\right), \quad k = 1, \dots, r,$$

and

$$CH_*(\mathrm{Inv}(\hat{N})) \cong CH_*(\mathrm{Inv}(N)).$$

Lifting to higher order modes

Theorem 2 (Mischaikow,Zgliczyński,01) *Let* $N \subset W$ *and* $\{a_k^\pm\}$ *be topologically self-consistent. Suppose that*

$$CH_j(\mathrm{Inv}(N)) \cong \begin{cases} \mathbb{Z} & \text{if } j = q \\ 0 & \text{otherwise}, \end{cases}$$

for some q*, then there exists a fixed point*

$$u^* \in N \times V$$

for the partial differential equation $\dot{u} = F(u)$*.*

Estimates

(i) $1 \leq k \leq m$: actual variables,

$$W = \prod_{k=1}^{m} [a_k^-, a_k^+];$$

(ii) $m < k \leq M$: explicit bounds (intervals)

(iii) $M < k$: asymptotic bounds,

$$[a_k^-, a_k^+] = \frac{C}{k^s}[-1, 1]$$

for some $C > 0$ and some integer $s > 1$.

Prototype estimates

- For $1 \le k \le m$,

$$\sum_{n=m-k+1}^{\infty} a_n a_{n+k} \subset \sum_{n=m-k+1}^{M-k} a_n a_{n+k} + C \sum_{n=M-k+1}^{\infty} \frac{|a_n|}{(k+n)^s}[-1,1] + \frac{C^2}{(k+M+1)^s (s-1) M^{s-1}}[-1,1]$$

- For $k > M$

$$\sum_{n=1}^{\infty} a_n a_{n+k} \subset \frac{C}{k^{s-1}(M+1)} \left(\frac{C}{(M+1)^{s-1}(s-1)} + \sum_{n=1}^{M} |a_n] \right) [-1,1]$$

Example

For $\nu = 0.75$ and $m = 2$ we obtain the Galerkin system

$$\begin{aligned}\dot{a}_1 &= \frac{1}{4}a_1 + 2a_1a_2 \\ \dot{a}_2 &= -8a_2 - 2a_1^2.\end{aligned}$$

Fixed points are $\bar{a}^{\pm} = (\pm\frac{1}{\sqrt{2}}, -\frac{1}{8})$.

The full equations reads

$$\begin{aligned}\dot{a}_1 &= \frac{1}{4}a_1 + 2\sum_{n=1}^{\infty} a_n a_{n+1} \\ \dot{a}_2 &= -8a_2 - 2a_1^2 + 4\sum_{n=1}^{\infty} a_n a_{n+2}.\end{aligned}$$

We choose

$$W = \bar{a}^+ + [-0.1, 0.1] \times [-0.1, 0.1]$$

and suitable bounds $a_k^\pm$, in particular

$$[a_k^-, a_k^+] = \frac{10285.3}{k^{10}}[-1, 1]$$

for $k > 10$.

By estimating the contributions of the neglected modes we obtain the inclusion

$$\begin{aligned} \dot{a}_1 &\in \frac{1}{4}a_1 + 2a_1a_2 + \varepsilon_1 \\ \dot{a}_2 &\in -8a_2 - 2a_1^2 + \varepsilon_2. \end{aligned}$$

with

$$\begin{aligned}\varepsilon_1 &= [-1 \cdot 10^{-2}, 8 \cdot 10^{-10}]\\ \varepsilon_2 &= [-2 \cdot 10^{-8}, 7 \cdot 10^{-2}].\end{aligned}$$

infinite dimensional map

$\downarrow$ Galerkin + truncation estimate

finite dimensional multivalued map

$\downarrow$ spatial discretization (GAIO)

combinatorial multivalued map (directed graph)

$\downarrow$ graph algorithms

combinatorial index pair

$\downarrow$ computational homology (CHomP)

Conley index for finite dimensional continuous selector

$\downarrow$ lifting

Conley index for original map

The map

The Kot-Schaffer growth-dispersal model for plants:

$$\Phi : L^2 \to L^2, \quad \Phi(a)(y) = \frac{1}{2\pi} \int_{-\pi}^{\pi} b(x,y) \;\; \mu \, a(x) \left(1 - \frac{a(x)}{c(x)}\right) dx,$$

$a, b, c \in L^2([-\pi, \pi]), \mu > 0, b(x,y) = b(x-y)$.

