Max Neunhöffer

Background Constructive recognition Reductions Composition trees

Aschbacher's Theorem

A variant for GL and its proof The Statement Reducible: \mathcal{D}_1 Not abs. irred: \mathcal{D}_3 G/Z is simple: \mathcal{D}_6 or \mathcal{D}_6 Clifford theory W not abs. irred: \mathcal{D}_3 V i_N not homogeneous: \mathcal{D}_4 V i_N not homogeneous: \mathcal{D}_4 V i_N a cases for N/Z: \mathcal{D}_6 - \mathcal{D}_6

Application

Classes D2 and I Class D7

Aschbacher's Theorem revisited with a view to Constructive Matrix Group Recognition

Max Neunhöffer



University of St Andrews

Auckland, 28.1.2009

Max Neunhöffer

Background Constructive recognition Reductions Composition trees

Aschbacher's Theorem

A variant for GL and its proof The Statement Reducible: \mathfrak{D}_1 Not abs. i.red.: \mathfrak{D}_3 Subfield: \mathfrak{D}_2 G/Z is simple: \mathfrak{D}_0 or \mathfrak{D}_9 Ciliford theory W not abs. i.red.: \mathfrak{D}_2 Vi_N not homogeneous: \mathfrak{D}_4 S cases for N/Z: \mathfrak{D}_8 - \mathfrak{D}_9

Application

Classes D2 and Da Class D7

Constructive recognition of matrix groups

Problem

Let \mathbb{F}_q be the field with q elements und

 $M_1,\ldots,M_k\in \operatorname{GL}_n(\mathbb{F}_q).$

- The group order |G| and
- an algorithm that, given $M \in GL_n(\mathbb{F}_q)$,
 - decides, whether or not $M \in G$, and,
 - if so, expresses *M* as word in the *M_i*.

Max Neunhöffer

Background Constructive recognition Reductions Composition trees

Aschbacher's Theorem

A variant for GL and its proof The Statement Reducible: \mathcal{D}_1 Not abs. irred.: \mathcal{D}_2 Subfield: \mathcal{D}_3 Cilliford theory Win ot abs. irred.: \mathcal{D}_2 Vi_N not homogeneous: \mathcal{D}_4 Vi_N homogeneous: \mathcal{D}_4 3 cases for N/2: \mathcal{D}_6 – \mathcal{D}_9

Application

Classes D₂ and D Class D₇

Constructive recognition of matrix groups

Problem

Let \mathbb{F}_q be the field with q elements und

 $M_1,\ldots,M_k\in \operatorname{GL}_n(\mathbb{F}_q).$

- The group order |G| and
- an algorithm that, given $M \in GL_n(\mathbb{F}_q)$,
 - decides, whether or not $M \in G$, and,
 - if so, expresses *M* as word in the *M_i*.
- The runtime should be bounded from above by a polynomial in *n*, *k* and log *q*.

Max Neunhöffer

Background Constructive recognition Reductions Composition trees

Aschbacher's Theorem

A variant for GL and its proof The Statement Reducible: \mathcal{D}_1 Not abs. irred: \mathcal{D}_2 Subfield: \mathcal{D}_2 G/Z is simple: \mathcal{D}_0 or \mathcal{D}_0 Ciliford theory W not abs. irred: \mathcal{D}_2 W not homogeneous: \mathcal{D}_4 Vi_N homogeneous: \mathcal{D}_4 3 cases for N/Z: \mathcal{D}_0 - \mathcal{D}_0

Application

Classes D2 and Da Class D7

Constructive recognition of matrix groups

Problem

Let \mathbb{F}_q be the field with q elements und

 $M_1,\ldots,M_k\in \operatorname{GL}_n(\mathbb{F}_q).$

- The group order |G| and
- an algorithm that, given $M \in GL_n(\mathbb{F}_q)$,
 - decides, whether or not $M \in G$, and,
 - if so, expresses *M* as word in the *M_i*.
- The runtime should be bounded from above by a polynomial in *n*, *k* and log *q*.
- A Monte Carlo Algorithm is enough.

