Finding normal subgroups

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- "High probability" means for the moment "higher than $1/[G : N]$".

Matrix groups . . .

Let $\mathbb{F}_q$ be the field with q elements and

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Constructive recognition

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$$M_1, \ldots, M_k \in \mathrm{GL}_n(\mathbb{F}_q).$$

Find for $G := \langle M_1, \ldots, M_k \rangle$:

- The group order $|G|$ and
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If this problem is solved, we call

$\langle M_1, \ldots, M_k \rangle$ recognised constructively.

✓

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- H is in some sense "smaller"
- or at least "easier to recognise constructively"
- e.g. $H \leq S_m$ or $H \leq \mathrm{GL}_{n'}(\mathbb{F}_{q'})$ with $n' \log q' < n \log q$

Recursion: composition trees

We get a tree:

$$\begin{array}{ccc} & G & \\ \nearrow & & \searrow \\ N & & H \end{array}$$

Up arrows: inclusions
Down arrows: homomorphisms

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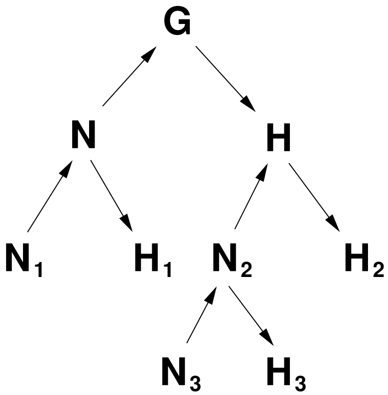

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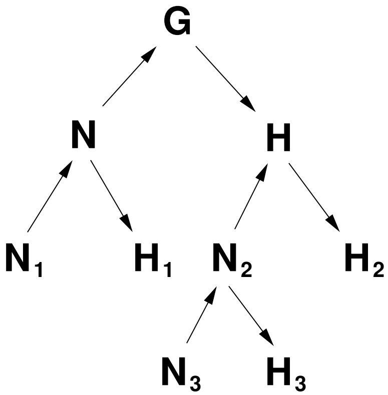

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Old idea, substantial improvements are still being made

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Since we can compute normal closures, our initial problem is exactly, what we need to do.

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The latter means that for $H < G$, we can compute some elements that generate with high probability the smallest normal subgroup of G containing H.

Blind descent (Babai, Beals)

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   - If some $c_i := [x, y_i] \neq 1$, then take $z := c_i$ as in 1.
   - Otherwise $g \in C_G(Y)$ but $g \notin Z(G)$, thus $Y \neq G$, we take $z := y$.

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5. $x := \textsc{BlindDescent}(x, y')$

Return x

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Input: $G = \langle g_1, \ldots, g_k \rangle$ and an involution $x \in G$.
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  if o is even **then**
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Note: If $xy = yx$ then $c = 1_G$ and $o = 1$ and $z = y$.

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But this happens rarely.

What does the Involution Jumper do?

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- It needs an involution to start with.

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Proof: $f(x, y) :=$

$$\begin{cases} [x,y]^k & \text{if } \mathrm{ORDER}([x,y]) = 2k \\ (y[x,y]^k)^l & \text{if } \mathrm{ORDER}([x,y]) = 2k+1 > 1 \text{ and} \\ & \quad \mathrm{ORDER}([y[x,y]^k]) = 2l \\ y^k & \text{if } xy = yx \text{ and } \mathrm{ORDER}(y) = 2k \end{cases}$$

and we have $f(x^g, y^g) = f(x, y)^g$ whenever f is defined. ✓

A Markov chain $\mathcal{M}$

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The transition is done as follows: At a class a^G:

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Theorem

The above Markov chain $\mathcal{M}$ is irreducible and aperiodic and thus has a stationary distribution in which every state has non-zero probability.

Back to the original question

Problem

Let $1 < N \lhd G = \langle g_1, \ldots, g_k \rangle$ be a finite group and N be a normal subgroup.
Produce a non-trivial element of N as a word in the g_i with "high probability".

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the IJ will eventually jump onto an involution class in N.

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Repeat K times:

1. $y := \textsc{RandomElement}(G)$
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3. For a few prime divisors p of o do:
   - $y' := y^{o/p}$ has order p
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4. $z := \textsc{InvolutionJumper}(G, z)$
5. $x := \textsc{BlindDescent}(x, z)$

Return x

Examples

In practice, the IJ works extremely well in many cases:

G	N	# hops*
$S_5 \wr S_{10}$	$S_5^{\times 10}$	1.91
$\mathrm{GL}(3,3) \wr S_6 < \mathrm{GL}(18,3)$	$\mathrm{GL}(3,3)^{\times 6}$	1.17
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Running Algorithm 2 (with $K = 5$) also works nicely:

G	N	succ.
$S_5 \wr S_{10}$	$S_5^{\times 10}$	100%
$\mathrm{GL}(3,3) \wr S_6 < \mathrm{GL}(18,3)$	$\mathrm{GL}(3,3)^{\times 6}$	100%
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(here we have done 100 runs)

Reductions for imprimitive matrix groups

Situation

Let $G \leq \mathrm{GL}_n(\mathbb{F}_q)$ acting linearly on $V := \mathbb{F}_q^{1\times n}$, such that V is irreducible. Assume there is N with $Z(G) < N \lhd G$ such that

$$V|_N = W_1 \oplus W_2 \oplus \cdots \oplus W_k,$$

all W_i are invariant under N, and G permutes the W_i transitively. Then there is a reduction $\varphi : G \to S_k$.

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all W_i are invariant under N, and G permutes the W_i transitively. Then there is a reduction $\varphi : G \to S_k$.

We use Algorithm 2, for the result x, do:

- compute the normal closure $M := \langle x^G \rangle$,
- use the MeatAxe to check whether $V|_M$ is reducible,
- if $x \in N$, we find a reduction.

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Fortunately: Centralisers of involutions seem to contain enough involutions.