# Actions, representations and various algebraic structures

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# Actions and representations

An action of $G$ on $X$ is a map

$$A : X \times G \to X, \quad (x, g) \mapsto x \cdot g$$

A representation of $G$ on $X$ is a map

$$R : G \to X^X = \{f : X \to X\}$$

The two concepts are the same:

given $A$, set

$$R(g) := (x \mapsto A(x, g)) = (x \mapsto x \cdot g)$$

given $R$, set

$$A(x, g) := R(g)(x)$$

# Group algebras — definition

Let $\mathbb{F}$ be a field and $G$ a finite group.

$\mathbb{F}G :=$ vector space with basis $G$, multiplication inherited from $G$ and distributive law:

$$\left(\sum_{g\in G} \lambda_g \cdot g\right) \cdot \left(\sum_{\tilde{g}\in G} \mu_{\tilde{g}} \cdot \tilde{g}\right) = \sum_{g,\tilde{g}\in G} \lambda_g \cdot \mu_{\tilde{g}} \cdot (g\tilde{g})$$

for $\lambda_g, \mu_{\tilde{g}} \in \mathbb{F}$.

$\mathbb{F}G := \{f : G \to \mathbb{F}\}$ with pointwise addition and convolution product:

$$(f \cdot h)(g) := \sum_{\tilde{g}\in G} f(g \cdot \tilde{g}^{-1}) \cdot h(\tilde{g})$$

for $f, h : G \to \mathbb{F}$.

$\mathbb{F}G :=$ associative $\mathbb{F}$-algebra with generators $G$ and relations $g \cdot \tilde{g} - (g\tilde{g}) = 0$ for $g, \tilde{g} \in G$.

# Group algebras — properties

$\mathbb{F}$: field, $G$: group, $\mathbb{F}G$: group algebra, $V$: $\mathbb{F}$-vector space.

There is a bijection between

$$\{\varphi : G \to \mathrm{GL}(V) \mid \varphi \text{ is a group homomorphism}\}$$

and

$$\{\psi : \mathbb{F}G \to \mathrm{End}_{\mathbb{F}}(V) \mid \psi \text{ is an algebra homomorphism}\}$$

Given $\varphi : G \to \mathrm{GL}(V)$, define

$$\psi\left(\sum_{g \in G} \lambda_g \cdot g\right) := \sum_{g \in G} \lambda_g \cdot \varphi(g)$$

(use finite presentation).

Given $\psi : \mathbb{F}G \to \mathrm{End}_{\mathbb{F}}(V)$, simply restrict $\varphi := \psi|_G$, since

$$\mathbf{1}_V = \psi(\mathbf{1}_G) = \psi(g \cdot g^{-1}) = \psi(g) \cdot \psi(g^{-1}) \qquad \text{for all } g \in G.$$

# Modules

**Definition ($G$-module or $\mathbb{F}G$-module)**

An $\mathbb{F}$-vector space $V$ together with

- a group homomorphism $\varphi : G \to \mathrm{GL}(V)$,
- or an algebra homomorphism $\psi : \mathbb{F}G \to \mathrm{End}_{\mathbb{F}}(V)$

is called a $G$-module over $\mathbb{F}$ or an $\mathbb{F}G$-module.

This is nothing but

an $\mathbb{F}$-vector space with an $\mathbb{F}$-linear action for $G$.

This is nothing but

an $\mathbb{F}$-linear representation for $G$.

# Kernels and faithfulness

Let $A : X \times G \to X$ be an action, or equivalently, let $R : G \to X^X$ be a representation.

Depending on the types of $G$ and $X$, it might make sense to speak of the kernel of the representation $R$ or not.

**Definition (Faithful representation/action)**

We call the representation $R$ (or the action $A$) faithful, if its kernel $\ker R$ is trivial.

Note: If a $G$-module $V$ over $\mathbb{F}$ is faithful, it does not necessarily follow that the corresponding $\mathbb{F}G$-module $V$ is faithful!

# Homomorphisms and isomorphisms

Let $A : X \times G \to X$ and $\tilde{A} : \tilde{X} \times G \to \tilde{X}$ be two actions.

## Definition ($G$-homomorphism)

A homomorphism $\varphi : X \to \tilde{X}$ is called a $G$-homomorphism or $G$-equivariant, if

$$\varphi(x \cdot g) = \varphi(x) \cdot g \quad \text{for all } x \in X \text{ and all } g \in G.$$

Equivalently, this means

$$\varphi(A(x, g)) = \tilde{A}(\varphi(x), g) \quad \text{for all } x \in X \text{ and all } g \in G.$$

Equivalently, this means that this diagram commutes:

$$\begin{tikzcd} X \times G \arrow[r, "A"] \arrow[d, "\varphi \times \mathrm{id}_G"'] & X \arrow[d, "\varphi"] \\ \tilde{X} \times G \arrow[r, "\tilde{A}"'] & \tilde{X} \end{tikzcd}$$

If $\varphi$ has a $G$-equiv. inverse, it is called a $G$-isomorphism.