Equivalent countable system

Using a basis of Fourier-modes $\varphi_k = \exp(ik\cdot)$ for L^2 one obtains the countable system of maps:

$$f_k(a) = \mu b_k \left[a_k - \sum_{j+l+n=k} c_j a_l a_n \right], \quad k \in \mathbb{Z},$$

a_k, b_k, c_k Fourier coefficients of a, b, c^{-1}.

Regularity of the solution

$$|\langle \Phi(a), \varphi_k \rangle| \leq C_{g,a} |b_k|$$

Line of reasoning

- Let $P_m : L^2 \to X_m = \operatorname{span}\{\varphi_0, \ldots, \varphi_{m-1}\}$ be the projection onto the first m modes and consider the finite dimensional map

$$f^{(m)} : X_m \to X_m, \quad f^{(m)} = P_m \circ f;$$

- What is the relation between the dynamics of f and of $f^{(m)}$?
- Write $f(a) = f(P_m a) + (f(a) - f(P_m a))$ and suppose that we can bound $f(a) - f(P_m a)$ on a compact subset

$$Z = W \times V, \quad W \subset X_m,$$

of L^2:

$$|f(a) - f(P_m a)| < \varepsilon^{(m)} \quad \text{for all } a \in Z.$$

- Now consider a *multivalued* map $F^{(m)} : W \rightrightarrows X_m$ with the property that for all $a \in Z$

$$P_m f(a) \in F^{(m)}(P_m a).$$

- Compute objects of interest for $F^{(m)}$ via a rigorous set-oriented approach in combination with the Conley-index theory:
  - cover the maximal invariant set of $F^{(m)}$ in W;
  - compute approximate locations of objects of interest (periodic points, connecting orbits, chain recurrent sets);
  - construct a corresponding index pair;
  - compute its Conley index;
- Lift the information on $F^{(m)}$, resp. $f^{(m)}$, to the full system Φ.

Finite dimensional multivalued map

$$F_k^{(m)}(a_0, \ldots, a_{m-1}) = \mu b_k \left[a_k - \sum_{\substack{j+l+n=k \\ 0 \le j,l,n \le m-1}} c_j a_l a_n \right] + \varepsilon_k^{(m)} [-1, 1],$$

$k = 0, 1, \ldots, m-1.$

The error $\varepsilon_k^{(m)}$ has been computed in such a way that

$$\left| f_k(a) - f_k^{(m)}(a_0, \ldots, a_{m-1}) \right| \in \varepsilon_k^{(m)}[-1, 1]$$

for all a in some compact set $Z = W \times V \subset L^2$.

Computing $\mathcal{F}$

- Write
$$f(x+h) = f(x) + Df(x)h + f^{nl}(x,h).$$
- For the box $B = B(c,r) \in \mathcal{B}$ (c: center, r: radius) compute $\varepsilon^{nl}(c)$ such that
$$\max_{|h|\leq r} \left| f^{nl}(c,h) \right| \leq \varepsilon^{nl}(c)$$
- For $x \in B$ set
$$F^{(m)}(x) = B(f(c), |Df(c)|r + \varepsilon^{nl}(c) + \varepsilon^{(m)})$$
- Finally define
$$\mathcal{F}(B(c,r)) = \{B' \in \mathcal{B} \mid F(c) \cap B' \neq \emptyset\}.$$

- Note: the set $\mathcal{F}(B)$ can be determined by a single depth first search of the tree:

$\mathcal{F}$ = $\texttt{cap}(B, C, k)$

$\quad$ $\texttt{if}$ $B \cap C \neq \emptyset$

$\quad\quad$ $\texttt{if depth}(B) = k$

$\quad\quad\quad$ $\mathcal{F} := \mathcal{F} \cup \{B\}$

$\quad\quad$ $\texttt{else}$

$\quad\quad\quad$ $\mathcal{F} := \mathcal{F} \cup \texttt{cap}(B^+, C, k) \cup \texttt{cap}(B^-, C, k)$