Max Neunhöffer

Background Constructive recognition Reductions Composition trees

Aschbacher's Theorem

A variant for GL and its proof The Statement Reducible: \mathcal{D}_1 Not abs. irred: \mathcal{D}_2 Subfield: \mathcal{D}_2 G/Z is simple: \mathcal{D}_0 or \mathcal{D}_0 Ciliford theory W not abs. irred: \mathcal{D}_2 W not homogeneous: \mathcal{D}_4 Vi_N homogeneous: \mathcal{D}_4 3 cases for N/Z: \mathcal{D}_0 - \mathcal{D}_0

Application

Classes D2 and Da Class D7

Constructive recognition of matrix groups

Problem

Let \mathbb{F}_q be the field with q elements und

 $M_1,\ldots,M_k\in \operatorname{GL}_n(\mathbb{F}_q).$

- The group order |G| and
- an algorithm that, given $M \in GL_n(\mathbb{F}_q)$,
 - decides, whether or not $M \in G$, and,
 - if so, expresses *M* as word in the *M_i*.
- The runtime should be bounded from above by a polynomial in *n*, *k* and log *q*.
- A Monte Carlo Algorithm is enough. (Verification!)

Max Neunhöffer

Background Constructive recognition Reductions Composition trees

Aschbacher's Theorem

A variant for GL and its proof The Statement Reducible: \mathcal{D}_1 Not abs. irred: \mathcal{D}_2 Subfield: \mathcal{D}_2 G/Z is simple: \mathcal{D}_0 or \mathcal{D}_0 Ciliford theory W not abs. irred: \mathcal{D}_2 W not homogeneous: \mathcal{D}_4 Vi_N homogeneous: \mathcal{D}_4 3 cases for N/Z: \mathcal{D}_0 - \mathcal{D}_0

Application

Classes D2 and Da Class D7

Constructive recognition of matrix groups

Problem

Let \mathbb{F}_q be the field with q elements und

 $M_1,\ldots,M_k\in \operatorname{GL}_n(\mathbb{F}_q).$

Find for $G := \langle M_1, \ldots, M_k \rangle$:

- The group order |G| and
- an algorithm that, given $M \in GL_n(\mathbb{F}_q)$,
 - decides, whether or not $M \in G$, and,
 - if so, expresses *M* as word in the *M_i*.
- The runtime should be bounded from above by a polynomial in *n*, *k* and log *q*.
- A Monte Carlo Algorithm is enough. (Verification!)

If this problem is solved, we call $\langle M_1, \ldots, M_k \rangle$ recognised constructively.

Max Neunhöffer

Background Constructive recognition Reductions

Aschbacher's Theorem

A variant for GL and its proof The Statement Reducible: \mathcal{D}_1 Not abs. irred: \mathcal{D}_3 Subfiel: \mathcal{D}_2 Gilford theory V not abs. irred: \mathcal{D}_3 or \mathcal{D}_3 Clifford theory V not abs. irred: \mathcal{D}_3 V $_{IN}$ not homogeneous: \mathcal{D}_2 V $_{IN}$ homogeneous: \mathcal{D}_2 S cases for N/2: \mathcal{D}_2 - \mathcal{D}_3

Application

Classes D2 and D Class D7

Reductions

Let $G := \langle M_1, \ldots, M_k \rangle \leq \operatorname{GL}_n(\mathbb{F}_q).$

Max Neunhöffer

Background Constructive recognition Reductions Composition trees

Aschbacher's Theorem

A variant for GL and its proof The Statement Reducible: \mathcal{D}_1 Not abs. irred: \mathcal{D}_3 Subfiel: \mathcal{D}_2 G/Z is simple: \mathcal{D}_0 or \mathcal{D}_0 Clifford theory W not abs. irred: \mathcal{D}_3 V $|_{V_1}$ not homogeneous: \mathcal{D}_2 V $|_{V_1}$ homogeneous: \mathcal{D}_2 d cases for W/C2, \mathcal{D}_2 -D-b

Application

Classes D2 and Da Class D7

Reductions

Let
$$G := \langle M_1, \ldots, M_k \rangle \leq \operatorname{GL}_n(\mathbb{F}_q).$$

A reduction is a group homomorphism

$$\begin{array}{rccc} \varphi & : & G & \to & H \\ & & M_i & \mapsto & P_i & \text{ for all } i \end{array}$$

