Computational Homology

Abstract Definition of Homology

A *free chain complex* $\mathcal{C} = \{(C_k, \partial_k) \mid k \in \mathbb{Z}\}$ consists of free abelian groups C_k, called *chains* and group homomorphisms $\partial_k : C_k \to C_{k-1}$, called *boundary operators*, satisfying

$$\partial_k \circ \partial_{k+1} = 0$$

for all $k \in \mathbb{Z}$

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$$Z_k := \ker \partial_k = \{z \in C_k \mid \partial_k z = 0\}$$

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The *homology groups* of $\mathcal{C}$ are

$$H_k := Z_k / B_k$$

Computing Homology

```
function homologyGroupOfChainComplex( array[ 0 : ] of matrix D)
 array[ −1 : ] of matrix V, W;
for k := 0 to lastIndex(D) do
  (W[k], V[k − 1]) := kernelImage(D[k]);
endfor;
V[lastIndex(D)] := 0;
 array[ 0 : ] of list H;
for k := 0 to lastIndex(D) do
  H[k] := quotientGroup(W[k], V[k]);
endfor;
return H;
```

function $\texttt{kernelImage}$(**matrix** $\texttt{D}$)
$\texttt{m} :=$ **numberOfRows**($\texttt{D}$);
$\texttt{n} :=$ **numberOfColumns**($\texttt{D}$);
$\texttt{DT} := \texttt{transpose}(\texttt{D})$;
$(\texttt{B}, \texttt{P}, \bar{\texttt{P}}, \texttt{k}) := \texttt{rowEchelon}(\texttt{DT})$;
$\texttt{BT} := \texttt{transpose}(\texttt{B})$;
$\texttt{PT} := \texttt{transpose}(\texttt{P})$;
return $(\texttt{PT}[1 : \texttt{n}, \texttt{k} + 1 : \texttt{n}], \texttt{BT}[1 : \texttt{m}, 1 : \texttt{k}])$;

P is invertible $\qquad Q = \begin{array}{cc} & \\ k & \left[\begin{array}{ccc} * & * & * \\ & * & * \\ & & * \\ & 0 & \end{array}\right] \end{array}$

$B = PD^T$ implies $\text{im}\, B^T = \text{im}\, D$

$B^T(P^T)^{-1} = D$ implies $\left\{P^T e_j \mid j = k+1, \ldots, n\right\} = \ker D$.

Cubical Sets

An *elementary interval* I is an interval in $\mathbb{R}$ of the form

$$I = [l, l+1] \quad \text{or} \quad I = [l, l] = [l] \quad \text{where } l \in \mathbb{Z}$$

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An *elementary cube* Q is a finite product of elementary intervals

$$Q = I_1 \times I_2 \times \cdots \times I_d \subset \mathbb{R}^d$$

The *dimension*, $\dim(Q)$ is the number of nondegenerate components of Q.

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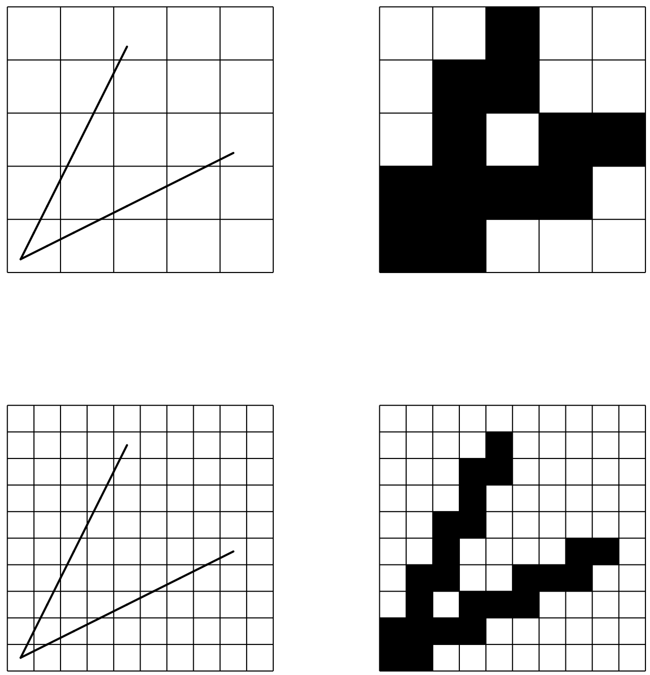
Let $\mathcal{K}$ denote the set of all elementary cubes.