$\quad$ $\texttt{return}$ $\mathcal{F}$

Control of round off via interval arithmetic (BIAS, Profil, b4m, GAIO);

Lifting to the full system

The compact set $Z = W \times V \subset L^2$ is of the form

$$Z = \prod_{k=0}^{\infty} [a_k^-, a_k^+].$$

Theorem 3 *Let $I^{(m)}$ be an isolating neighborhood for $F^{(m)}$. If*

$$f_k(Z) \subset (a_k^-, a_k^+), \quad k \geq m,$$

then

$$I = I^{(m)} \times \prod_{k=m}^{\infty} [a_k^-, a_k^+]$$

is an isolating neighborhood for Φ. In particular, the Conley index for a corresponding index pair is the same as for $I^{(m)}$.

Truncation estimates

Consider a polynomial nonlinearity $c(x)a(x)^p$ in Φ. The corresponding terms in the associated countable system read

$$a_k \mapsto \sum_{n_0,\ldots,n_{p-1}\in\mathbb{Z}} c_{n_0} a_{n_1} \ldots a_{n_{p-1}} a_{k-(n_0+\cdots+n_{p-1})}.$$

Regularity assumptions. Suppose that for some constants $A, B, C > 0$, $b, s > 1$, $|a_k| \le \frac{A}{s^{|k|}}, \quad |b_k| \le \frac{B}{b^{|k|}}, \quad |c_k| \le \frac{C}{s^{|k|}}, \quad k \in \mathbb{Z}$, then

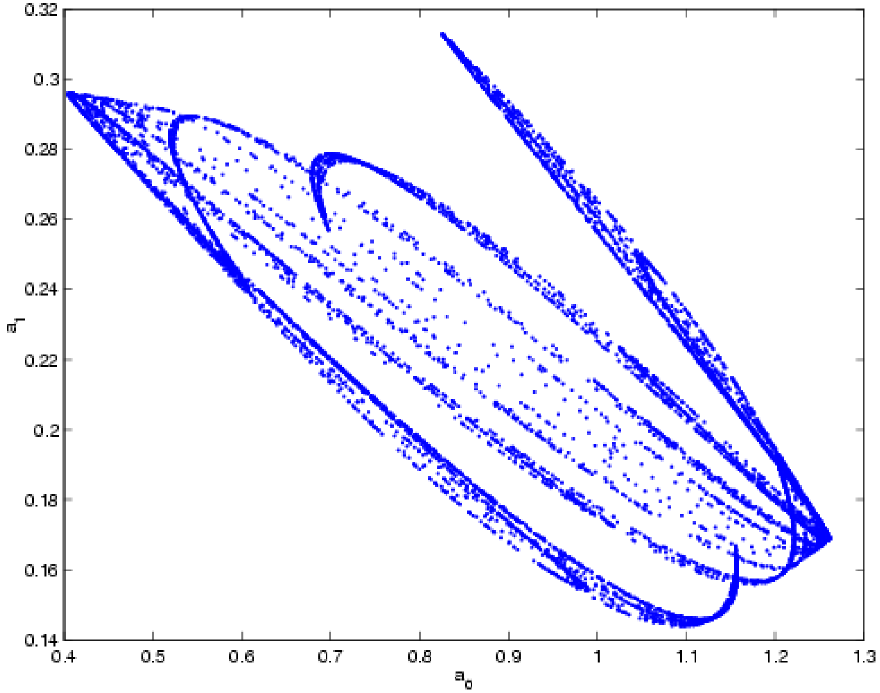
$$\Big|\sum_{n_1,\ldots,n_{p-1}\in\mathbb{Z}} c_{n_0} a_{n_1} \ldots a_{n_{p-1}} a_{k-(n_1+\ldots+n_{p-1})}\Big| \le \frac{\alpha^p A^p C}{s^{|k|}} \left(\frac{b}{\beta}\right)^{|k|}$$