Max Neunhöffer

Background Constructive recognition Reductions Composition trees

Aschbacher's Theorem

A variant for GL and its proof The Statement Reducible: \mathcal{D}_1 Not abs. irred: \mathcal{D}_3 Subfiel: \mathcal{D}_2 G/Z is simple: \mathcal{D}_0 or \mathcal{D}_0 Clifford theory W not abs. irred: \mathcal{D}_3 or \mathcal{D}_3 V_{IN} not homogeneous: \mathcal{D}_2 V_{IN} nongeneous: \mathcal{D}_2 a cases for N/2: \mathcal{D}_2 - \mathcal{D}_2 - \mathcal{D}_3

Application

Classes D₂ and D. Class D₇

Reductions

Let $G := \langle M_1, \ldots, M_k \rangle \leq \operatorname{GL}_n(\mathbb{F}_q).$

A reduction is a group homomorphism

$$\begin{array}{rccc} \varphi & : & G & \to & H \\ & & M_i & \mapsto & P_i & \text{ for all } i \end{array}$$

with the following properties:

• $\varphi(M)$ is explicitly computable for all $M \in G$

Max Neunhöffer

Background Constructive recognition Reductions Composition trees

Aschbacher's Theorem

A variant for GL and its proof The Statement Reductive: \mathcal{D}_1 Not abs. $irred: \mathcal{D}_3$ Subfiel: \mathcal{D}_2 G/Z is simple: \mathcal{D}_0 or \mathcal{D}_9 Clifford theory W not abs. $irred: \mathcal{D}_3$ V/w, not homogeneous: \mathcal{D}_4 V/w, homogeneous: \mathcal{D}_4 S cases for V/Z: \mathcal{D}_9 - \mathcal{D}_9

Application

Classes D₂ and I Class D₇

Reductions

Let
$$G := \langle M_1, \ldots, M_k \rangle \leq \operatorname{GL}_n(\mathbb{F}_q).$$

A reduction is a group homomorphism

$$\begin{array}{rccc} \varphi & : & G & \to & H \\ & & M_i & \mapsto & P_i & \text{ for all } i \end{array}$$

- $\varphi(M)$ is explicitly computable for all $M \in G$
- φ is surjective: $H = \langle P_1, \ldots, P_k \rangle$

Max Neunhöffer

Background Constructive recognition Reductions Composition trees

Aschbacher's Theorem

A variant for GL and its proof The Statement Reducible: \mathcal{D}_1 Not abs. irred: \mathcal{D}_2 Subfield: \mathcal{D}_3 Gilfiord theory Win ot abs. irred: \mathcal{D}_2 Vi_N not homogeneous: \mathcal{D}_4 Vi_N homogeneous: \mathcal{D}_4 3 cases for N/2: \mathcal{D}_6 – \mathcal{D}_9

Application

Classes D2 and D Class D7

Reductions

Let $G := \langle M_1, \ldots, M_k \rangle \leq \operatorname{GL}_n(\mathbb{F}_q).$

A reduction is a group homomorphism

$$\begin{array}{rccc} \varphi & : & G & \to & H \\ & & M_i & \mapsto & P_i & \text{ for all } i \end{array}$$

- $\varphi(M)$ is explicitly computable for all $M \in G$
- φ is surjective: $H = \langle P_1, \ldots, P_k \rangle$
- *H* is in some sense "smaller"
- or at least "easier to recognise constructively"

Max Neunhöffer

Background Constructive recognition Reductions Composition trees

Aschbacher's Theorem

A variant for GL and its proof The Statement Reducible: \mathfrak{D}_1 Not abs. irred: \mathfrak{D}_2 Subfiel: \mathfrak{D}_2 Clifford theory W not abs. irred: \mathfrak{D}_2 or \mathfrak{D}_3 Clifford theory W not abs. irred: \mathfrak{D}_2 Vi_N not homogeneous: \mathfrak{D}_4 S cases for N/2: \mathfrak{D}_6 – \mathfrak{D}_5

Application

Classes D₂ and D Class D₇

Reductions

Let $G := \langle M_1, \ldots, M_k \rangle \leq \operatorname{GL}_n(\mathbb{F}_q)$.

