Actions and Representations

An **action** of G on X is a map

$$A : X \times G \to X, \quad (x, g) \mapsto x \cdot g$$

A **representation** of G on X is a map

$$H : G \to X^X = \{f : X \to X\}$$

The two concepts are **the same**:

$$\text{given } A, \text{set } R(g) := (x \mapsto A(x, g)) = (x \mapsto x \cdot g) \qquad \text{and} \qquad \text{given } R, \text{set } A(x, g) := R(g)(x)$$

#	X	G	rules	homomorphism	name
(1)	set	semigroup	$(x \cdot g) \cdot \tilde{g} = x \cdot (g \cdot \tilde{g})$	$G \to T_X$ (hom. of semigroups)	transformation rep.
(2)	set	monoid	$(x \cdot g) \cdot \tilde{g} = x \cdot (g \cdot \tilde{g})$ and $x \cdot \mathbf{1} = x$	$G \to T_X$ (hom. of monoids)	transformation rep.
(3)	set	group	$(x \cdot g) \cdot \tilde{g} = x \cdot (g \cdot \tilde{g})$ and $x \cdot \mathbf{1} = x$	$G \to S_X$ (hom. of groups)	permutation rep.
(4)	$\mathbb{F}$-vector space	semigroup	as (1) and $(x \mapsto xg)$ is linear	$G \to \mathrm{End}_{\mathbb{F}}(X)$ (hom. of semigroups)	linear rep.
(5)	$\mathbb{F}$-vector space	monoid	as (2) and $(x \mapsto xg)$ is linear	$G \to \mathrm{End}_{\mathbb{F}}(X)$ (hom. of monoids)	linear rep.
(6)	$\mathbb{F}$-vector space	group	as (3) and $(x \mapsto xg)$ is linear	$G \to \mathrm{GL}(X)$ (hom. of groups)	linear rep.
(7)	$\mathbb{F}$-vector space	$\mathbb{F}$-algebra	as (6) and $x \cdot (g - \tilde{g}) = x \cdot g - x \cdot \tilde{g}$ and $x \cdot (\lambda \cdot g) = (\lambda \cdot x) \cdot g$ for $\lambda \in \mathbb{F}$	$G \to \mathrm{End}_{\mathbb{F}}(X)$ (hom. of algebras)	linear/matrix rep.
(8)	$\mathbb{C}$-vector space	group	as (6) and $(x \mid y) = (xg \mid yg)$	$G \to \mathrm{U}(X)$ (hom. of groups)	unitary rep.
(9)	L-free L-module	group	as (3) and $(x \mapsto xg)$ is L-linear	$G \to \mathrm{End}_L(X)^\times$ (hom. of groups)	integral linear rep.
(10)	graph	group	as (3) and $(x, y) \in E$ iff $(xg, yg) \in E(X)$	$G \to \mathrm{Aut}(X)$ (hom. of groups)	"graph rep."
(11)	$\mathbb{F}$-projective space	group	as (3) and $(x \mapsto xg)$ is $\mathbb{F}$-projective	$G \to \mathrm{PGL}(X)$ (hom. of groups)	projective rep.
(12)	$\mathbb{F}$-vector space	Lie-algebra	$(x \mapsto xg)$ is $\mathbb{F}$-linear and $x \cdot (\lambda \cdot g) = (\lambda \cdot x) \cdot g$ for $\lambda \in \mathbb{F}$ and $x \cdot (g - \tilde{g}) = x \cdot g - x \cdot \tilde{g}$ and $x \cdot [g, \tilde{g}] = (x \cdot g) \cdot \tilde{g} - (x \cdot \tilde{g}) \cdot g$	$G \to \mathrm{Lie}(\mathrm{End}_{\mathbb{F}}(X))$ (hom. of Lie algebras)	Lie algebra rep.

where $\mathbb{F}$ is a field and L is a commutative ring and $X = (V, E)$ if it is a graph.