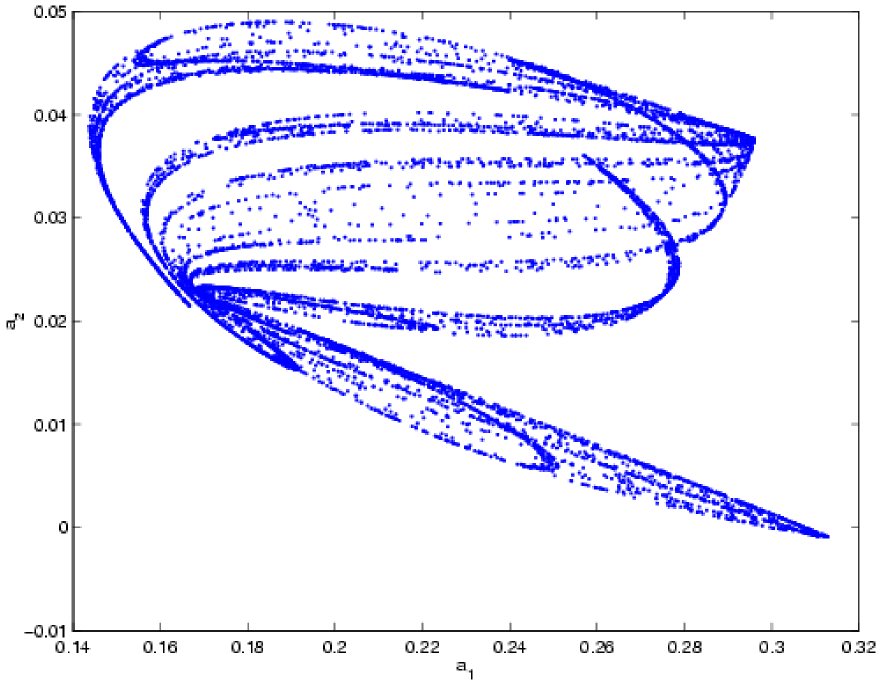
where β is such that $b/s < \beta < b$ and $\alpha = \alpha(s, b, \beta)$.

Example computation

We consider the parameters $\mu = 3.5, b_k = 2^{-k}, c_0 = 0.8, c_1 = -0.2$ and $c_k = 0$ for $k > 1$.

(i) Running a simulation for $m = 50$:

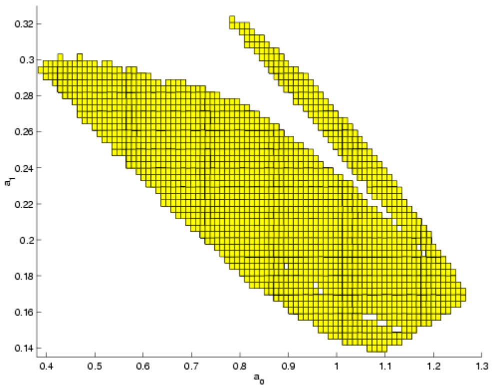


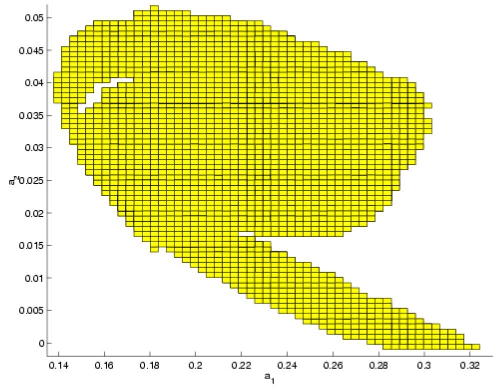


(ii) $\rightsquigarrow$ exponential estimate for the a_k; initial bounds:

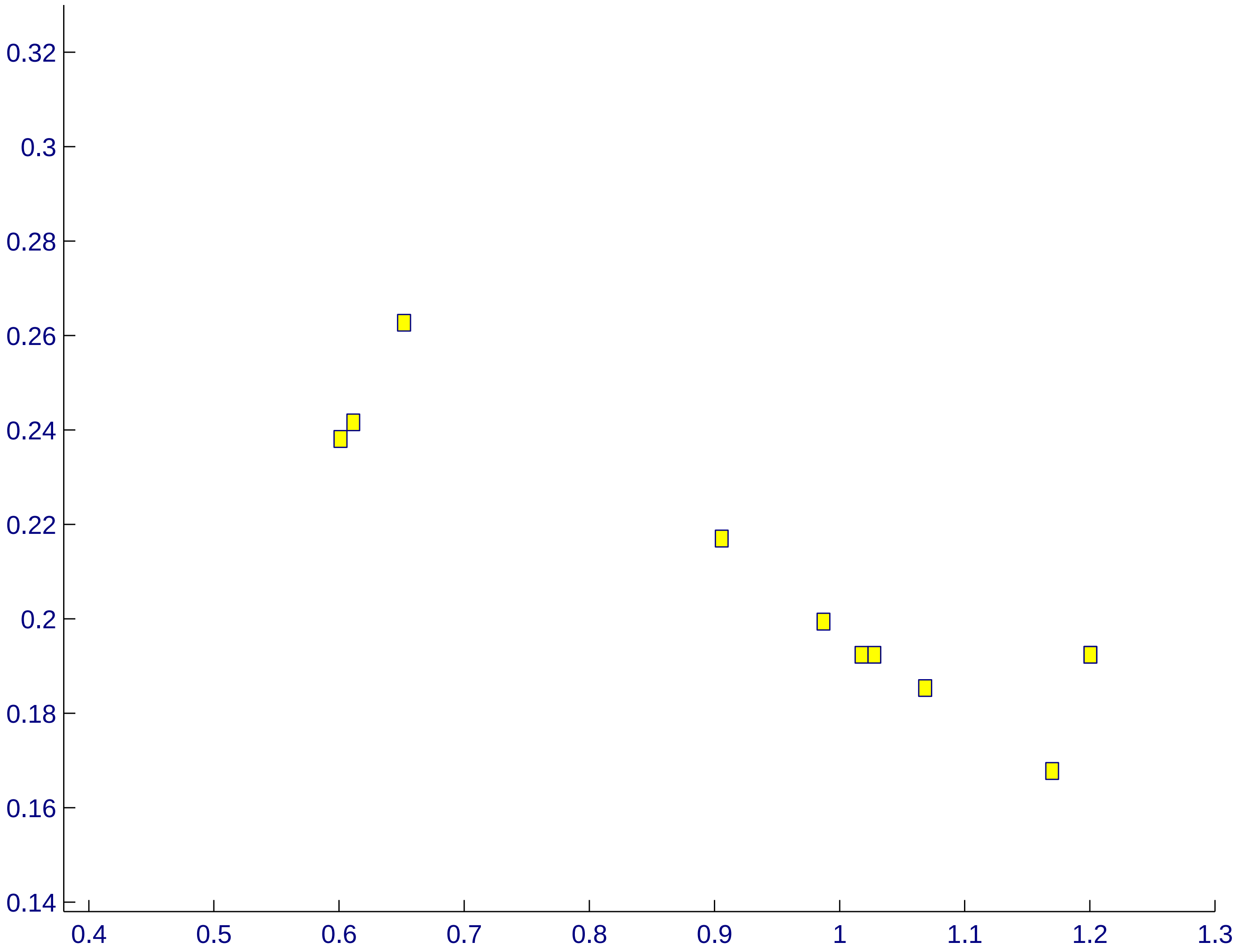
k	a_k^-	a_k^+
0	0.2	1.5
1	0.05	0.5
2	-0.001	0.1
$2 < k < M$	-2^{-k}	2^{-k}

(iii) Covering of the maximal invariant set in the chosen region:

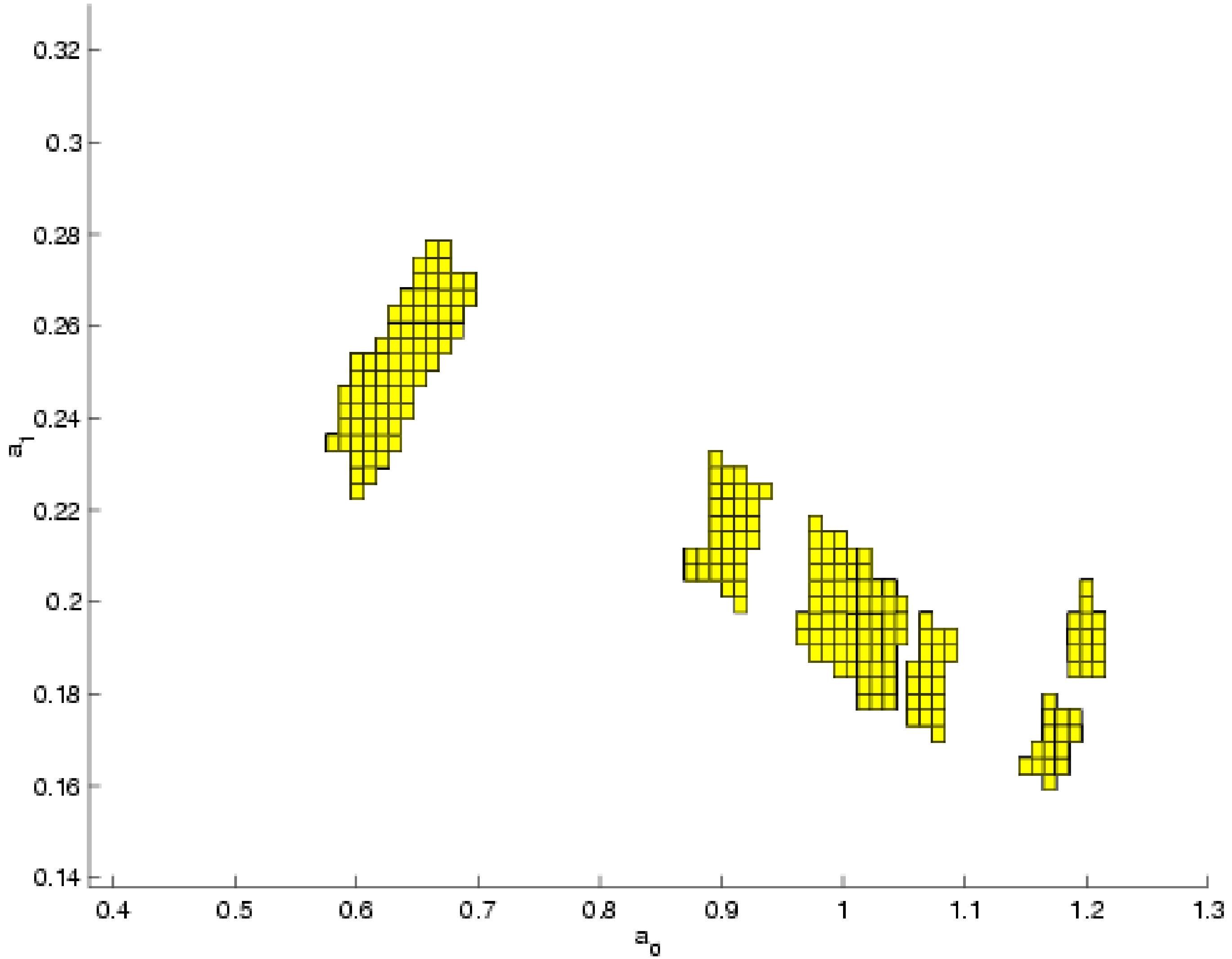




(iv) Connecting orbit from a fixed point to a period two point:



(v) Isolating neighborhood:



(vi) Homology of the corresponding index pair:

$$H_*(N_1, N_0) \cong (0,\ \mathbb{Z}^8,\ 0,\ 0,\ \dots)$$

and the map in homology:

$$F_1 := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Theorem 4 *The map Φ possesses an orbit*

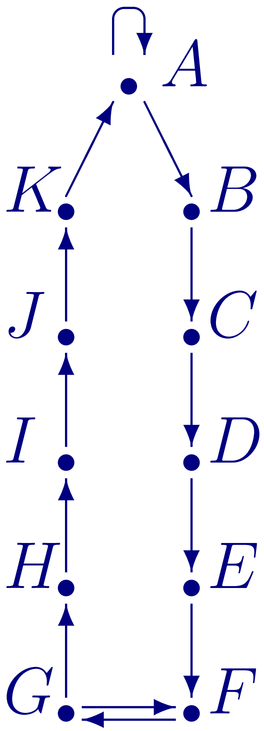
$$(a_j)_{j\in\mathbb{Z}}, \quad a_j \in L^2([-\pi,\pi]),$$