# Subacts

Let $G$ act on $X$, i.e. $A : X \times G \to X$.

## Definition ($G$-invariant subset, Subact)

A subset $Y \subseteq X$ is called $G$-invariant, if

$$y \cdot g \in Y \qquad \text{for all } y \in Y \text{ and all } g \in G.$$

The restriction $A|_{Y \times G}$ is then a map to $Y$ and $G$ acts on $Y$.
If $Y \subseteq X$ is also a substructure of $X$, we call $Y$ a subact (or submodule resp.).

Recall: A permutation representation was called transitive if it has no proper subacts.

## Definition (Irreducible/simple module)

An $\mathbb{F}G$-module $M$ is called irreducible or simple, if it has no submodules except 0 and $M$ itself.

# Factor acts

Let $G$ act on $X$, i.e. $A : X \times G \to X$.

## Definition ($G$-invariant partition, factor act)

Let $X = \dot{\bigcup}_{i \in I} Y_i$ be partitioned such that

$$\forall\, i \in I \text{ and } g \in G, \text{ we have } Y_i \cdot g \subseteq Y_j \text{ for some } j \in I.$$

We say that the partition is $G$-invariant and get an action on the set of parts $Y := \{Y_i \mid i \in I\}$:

$$Y_i * g := Y_j \quad \text{if} \quad Y_i \cdot g \subseteq Y_j.$$

Recall: We call a permutation action primitive, if it has no non-trivial factor acts.

Note: We usually want extra conditions to ensure that $Y$ has the same algebraic structure as $X$ and the new action is a homomorphism of such structures for all $g$.

# Extensions and direct sums

This is only about modules!

Let

$$0 \longrightarrow W \xrightarrow{i} V \xrightarrow{\pi} U \cong V/W \longrightarrow 0$$

be a module $V$ with a non-trivial submodule.

This sequence may or may not be split:

$$\begin{tikzcd} 0 \arrow[r] & W \arrow[r, "i"] & V \arrow[r, "\pi"] & U \arrow[l, bend left, "r"] \arrow[r] & 0 \end{tikzcd},$$

i.e. there is $r : U \to W$ with $\pi \circ r = \mathrm{id}_U$.

If and only if it is split, the module $V$ is isomorphic to the direct sum

$$V \cong W \oplus U.$$

# Indecomposability and semisimplicity

## Definition (Indecomposable module)

An $\mathbb{F}G$-module $V$ is called indecomposable if it is not isomorphic to a direct sum of two proper submodules. Otherwise it is called decomposable.

## Lemma (Decomposable implies reducible)

*A decomposable module is reducible.*

## Definition (Semisimple modules and algebras)

A module is called semisimple, if it is isomorphic to a direct sum of simple modules.
An $\mathbb{F}$-algebra $\mathcal{A}$ is called semisimple, if every $\mathcal{A}$-module is semisimple.

# Ordinary representation theory of groups

For a finite group, the group algebra $\mathbb{C}G$ is semisimple.

The ordinary representation theory of groups solves:

## Problem (Classification of simple modules)

*Classify the isomorphism types of simple $\mathbb{C}G$-modules, i.e. classify irreducible $\mathbb{C}G$-modules up to isomorphism.*

## Lemma (Characters)

*Two representations*

$$R_1 : G \to \mathrm{GL}(V) \qquad \text{and} \qquad R_2 : G \to \mathrm{GL}(W)$$

*afforded by two $\mathbb{C}G$-modules $V$ and $W$ are isomorphic, if and only if their characters $\chi_1 = \mathrm{Tr} \circ R_1$ and $\chi_2 = \mathrm{Tr} \circ R_2$ are equal.*
*The two characters $\chi_i : G \to \mathbb{C}$ are class functions.*

# Research problems in ordinary rep. theory

Already done:

- Character tables of symmetric groups.
- Character tables of alternating groups.
- The ATLAS (character tables of simple groups).
- Some generic character tables.

Still to do:

- Determine character tables for more groups.
- Determine more generic tables for whole families of groups.
- Devise better algorithms to compute tables.

# Modular representation theory of groups

$\mathbb{F}$: field with $\mathrm{char}(\mathbb{F}) \mid |G|$, then $\mathbb{F}G$ is not semisimple.

