A Connection Matrix Structure Theorem

S is an attractor for a flow $\varphi : \mathbb{R} \times X \to X$.

A Morse decomposition of S is given by

$$\mathcal{M}(S) = \left\{ M(p) \mid p \in \left\{ 0^{\pm} < 1^{\pm} < 2 \right\} \right\}$$

$$CH_k(M(p^*)) \cong \begin{cases} \mathbb{Z} & \text{if } k = p, \\ 0 & \text{otherwise.} \end{cases}$$

$$\Delta = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Technical Lemma

(ii) There exists a time reparameterized flow $\tilde{\varphi} : \mathbb{R} \times X \to X$ of φ.

(ii) There exists sets $N_{p^\pm}$, $L^+_{p^\pm}$, $L^-_{p^\pm}$, $p = 0, 1, 2$ (let $2 = 2^+$):

- $N_{p^\pm}$ are isolating blocks for $M(p^\pm)$ with entrance set $L^-_{p^\pm}$ and exit set $L^+_{p^\pm}$
- $\tilde{\varphi}(\mathbb{R}, x) \cap N_{p^\pm} = \tilde{\varphi}(I_{p^\pm}(x), x)$, where $I_{p^\pm}$ is a closed interval.

Remark: Since $I_{p^\pm}$ is a closed interval we write $I_{p^\pm} = [a_{p^\pm}(x), b_{p^\pm}(x)]$ where if $I_{p^\pm} = \emptyset$, then $a_{p^\pm}(x), b_{p^\pm}(x)$ are not defined, and it is possible that $a_{p^\pm}(x) = -\infty$ or $b_{p^\pm}(x) = \infty$

Technical Lemma

(iii) Let $\Theta_{p^\pm} := \left\{x \in S \mid I_{p^\pm}(x) \neq \emptyset\right\}$. Then,

$$a_{p^\pm}(x), b_{p^\pm} : \Theta_{p^\pm} \to [-\infty, \infty]$$

are continuous

(iv) There exists a Lyapunov function $\widetilde{V} : S \to [0, 2]$ such that

- if $x \in M(p^\pm) \cup (L^+_{p^\pm} \cap L^-_{p^\pm})$ then $\widetilde{V}(x) = p$.
- if $\widetilde{\varphi}([0, t], x) \cap \bigcup_{p=0}^{2} N_{p^\pm} = \emptyset$, then $\widetilde{V}(\widetilde{\varphi}(t, x)) = \widetilde{V}(x) - t$.

Constructing a map $\widetilde{f} : S \to \partial([0,2] \times [-1,1]^2)$

Define

$$\lambda_{p^\pm}(x) := \begin{cases} \infty & \text{if } b_{p^\pm}(x) = \infty \text{ or } a_{p^\pm}(x) = -\infty, \\ 0 & \text{if } I_{p^\pm}(x) = \emptyset \\ b_{p^\pm}(x) - a_{p^\pm}(x) & \text{otherwise} \end{cases}$$

Define

$$\tau_p(x) = \frac{2}{\pi} \tan^{-1} \left(\lambda_{p^+}(x) - \lambda_{p^-}(x) \right)$$

Lemma: $\tau_p : \Theta_p \to [-1, 1]$ is continuous.

Define $V : S \to [0, 2]$ obtained by modifying $\widetilde{V}$.

Define $\widetilde{f}(x) = (V(x), \tau_0(x), \tau_1(x))$