A reduction is a group homomorphism

$$\begin{array}{rcl} \varphi & : & G & \to & H \\ & & M_i & \mapsto & P_i & \text{ for all } i \end{array}$$

- $\varphi(M)$ is explicitly computable for all $M \in G$
- φ is surjective: $H = \langle P_1, \ldots, P_k \rangle$
- *H* is in some sense "smaller"
- or at least "easier to recognise constructively"
- e.g. $H \leq S_m$ or $H \leq \operatorname{GL}_{n'}(\mathbb{F}_{q'})$ with $n' \log q' < n \log q$

Max Neunhöffer

Background Constructive recognition Reductions Composition trees

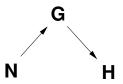
Aschbacher's Theorem

A variant for GL and its proof The Statement Reducible: \mathcal{D}_1 Not abs. irred: \mathcal{D}_3 Subfiel: \mathcal{D}_2 G/Z is simple: \mathcal{D}_3 or \mathcal{D}_3 Clifford theory W not abs. irred: \mathcal{D}_3 V /_N not homogeneous: \mathcal{D}_2 V /_N no homogeneous: \mathcal{D}_2 a cases for N/2: \mathcal{D}_2 - \mathcal{D}_3

Application

Classes D2 and D Class D7

Recursive reduction: composition trees We get a tree:



Up arrows: inclusions Down arrows: homomorphisms

Max Neunhöffer

Background Constructive recognition Reductions Composition trees

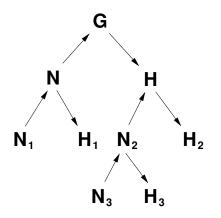
Aschbacher's Theorem

A variant for GL and its proof The Statement Reducible: \mathcal{D}_1 Not abs. irred: \mathcal{D}_3 Subfiel: \mathcal{D}_2 G/Z is simple: \mathcal{D}_0 or \mathcal{D}_0 Clifford theory W not abs. irred: \mathcal{D}_3 or \mathcal{D}_3 V_{IN} not homogeneous: \mathcal{D}_2 V_{IN} nongeneous: \mathcal{D}_2 a cases for N/2: \mathcal{D}_2 - \mathcal{D}_2 - \mathcal{D}_3

Application

Classes D2 and D Class D7

Recursive reduction: composition trees We get a tree:



Up arrows: inclusions Down arrows: homomorphisms

Max Neunhöffer

Background Constructive recognition Reductions Composition trees

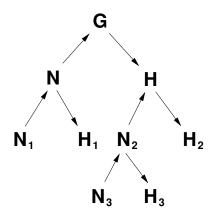
Aschbacher's Theorem

A variant for GL and its proof The Statement Reducible: \mathcal{D}_1 Not abs. irred: \mathcal{D}_3 Subfiel: \mathcal{D}_2 G/Z is simple: \mathcal{D}_3 or \mathcal{D}_3 Clifford theory W not abs. irred: \mathcal{D}_3 V /_N not homogeneous: \mathcal{D}_2 V /_N no homogeneous: \mathcal{D}_2 a cases for N/2: \mathcal{D}_2 - \mathcal{D}_3

Application

Classes D₂ and £ Class D₇

Recursive reduction: composition trees We get a tree:



Up arrows: inclusions Down arrows: homomorphisms

Old idea, improvements are still being made

Max Neunhöffer

Background Constructive recognition Reductions Composition trees

Aschbacher's Theorem

A variant for GL and its proof The Statement Reducible: \mathcal{D}_1 Not abs. irred: \mathcal{D}_3 Subfiel: \mathcal{D}_2 G/Z is simple: \mathcal{D}_0 or \mathcal{D}_0 Clifford theory W not abs. irred: \mathcal{D}_3 or \mathcal{D}_3 V_{IN} not homogeneous: \mathcal{D}_2 V_{IN} nongeneous: \mathcal{D}_2 a cases for N/2: \mathcal{D}_2 - \mathcal{D}_2 - \mathcal{D}_3

Application

Classes D₂ and D₄ Class D₇

Which theorem of Aschbacher do I mean?