A set $X \subset \mathbb{R}^d$ is *cubical* if it can be written as a finite union of elementary cubes. If X is cubical, let

$$\begin{aligned} \mathcal{K}(X) &:= \{Q \in \mathcal{K} \mid Q \subset X\} \\ \mathcal{K}_k(X) &:= \{Q \in \mathcal{K}(X) \mid \dim Q = k\} \end{aligned}$$

Cubical Approximations



Cubical Chain Complex

Let X be a cubical set. The set of *elementary k-chains* of X is

$$\widehat{\mathcal{K}}_k(X) := \left\{ \widehat{Q} \mid Q \in \mathcal{K}_k(X) \right\}$$

The *cubical k-chains* of X, denoted by $C_k(X)$, is the free abelian group generated by $\widehat{\mathcal{K}}_k(X)$.

A k-chain has the form

$$c = \sum_{i=1}^{m} \alpha_i \widehat{Q}_i \qquad \alpha_i \in \mathbb{Z}$$

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The *support* of c is

$$|c| = \bigcup_{\alpha_i \neq 0} Q_i$$

Products of Cubical Chains

Let X be a cubical set. Let $c = \sum_{i=1}^{m} \alpha_i \widehat{Q}_i$ and $c' = \sum_{i=1}^{m} \beta_i \widehat{Q}_i$ be k-chains. The *scalar product* is

$$\langle c, c' \rangle := \sum_{i=1}^{m} \alpha_i \beta_i$$

and the *cubical product* is

$$c \diamond c' := \sum_{i=1}^{m} \sum_{j=1}^{m} \alpha_i \beta_j \widehat{Q_i \times Q_j}$$

Cubical Boundary Operator

Goal: Define $\partial_k : C_k(X) \to C_{k+1}(X)$ such that $\partial_k \circ \partial_{k+1} = 0$.

Notation: Write $\partial : C_k(X) \to C_{k+1}(X)$ where $\partial^2 = 0$.

Remark: Only need to define ∂ on $\widehat{\mathcal{K}}_k(X)$.

Cubical Boundary Operator

Let Q be an elementary interval, i.e. $Q = [l]$ or $Q = [l, l+1]$
Define

$$\partial\widehat{Q} := \begin{cases} 0 & \text{if } Q = [l] \\ \widehat{[l+1]} - \widehat{[l]} & \text{if } Q = [l, l+1] \end{cases}$$

Let $Q \in \mathcal{K}(X)$, then $Q = I_1 \times I_2 \times \cdots \times I_d = I_1 \times P = I \times P$. Hence,

$$\widehat{Q} = \widehat{I} \diamond \widehat{P}$$

Define

$$\partial\widehat{Q} = \partial\widehat{I} \diamond \widehat{P} + (-1)^{\dim I}\widehat{I} \diamond \partial\widehat{P}$$

Lemma: $\partial^2 = 0$

Cubical Homology

Let X be a cubical set. The associated *cubical chain complex* is

$$\mathcal{C}(X) := \{(C_k(X), \partial_k) \mid k \in \mathbb{Z}\}$$

The associated *cubical homology groups* are

$$H_k(X) := Z_k(X)/B_k(X) = \ker \partial_k / \operatorname{im} \partial_{k+1}$$

Elementary Collapse

Let X be a cubical set and let $Q, P \in \mathcal{K}(X)$. If $Q \subset P$, then Q is a *face* of P. If $Q \neq P$, then Q is a *proper face* of P. A face that is a proper face of exactly one elementary cube in X is a *free face*.

Let $Q, P \in \mathcal{K}(X)$. Assume Q is a free face in X and $Q \subset P$. Let

$$\mathcal{K}'(X) := \mathcal{K}(X) \setminus \{Q, P\} \quad \text{and} \quad X' := \bigcup_{R \in \mathcal{K}'(X)} R$$

Theorem: $H_*(X) \cong H_*(X')$

A cubical set X is *acyclic* if

$$H_k(X) \cong \begin{cases} \mathbb{Z} & \text{if } k=0 \\ 0 & \text{otherwise} \end{cases}$$

Elementary Collapse

Let $A \subset X$. A *strong deformation retraction* of X onto A is a continuous map $h : X \times [0,1] \to X$ satisfying

$$
\begin{aligned}
h(x,0) &= x \quad \text{for all} \quad x \in X \\
h(x,1) &\in A \quad \text{for all} \quad x \in X \\
h(a,1) &= a \quad \text{for all} \quad a \in A
\end{aligned}
$$

Topological Remark: Elementary collapse can be viewed as a discrete example of a strong deformation retraction.

