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Control Theory I – Tutorial 1

To be handed in till: Monday, Apr. 27, To be discussed on: Tuesday, Apr. 28

1. (2+2+2+2)
   Consider the pendulum equations (all constants have been set to 1 for simplicity)

$$\begin{aligned}\dot{x}_1(t) &= x_2(t)\\ \dot{x}_2(t) &= \sin(x_1(t)) + u(t).\end{aligned} \qquad (1)$$

   Our goal is to compute a control function $u(\cdot)$ that steers the system from $x(0) = x_0$ to $x(\tau) = 0$, where $x_0 \in \mathbb{R}^2$ and $\tau > 0$ are given. For this, we will use a simple but effective trick as described below:

   (a) Set $v := \sin(x_1) + u$. The initial value problem $\dot{x}_1 = x_2$, $\dot{x}_2 = v$, $x(0) = x_0$ can be solved exactly.

   (b) Make the ansatz $v(t) := a + bt$. Use $x(\tau) \stackrel{!}{=} 0$ to express a, b in terms of x_{01}, x_{02}, τ.

   (c) Compute $u(\cdot)$ and show that it really "does it" (use the fact that the solution of (1) with $x(0) = x_0$ is unique).

   (d) Test your results numerically for $\tau = 1$, $x_0 = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}$.

2. (2+2)

   (a) Show that the solutions of the quadratic equation $\lambda^2 + p\lambda + q = 0$, where $p, q \in \mathbb{R}$, have a negative real part if and only if $p > 0$ and $q > 0$.
   Remark: The equation $\lambda^3 + \lambda^2 + \lambda + 1 = 0$ shows that the situation is more complicated for cubic or higher order equations.

   (b) Now consider the linearization of the pendulum equations around the upright position $\bar{x} = 0$, which is given by

$$\begin{aligned}\dot{x}_1(t) &= x_2(t)\\ \dot{x}_2(t) &= x_1(t) + u(t).\end{aligned}$$

   We would like to choose $u(\cdot)$ such that the controlled pendulum is asymptotically stable at $\bar{x} = 0$. (Hint: $\dot{x} = Ax$ is asymptotically stable at $\bar{x} = 0$ if and only if all eigenvalues of A have a negative real part.)
   Naive idea: Recall that $x_1 = \theta$, $x_2 = \dot{\theta}$. Introducing coordinates, let the position of the point mass be given by $\begin{bmatrix} \sin(\theta) \\ \cos(\theta) \end{bmatrix}$. We consider small variations of θ around zero. If $\theta > 0$, the pendulum is "to the right" of the upright position, and we should apply a counter-clockwise torque $u < 0$.

Conversely, if $\theta < 0$, we should apply a clockwise torque $u > 0$. This leads to the ansatz $u(t) := a\theta(t) = ax_1(t)$, where $a < 0$.

Show that it is not possible to choose a such that the pendulum becomes asymptotically stable. Similarly, show that $u(t) := b\dot{\theta}(t) = bx_2(t)$ will also not work. Finally, let's try $u(t) := a\theta(t) + b\dot{\theta}(t) = ax_1(t) + bx_2(t)$. Derive conditions on $a, b \in \mathbb{R}$ such that the resulting controlled system will be asymptotically stable.

3. (2+3+3)

The Fibonacci difference equation is given by

$$y(t+2) - y(t+1) - y(t) = 0 \quad \text{for } t \in \mathbb{N}.$$

Show that its general solution is

$$y(t) = c_1(\tfrac{1+\sqrt{5}}{2})^t + c_2(\tfrac{1-\sqrt{5}}{2})^t \quad \text{for } t \in \mathbb{N}.$$

(The term "general solution" means: every solution has this form for a suitable choice of the constants $c_1, c_2 \in \mathbb{R}$. In fact, c_1, c_2 are uniquely determined by $y(0), y(1)$. The particular case $y(0) = y(1) = 1$ yields the classical Fibonacci sequence.)

(a) First approach: Making the inspired guess $y(t) = \lambda^t$, we obtain the "characteristic equation" $\lambda^2 - \lambda - 1 = 0$, which yields $\lambda_{1,2} = \frac{1\pm\sqrt{5}}{2}$. Thus

$$\{y \in \mathbb{R}^{\mathbb{N}} \mid \exists c_1, c_2 \in \mathbb{R} : y(t) = c_1\lambda_1^t + c_2\lambda_2^t \text{ for all } t \in \mathbb{N}\} \subseteq$$

$$\{y \in \mathbb{R}^{\mathbb{N}} \mid y(t+2) - y(t+1) - y(t) = 0 \text{ for all } t \in \mathbb{N}\}.$$

Now show that both sets are real vector spaces and use a dimensional argument.

(b) Second approach: Introducing $x(t) := \begin{bmatrix} y(t) \\ y(t+1) \end{bmatrix}$ for $t \in \mathbb{N}$, the Fibonacci difference equation can be rewritten as $x(t+1) = Ax(t)$ for some matrix $A \in \mathbb{R}^{2\times 2}$. A straightforward induction argument shows that $x(t) = A^t x_0$, where $x_0 \in \mathbb{R}^2$. Compute the eigenvalues of A and conclude that the matrix A is diagonizable, that is, there exists an invertible matrix $T \in \mathbb{R}^{2\times 2}$ such that

$$T^{-1}AT = \Lambda := \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}.$$

Thus $A^t = T\Lambda^t T^{-1}$, where Λ^t is fairly easy to compute ...

(c) Use a similar argument as in (b) for finding the general solution of

$$y(t+2) + 2y(t+1) + y(t) = 0.$$