Max Neunhöffer

Background Constructive recognition Reductions Composition trees

Aschbacher's Theorem

A variant for GL and its proof The Statement Reducible: \mathcal{D}_1 Not abs. irred: \mathcal{D}_3 Subfiel: \mathcal{D}_2 G/Z is simple: \mathcal{D}_0 or \mathcal{D}_0 Clifford theory W not abs. irred: \mathcal{D}_3 or \mathcal{D}_1 V_{IN} not homogeneous: \mathcal{D}_2 V_{IN} no homogeneous: \mathcal{D}_2 S cases for N/2: \mathcal{D}_2 - \mathcal{D}_3

Application

Classes D₂ and D₄ Class D₇

Which theorem of Aschbacher do I mean?

Theorem (Aschbacher 1984)

Let G_0 be a simple classical group over a finite field and $G_0 \le G \le Aut(G_0)$. Let H < G such that $HG_0 = G$.

Max Neunhöffer

Background Constructive recognitio Reductions Composition trees

Aschbacher's Theorem

A variant for GL and its proof The Statement Reducible: \mathfrak{D}_1 Not abs. irred: \mathfrak{D}_3 Subfield: \mathfrak{D}_5 G/Z is simple: \mathfrak{D}_6 or \mathfrak{D}_9 Clifford theory W not abs. irred: \mathfrak{D}_3 V $_{1N}$ not homogeneous: \mathfrak{D}_2 V $_{1N}$ not mongeneous: \mathfrak{D}_2 V $_{1N}$ not mongeneous: \mathfrak{D}_2 S cases for N/2: \mathfrak{B}_7 - \mathfrak{D}_9

Application

Classes D₂ and D Class D₇

Which theorem of Aschbacher do I mean?

Theorem (Aschbacher 1984)

Let G_0 be a simple classical group over a finite field and $G_0 \leq G \leq Aut(G_0)$. Let H < G such that $HG_0 = G$. Define geometrically classes C_1 to C_8 of subgroups of G.

Max Neunhöffer

Background Constructive recognitio Reductions Composition trees

Aschbacher's Theorem

A variant for GL and its proof The Statement Reducible: \mathfrak{D}_1 Not abs. irred: \mathfrak{D}_2 G/Z is simple: \mathfrak{D}_3 or \mathfrak{D}_9 G/Z is simple: \mathfrak{D}_3 or \mathfrak{D}_9 G/Ifford theory W not abs. irred: \mathfrak{D}_2 V I_N not homogeneous: \mathfrak{D}_2 V I_N not homogeneous: \mathfrak{D}_2 X I_N as cases for N/Z: \mathfrak{B}_9 - \mathfrak{D}_9

Application

Classes D2 and Da Class D7

Which theorem of Aschbacher do I mean?

Theorem (Aschbacher 1984)

Let G_0 be a simple classical group over a finite field and $G_0 \le G \le \operatorname{Aut}(G_0)$. Let H < G such that $HG_0 = G$. Define geometrically classes C_1 to C_8 of subgroups of G. Then either H is a subgroup of at least one of the groups in classes C_1 to C_8 , or the following hold:

Max Neunhöffer

Background Constructive recognition Reductions Composition trees

Aschbacher's Theorem

A variant for GL and its proof The Statement Reducible: \mathcal{D}_1 Not abs. i.red.: \mathcal{D}_3 Subfield: \mathcal{D}_5 G/Z is simple: \mathcal{D}_6 or \mathcal{D}_5 Clifford theory W not abs. i.red.: \mathcal{D}_3 V_{1/N} not homogeneous: \mathcal{D}_4 V_{1/N} not homogeneous: \mathcal{D}_4 3 cases for N/Z: \mathcal{D}_5 - \mathcal{D}_6

Application

Classes D2 and Da Class D7

Which theorem of Aschbacher do I mean?

Theorem (Aschbacher 1984)

Let G_0 be a simple classical group over a finite field and $G_0 \le G \le \operatorname{Aut}(G_0)$. Let H < G such that $HG_0 = G$. Define geometrically classes C_1 to C_8 of subgroups of G. Then either H is a subgroup of at least one of the groups in classes C_1 to C_8 , or the following hold:

• There is a non-abelian simple group H_0 with $H_0 \le H \le \operatorname{Aut}(H_0)$.