Computational Remark: Elementary collapse suggests a method for computing homology.

Maps between Homology Groups

Let $\mathcal{C} = \{C_k, \partial_k\}$ and $\mathcal{C}' = \{C'_k, \partial'_k\}$ be chain complexes. A sequence of homomorphisms $\varphi_k : C_k \to C'_k$ is a *chain map* $\varphi : \mathcal{C} \to \mathcal{C}'$ if, for every $k \in \mathbb{Z}$,

$$\partial'_k \varphi_k = \varphi_{k-1} \partial_k.$$

Define $\varphi_{k*} : H_k(\mathcal{C}) \to H_k(\mathcal{C}')$ by

$$\varphi_{k*}([z]) := [\varphi_k(z)],$$

where $z \in Z_k$

Inclusion Map

Consider a chain complex $\mathcal{C} = \{C_k, \partial_k\}$ with a subchain complex $\mathcal{C}' = \left\{C'_k, \partial_k\right\}$. Let $\iota_k : C'_k \to C_k$ be the inclusion map given by

$$\iota_k c' = c'$$

for every $c' \in C'_k$. Since

$$\partial_k \iota_k c' = \partial_k c' = \iota_{k-1} \partial_k c',$$

ι is a chain map. Therefore,

$$\iota_* : H_*(\mathcal{C}') \to H_*(\mathcal{C})$$

is defined.

Projection Map

Consider the elementary cube $Q = [0,1]^d$ and the projection $p : Q \to Q$ given by

$$p(x_1, x_2, x_3, \ldots, x_d) := (0, x_2, x_3, \ldots, x_d).$$

We want to associate with p a chain map $\pi : \mathcal{C}(Q) \to \mathcal{C}(Q)$. Any face E of Q can be written as

$$E = E_1 \times P \qquad \text{where} \quad E_1 \in \{[0,1],[0],[1]\}$$

Observe

$$E' := p(E) = [0] \times P.$$

Define

$$\pi_k(\widehat{E}) := \begin{cases} \widehat{E'} & \text{if } E_1 = [0] \text{ or } E_1 = [1], \\ 0 & \text{otherwise.} \end{cases}$$

π is a chain map (case 1)

By definition

$$\partial \widehat{E} = \partial \widehat{E_1} \diamond \widehat{P} + (-1)^{\mathrm{dim}(E_1)} \widehat{E_1} \diamond \partial \widehat{P}.$$

If $E_1 = [0]$ or $E_1 = [1]$,

$$\begin{aligned} \pi \partial \widehat{E} &= \pi \left(\partial \widehat{E_1} \diamond \widehat{P} + \widehat{E_1} \diamond \partial \widehat{P} \right) \\ &= \pi (\widehat{E_1} \diamond \partial \widehat{P}) \\ &= \widehat{[0]} \diamond \partial \widehat{P} \end{aligned}$$

and, consequently,

$$\partial \pi \widehat{E} = \partial (\widehat{[0]} \diamond \widehat{P}) = \widehat{[0]} \diamond \partial \widehat{P} = \pi \partial \widehat{E}.$$

π is a chain map (case 2)

If $E_1 = [0, 1]$, then $\pi\widehat{E} = 0$, by definition.

On the other hand,

$$\begin{aligned}
\pi\partial\widehat{E} &= \partial\widehat{E_1} \diamond \widehat{P} + (-1)^{\dim(E_1)}\widehat{E_1} \diamond \partial\widehat{P} \\
&= \pi((\widehat{[1]} - \widehat{[0]}) \diamond \widehat{P} - \widehat{E_1} \diamond \widehat{\partial P}) \\
&= \widehat{[0]} \diamond \widehat{P} - \widehat{[0]} \diamond \widehat{P} - 0 \\
&= 0 \\
&= \partial\pi\widehat{E}
\end{aligned}$$

Thus $\pi : \mathcal{C}(Q) \to \mathcal{C}(Q)$ is a chain map.

Let $\varphi, \psi : \mathcal{C} \to \mathcal{C}'$ be chain maps. A collection of group homomorphisms

$$D_k : C_k \to C'_{k+1}$$

is a *chain homotopy* between φ and ψ if, for all $k \in \mathbb{Z}$,

$$\partial'_{k+1} D_k + D_{k-1} \partial_k = \psi_k - \varphi_k.$$

Theorem: If φ and ψ are chain homotopic, then $\varphi_* = \psi_*$.