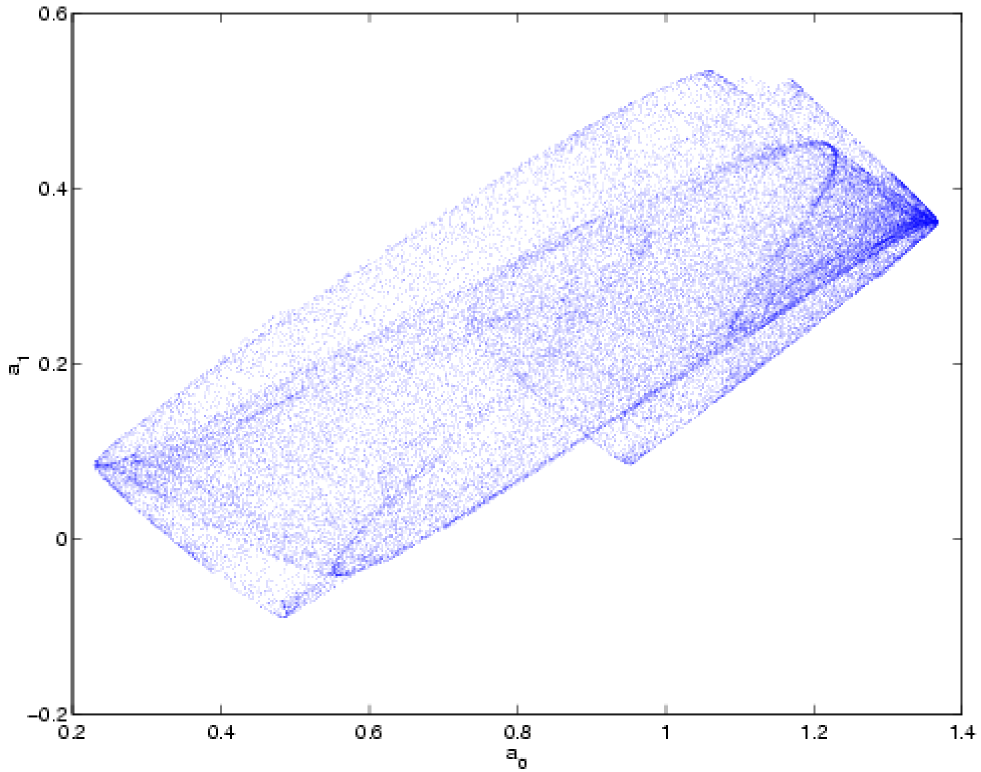
connecting a neighborhood of a fixed point $p_1 \in L^2([-\pi,\pi])$ of Φ to a neighborhood of a period two point $p_2 \in L^2([-\pi,\pi])$ of Φ, such that for the coordinates $(p_1),(p_2)$ and (a_j), $j \in \mathbb{Z}$,

$$(p_1),(p_2),(a_j) \in |\mathcal{I}^{(12)}| \times \prod_{k=12}^{49} [a_k^-, a_k^+] \times \prod_{k=50}^{\infty} \frac{1}{2^k}[-1,1], \quad j \in \mathbb{Z}.$$

Here the $a_k^\pm$ are the final bounds.

2. Example computation





Theorem. For the parameter values [...] there is an invariant set, contained in [...], on which Φ is semi-conjugate to the subshift given by the transition graph.

Software

- CHomP — Computational Homology Program

  `http://http://www.math.gatech.edu/~chom/`

  Tomasz Kaczynski, Konstantin Mischaikow, Marian Mrozek, Pawel Pilarczyk.

- GAIO — Global analysis of invariant objects

  `http://www.upb.de/math/~agdellnitz/gaio`

  Michael Dellnitz, O.J.

- Scripts for these computations:

  `http://www.upb.de/math/~junge/kot_schaffer/code`