The modular rep. theory of groups strives to solve:

### Problem (Classification of simple modules)

*Classify the isomorphism types of simple $\mathbb{F}G$-modules, i.e. classify irreducible $\mathbb{F}G$-modules up to isomorphism.*

### Problem (Classification of indecomposable modules)

*Classify the isomorphism types of indecomposable $\mathbb{F}G$-modules.*

### Lemma (Brauer characters)

*Two irreducible representations $R_1 : G \to \mathrm{GL}(V)$ and $R_2 : G \to \mathrm{GL}(W)$ afforded by two $\mathbb{F}G$-modules $V$ and $W$ are isomorphic, if and only if their Brauer characters $\psi_1$ and $\psi_2$ are equal.*
*The two Brauer characters $\psi_i$ take values in $\mathbb{C}$!*

# Research problems in modular rep. theory

Already done:

- Brauer tables of some small symmetric groups ($n \leq 18$).
- Brauer tables of some small alternating groups.
- Modular ATLAS (Brauer tables of simple groups). 1992 by Hiß, Jansen, Lux and Parker: groups up to page 100 in the ATLAS, now some more.

Still to do:

- Determine Brauer tables for more groups.
- Complete the Modular ATLAS.
- Classify simple modules of $\mathbb{F}S_n$.
- Compute the 2-modular Brauer table of the Monster.
- Find an algorithm to compute a Brauer table???
- Classify indecomposable $\mathbb{F}G$-modules???

# Permutation groups

## Problem (Permutation group algorithms)

*Given $G := \langle g_1, \ldots, g_k \in S_n \rangle \leq S_n$ on a computer.*
*Find efficient algorithms to compute with and in $G$:*

- *Test membership of $\pi \in S_n$ in $G$.*
- *Find the group order $|G|$.*
- *Decide whether $G = A_n$ or $G = S_n$ or none.*
- *Find orbits and blocks of primitivity.*
- *Find a presentation.*
- *Find the centre of $G$.*
- ...

All of this is done and works well in nearly linear time:

runtime is bounded by $C \cdot n \cdot k \cdot \log^D(|G|)$.

# Open questions for permutation groups

Still to do (in nearly linear time):

- Compute the centraliser $C_G(H)$ for some $H < S_n$.
- Compute the derived subgroup $G'$.
- Compute intersections of $G, H < S_n$.
- Compute conjugacy classes of permutation groups.
- Test $G, H < S_n$ for conjugacy.

# Matrix and projective groups

**Problem (Matrix group algorithms)**

*Given* $G := \langle M_1, \ldots, M_k \in \mathrm{GL}(\mathbb{F}_q^n) \rangle \leq \mathrm{GL}(\mathbb{F}_q^n)$ *on a computer.*
*Ultimate goal: Answer similar questions as for permutation groups.*

This is largely unsolved!

**Problem (Projective group algorithms)**

*Given* $G := \left\langle \bar{M}_1, \ldots, \bar{M}_k \in \mathrm{PGL}(n, q) \right\rangle \leq \mathrm{PGL}(n, q)$ *on a computer.*
*Ultimate goal: Answer similar questions as for permutation groups.*

# Constructive recognition

## Problem (Constructive recognition)

*Let $\mathbb{F}_q$ be the field with $q$ elements und*

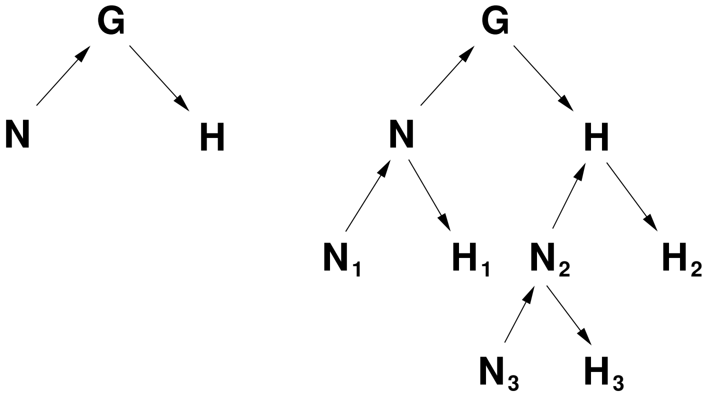
$$M_1, \ldots, M_k \in \mathrm{GL}(\mathbb{F}_q^n).$$

*Find for $G := \langle M_1, \ldots, M_k \rangle$:*

- *The group order $|G|$ and*
- *an algorithm that, given $M \in \mathrm{GL}(\mathbb{F}_q^n)$,*
  - *decides, whether or not $M \in G$, and,*
  - *if so, expresses $M$ as word in the $M_i$.*
- *The runtime should be bounded from above by a polynomial in $n$, $k$ and $\log q$.*
- *A Monte Carlo Algorithmus is enough. (Verification!)*

# Recursion: composition trees

We get a tree:




Up arrows: inclusions

Down arrows: homomorphisms

Old idea, improvements are still being made

# Enumerating large orbits

Orbit enumerations play an important role in

- modular representation theory,
- permutation group algorithms,
- matrix and projective group algorithms,
- combinatorics,
- finite geometry.

To get a feeling:

- To enumerate an orbit of 1140000 vectors in $\mathbb{F}_2^{760}$ needs around 90 seconds.
- To enumerate 95% of the same orbit with better tricks takes 1.1 seconds.

Finding better ways to enumerate orbits is a current research topic.