Internal Elementary Reductions

Let $\mathcal{C} = \{C_k, \partial_k\}$ be a chain complex. A pair of generators (a, b) such that $a \in C_{m-1}$, $b \in C_m$ and $\langle \partial b, a \rangle = \pm 1$ is called a *reduction pair*.

Internal Elementary Reductions

Let $\mathcal{C} = \{C_k, \partial_k\}$ be a chain complex. A pair of generators (a, b) such that $a \in C_{m-1}$, $b \in C_m$ and $\langle \partial b, a \rangle = \pm 1$ is called a *reduction pair*.

A reduction pair induces a collection of group homomorphisms

$$\pi_k c := \begin{cases} c - \frac{\langle c,a \rangle}{\langle \partial b,a \rangle} \partial b & \text{if } k = m-1 \\ c - \frac{\langle \partial c,a \rangle}{\langle \partial b,a \rangle} b & \text{if } k = m \\ c & \text{otherwise} \end{cases}$$

where $c \in C_k$.

Theorem: The map $\pi : \mathcal{C} \to \mathcal{C}' := \pi(\mathcal{C}) \subset \mathcal{C}$ is a chain map. Furthermore, $\pi_k : C_k \to C'_k$ induces an isomorphism on homology.

Chain Complex Reduction Algorithm

Remark: There is a complication associated with internal elementary reductions.

Let $(\mathcal{C}, \partial)$ be a free abelian chain complex with basis W_k of C_k. Let $(a, b) \in W_{m-1} \times W_m$ be a reduction pair.

Under the internal elementary reduction $\pi_k : C_k \to C'_k$. The boundary operator remains the same, but the new basis, $W'_m = \{b'_1, b'_2, \ldots, b'_{d_m}\}$, is related to $W_m = \{b_1, b_2, \ldots, b_{d_m}, b\}$ by

$$b'_i = b_i - \frac{\langle \partial b_i, a \rangle}{\langle \partial b, a \rangle} b.$$

We can define a new chain complex $\bar{\mathcal{C}}$ whose basis is a subbasis of $\mathcal{C}$, but for which the boundary operator is modified. Define

$$\bar{W}_k := \begin{cases} \left\{b_1, b_2, \ldots, b_{d_m}\right\} & \text{if } k = m, \\ \left\{a_1, a_2, \ldots, a_{d_{m-1}}\right\} & \text{if } k = m-1, \\ W_k & \text{otherwise.} \end{cases}$$

Define $\eta = \bar{\mathcal{C}} \to \mathcal{C}'$ by

$$\eta_k(c) := \begin{cases} c - \frac{\langle \partial c, a \rangle}{\langle \partial b, a \rangle} b & \text{if } k = m, \\ c & \text{otherwise.} \end{cases}$$

Then

$$\bar{\partial} := \eta^{-1} \partial \eta$$

Theorem: $H_*(\mathcal{C}) \cong H_*(\mathcal{C}') \cong H_*(\bar{\mathcal{C}})$

Relative Homology

Let $\mathcal{C} = \{C_k, \partial_k\}$ be a chain complex. A chain complex $\mathcal{D} = \left\{D_k, \partial'_k\right\}$ is a *chain subcomplex* if

- D_k is a subgroup of C_k for all $k \in \mathbb{Z}$
- $\partial'_k = \partial_k \mid_{D_k}$

The relative chain complex is $(\mathcal{C}, \mathcal{D}) := \{C_k/D_k, \partial_k\}$

The relative cycles and boundaries are

$$\begin{aligned} Z_k(\mathcal{C}, \mathcal{D}) &:= \ker \partial_k : C_k/D_k \to C_{k-1}/D_{k-1} \\ B_k(\mathcal{C}, \mathcal{D}) &:= \operatorname{im} \partial_{k+1} : C_{k+1}/D_{k+1} \to C_k/D_k \end{aligned}$$

and the relative homology groups are

$$H_k(\mathcal{C}, \mathcal{D}) := Z_k(\mathcal{C}, \mathcal{D})/B_k(\mathcal{C}, \mathcal{D})$$

Exact Sequences

A sequence (finite or infinite) of groups and homomorphisms

$$\ldots \to G_3 \xrightarrow{\psi_3} G_2 \xrightarrow{\psi_2} G_1 \to \ldots$$

is *exact* at G_2 if

$$\text{im}\,\psi_3 = \ker \psi_2.$$

It is an *exact sequence* if it is exact at every group.

A *short exact sequence* is an exact sequence of the form

$$0 \to G_3 \xrightarrow{\psi_3} G_2 \xrightarrow{\psi_2} G_1 \to 0.$$

Connecting Homomorphism

Let $\mathcal{A} = \{A_k, \partial_k^{\mathcal{A}}\}$, $\mathcal{B} = \{B_k, \partial_k^{\mathcal{B}}\}$, and $\mathcal{C} = \{C_k, \partial_k^{\mathcal{C}}\}$ be chain complexes. Let $\varphi : \mathcal{A} \to \mathcal{B}$ and $\psi : \mathcal{B} \to \mathcal{C}$ be chain maps. The sequence

$$0 \to \mathcal{A} \xrightarrow{\varphi} \mathcal{B} \xrightarrow{\psi} \mathcal{C} \to 0$$

is a *short exact sequence of chain complexes* if for every k

$$0 \to A_k \xrightarrow{\varphi_k} B_k \xrightarrow{\psi_k} C_k \to 0$$

is a short exact sequence.

Theorem: Let

$$0 \to \mathcal{A} \xrightarrow{\varphi} \mathcal{B} \xrightarrow{\psi} \mathcal{C} \to 0$$

be a short exact sequence of chain complexes. Then for each k there exists a homomorphism

$$\partial_* : H_{k+1}(\mathcal{C}) \to H_k(\mathcal{A})$$

such that

$$\ldots \to H_{k+1}(\mathcal{A}) \xrightarrow{\varphi_*} H_{k+1}(\mathcal{B}) \xrightarrow{\psi_*} H_{k+1}(\mathcal{C}) \xrightarrow{\partial_*} H_k(\mathcal{A}) \to \ldots$$

is a long exact sequence.

The map ∂_* is called *connecting homomorphism*.

Exact Sequence of a Triple

Theorem: Let $N_0 \subset N_1 \subset N_2$ be cubical sets. Then there exists a long exact sequence

$$\to H_{k+1}(N_1, N_0) \xrightarrow{i_*} H_{k+1}(N_2, N_1) \xrightarrow{j_*} H_{k+1}(N_2, N_1) \xrightarrow{\partial_*} H_k(N_1, N_0) \to$$

Let I be an elementary interval. The associated *elementary cell* is

$$\overset{\circ}{I} := \begin{cases} (l, l+1) & \text{if } I = [l, l+1], \\ [l] & \text{if } I = [l, l]. \end{cases}$$

We extend this definition to a general elementary cube $Q = I_1 \times I_2 \times \ldots \times I_d \subset \mathbb{R}^d$ by defining the associated *elementary cell* as

$$\overset{\circ}{Q} := \overset{\circ}{I}_1 \times \overset{\circ}{I}_2 \times \ldots \times \overset{\circ}{I}_d.$$

Let $A \subset \mathbb{R}^d$ be a bounded set. Then the *open hull* of A is

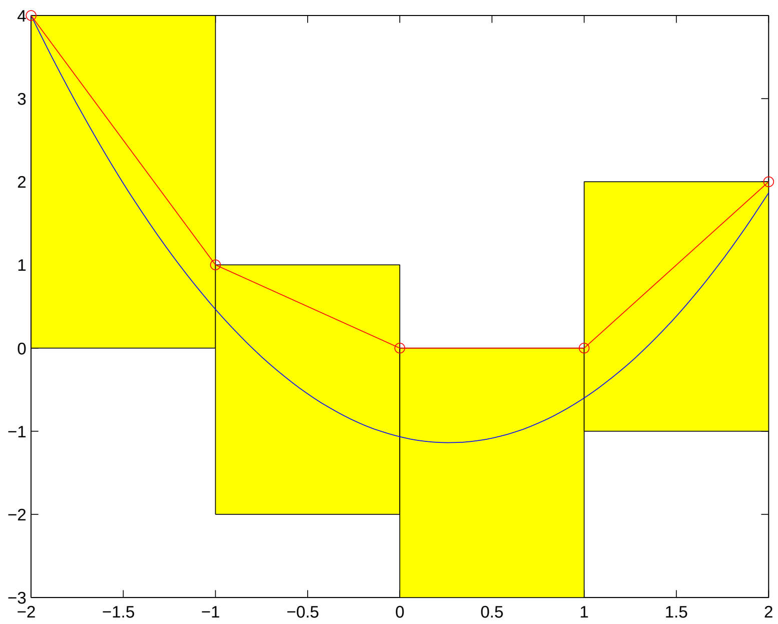
$$\text{oh}\,(A) := \bigcup \{\overset{\circ}{Q} \mid Q \in \mathcal{K}, Q \cap A \neq \emptyset\},$$

and the *closed hull* of A is

$$\text{ch}\,(A) := \bigcup \{Q \mid Q \in \mathcal{K}, \overset{\circ}{Q} \cap A \neq \emptyset\}.$$

Cubical Approximation of Functions

Let $f : X \to Y$ be a continuous map between cubical sets.



Goal: Approximate f by an acyclic-valued mutivalued cubical map $F : X \rightrightarrows Y$ such that $f(x) \in F(x)$.

Cubical Maps

Let X and Y be cubical sets. A *multivalued map* $F : X \rightrightarrows Y$ from X to Y is a function from X to subsets of Y, that is, for every $x \in X$, $F(x) \subset Y$.

Let X and Y be cubical sets. A multivalued map $F : X \rightrightarrows Y$ is *cubical* if

- For every $x \in X$, $F(x)$ is a cubical set.
- For every $Q \in \mathcal{K}(X)$, $F|_{\overset{\circ}{Q}}$ is constant, that is, if $x, x' \in \overset{\circ}{Q}$, then $F(x) = F(x')$.

Assume now that for a continuous map $f : X \to Y$, bounds on $f(Q)$ for $Q \in \mathcal{K}_{\mathsf{max}}(X)$ are given in the form of a combinatorial multivalued map $\mathcal{F} : \mathcal{K}_{\mathsf{max}}(X) \rightrightarrows \mathcal{K}(Y)$

$$\lfloor \mathcal{F} \rfloor(x) := \bigcap \{ |\mathcal{F}(Q)| \mid x \in Q \in \mathcal{K}_{\mathsf{max}}(X) \}$$

$$\lceil \mathcal{F} \rceil(x) := \bigcup \{ |\mathcal{F}(Q)| \mid x \in Q \in \mathcal{K}_{\mathsf{max}}(X) \}$$

Let $F : X \rightrightarrows Y$, $A \subset X$, and $B \subset Y$. The *image* of A is defined by

$$F(A) := \bigcup_{x \in A} F(x).$$

The *weak preimage* of B under F is

$$F^{*-1}(B) := \{x \in X \mid F(x) \cap B \neq \emptyset\},$$

while the *preimage* of B is

$$F^{-1}(B) := \{x \in X \mid F(x) \subset B\}.$$

A multivalued map F is *upper semicontinuous* if $F^{-1}(U)$ is open and it is *lower semicontinuous* if $F^{*-1}(U)$ is open for any open set $U \subset Y$.

Proposition: The map $\lfloor\mathcal{F}\rfloor$ is lower semicontinuous and the map $\lceil\mathcal{F}\rceil$ is upper semicontinuous.

Chain Selectors

A cubical multivalued map $F : X \rightrightarrows Y$ is called *acyclic-valued* if for every $x \in X$ the set $F(x)$ is acyclic.

Let $F : X \rightrightarrows Y$ be a cubical multivalued map. A chain map $\varphi : C(X) \to C(Y)$ satisfying the conditions:

$$
\begin{aligned}
\left|\varphi(\widehat{Q})\right| &\subset F(\overset{\circ}{Q}) \text{ for all } Q \in \mathcal{K}(X), \\
\varphi(\widehat{Q}) &\in \widehat{\mathcal{K}}_0(F(Q)) \text{ for any vertex } Q \in \mathcal{K}_0(X),
\end{aligned}
$$

is called a *chain selector* of F.

Theorem: Assume $F : X \rightrightarrows Y$ is a lower semicontinuous acyclic-valued cubical map. Then there exists a chain selector $\varphi : C(X) \to C(Y)$.

Proposition: Assume $F : X \rightrightarrows Y$ is a lower semicontinuous cubical map and φ is a chain selector for F. Then, for any $c \in C(X)$,

$$|\varphi(c)| \subset F(|c|).$$

Theorem: Let $\varphi, \psi : C(X) \to C(Y)$ be chain selectors for the lower semicontinous acyclic-valued cubical map $F : X \rightrightarrows Y$. Then φ is chain homotopic to ψ and hence they induce the same homomorphism in homology.

Definition: Let $F : X \rightrightarrows Y$ be a lower semicontinous acyclic-valued cubical map. Let $\varphi, \psi : C(X) \to C(Y)$ be chain selector for F. Then $F_* : H_*(X) \to H_*(Y)$ is defined by

$$F_* = \varphi_*.$$

Cubical Approximation of Maps

Let X and Y be cubical sets and let $f : X \to Y$ be a continuous function. A *cubical representation* of f is a lower semicontinuous multivalued cubical map $F : X \rightrightarrows Y$ such that

$$f(x) \in F(x) \qquad \forall\, x \in X.$$

The *minimal representation* of f is $M_f : X \rightrightarrows Y$ defined by

$$M_f(x) := \mathrm{ch}\,(f(\mathrm{ch}\,(x)))$$

Definition: Let $F : X \rightrightarrows Y$ be an acyclic-valued cubical representation of $f : X \to Y$. Then $f_* : H_*(X) \to H_*(Y)$ is defined by

$$f_* = F_*.$$

Computing Homology Maps

Consider a continuous map $f : X \to Y$.

Let Γ_f be the graph of f.

Let $\pi_X : \Gamma_F \to X$ and $\pi_Y : \Gamma_F \to Y$ be projection maps.

Observe that

$$f = \pi_Y \circ \pi_X^{-1}$$

and hence

$$f_* = \pi_{Y*} \circ \pi_{X*}^{-1}.$$

Consider $F = \lceil \mathcal{F} \rceil : X \rightrightarrows Y$ an acyclic-valued cubical representation of f.

Let $\Gamma_F \subset X \times Y$ denote the graph of F.

We are using an upper semi-continuous multivalued map, hence the graph is a cubical set.

Let $\pi_X : \Gamma_F \to X$ and $\pi_Y : \Gamma_F \to Y$ be projection maps (these map cubes to cubes).

Because $F : X \rightrightarrows Y$ is acyclic, $\pi_{X*} : H_*(\Gamma_F) \to H_*(X)$ is an isomorphism.

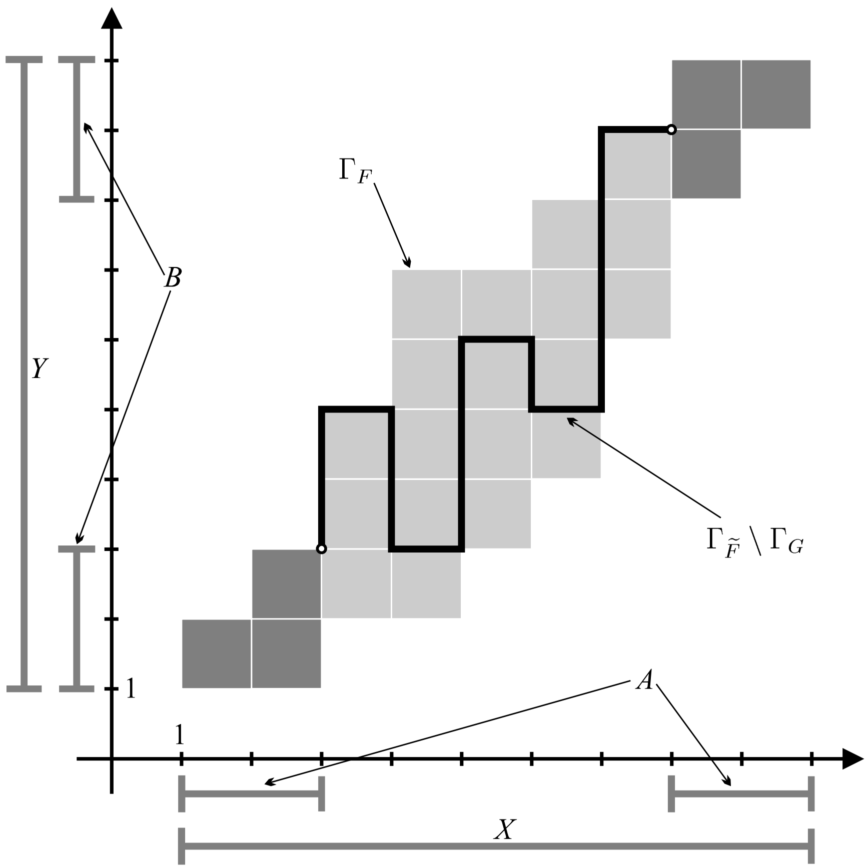
Theorem: $f_* = \pi_{Y*} \circ \pi_{X*}^{-1}$

Computing Relative Homology Maps

Consider $f : (X, A) \to (Y, B)$ where $X, A \subset \mathbf{R}^n$ and $Y, B \subset \mathbf{R}^m$ are full cubical sets.

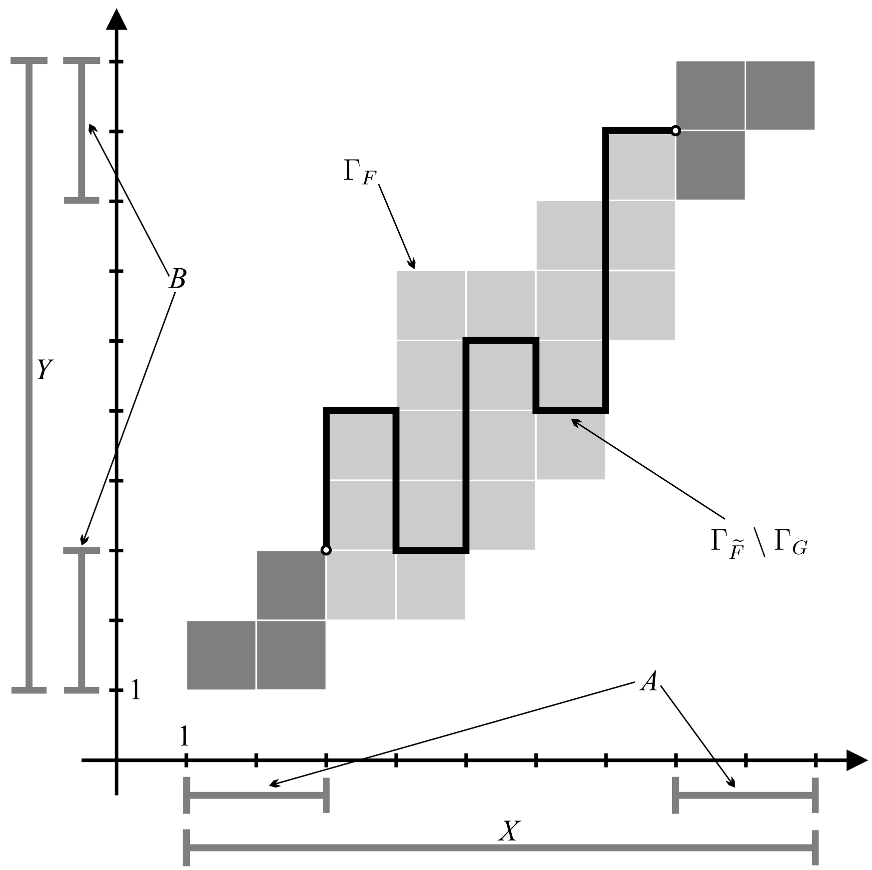
A combinatorial multivalued map $\mathcal{F} : \mathcal{X} \rightrightarrows \mathcal{Y}$ is a *representation* of $f : (X, A) \to (Y, B)$ if $\mathcal{F}$ is a representation of $f\colon X \to Y$ and $\mathcal{F}(\mathcal{A}) \subset \mathcal{B}$.

Remark: Given a representation $\mathcal{F}$ of $f\colon (X, A) \to (Y, B)$ it does not follow that $F(A) \subset B$, where $F = \lceil \mathcal{F} \rceil$.

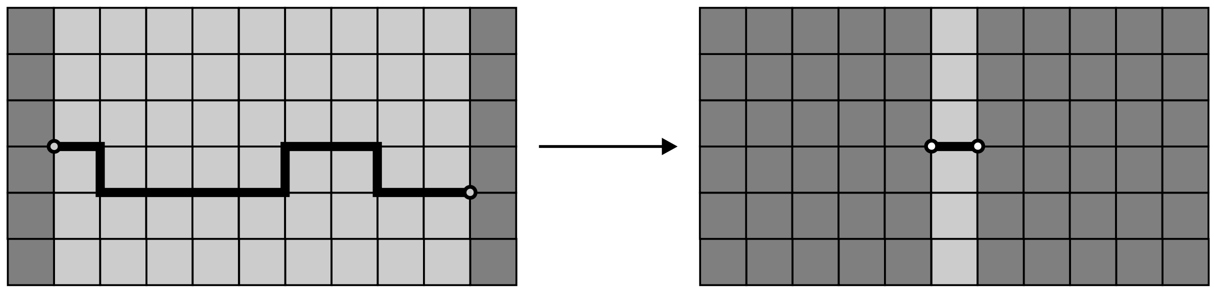


A pair (F, G) of multivalued maps is a representation of $f\colon (X, A) \to (Y, B)$ if $F\colon X \to Y$ is a representation of $f\colon X \to Y$ and $G\colon A \to B$ is a represenatation of $f|_A\colon A \to B$, and $G \subset F$.

Computing Homology Maps - Algorithms



Expand the size of A and remove.



Reduce the size of X.