Max Neunhöffer

Background Constructive recognition Reductions Composition trees

Aschbacher's Theorem

A variant for GL and its proof The Statement Reductive: \mathcal{D}_1 Not abs. $irred: \mathcal{D}_3$ Subfiel: \mathcal{D}_2 G/Z is simple: \mathcal{D}_0 or \mathcal{D}_9 Clifford theory W not abs. $irred: \mathcal{D}_3$ V/w, not homogeneous: \mathcal{D}_4 V/w, homogeneous: \mathcal{D}_4 S cases for V/Z: \mathcal{D}_9 - \mathcal{D}_9

Application

Classes D₂ and D Class D₇

Which theorem of Aschbacher do I mean?

Theorem (Aschbacher 1984)

Let G_0 be a simple classical group over a finite field and $G_0 \le G \le \operatorname{Aut}(G_0)$. Let H < G such that $HG_0 = G$. Define geometrically classes C_1 to C_8 of subgroups of G. Then either H is a subgroup of at least one of the groups in classes C_1 to C_8 , or the following hold:

- There is a non-abelian simple group H_0 with $H_0 \le H \le \operatorname{Aut}(H_0)$.
- The natural H-module V is absolutely irreducible.

Max Neunhöffer

Background Constructive recognition Reductions Composition trees

Aschbacher's Theorem

A variant for GL and its proof The Statement Reducible: \mathcal{D}_1 Not abs. $irred: \mathcal{D}_3$ Subfiel: \mathcal{D}_2 G/Z is simple: \mathcal{D}_0 or \mathcal{D}_9 Ciliford theory Win to homogeneous: \mathcal{D}_4 Vily, homogeneous: \mathcal{D}_4 S cases for V/Z: \mathcal{D}_9 – \mathcal{D}_9

Application

Classes D₂ and D Class D₇

Which theorem of Aschbacher do I mean?

Theorem (Aschbacher 1984)

Let G_0 be a simple classical group over a finite field and $G_0 \le G \le \operatorname{Aut}(G_0)$. Let H < G such that $HG_0 = G$. Define geometrically classes \mathcal{C}_1 to \mathcal{C}_8 of subgroups of G. Then either H is a subgroup of at least one of the groups in classes \mathcal{C}_1 to \mathcal{C}_8 , or the following hold:

- There is a non-abelian simple group H_0 with $H_0 \le H \le \operatorname{Aut}(H_0)$.
- The natural H-module V is absolutely irreducible.
- This representation for H cannot be realised over a smaller field.

Max Neunhöffer

Background Constructive recognition Reductions Composition trees

Aschbacher's Theorem

A variant for GL and its proof The Statement Reducible: \mathcal{D}_1 Not abs. $irred: \mathcal{D}_3$ Subfiel: \mathcal{D}_2 G/Z is simple: \mathcal{D}_0 or \mathcal{D}_9 Ciliford theory Win to homogeneous: \mathcal{D}_4 Vily, homogeneous: \mathcal{D}_4 S cases for V/Z: \mathcal{D}_9 – \mathcal{D}_9

Application

Classes D₂ and D Class D₇

Which theorem of Aschbacher do I mean?

Theorem (Aschbacher 1984)

Let G_0 be a simple classical group over a finite field and $G_0 \le G \le \operatorname{Aut}(G_0)$. Let H < G such that $HG_0 = G$. Define geometrically classes \mathcal{C}_1 to \mathcal{C}_8 of subgroups of G. Then either H is a subgroup of at least one of the groups in classes \mathcal{C}_1 to \mathcal{C}_8 , or the following hold:

- There is a non-abelian simple group H_0 with $H_0 \le H \le \operatorname{Aut}(H_0)$.
- The natural H-module V is absolutely irreducible.
- This representation for H cannot be realised over a smaller field.

There is a number of simplifying lies on this slide!

Max Neunhöffer

Background Constructive recognition Reductions Composition trees

Aschbacher's Theorem

A variant for GL and its proof The Statement Reducible: \mathcal{D}_1 Not abs. irred: \mathcal{D}_3 Subfiel: \mathcal{D}_6 or \mathcal{D}_9 Clifford theory W not abs. irred: \mathcal{D}_3 or \mathcal{D}_9 Vi_N not homogeneous: \mathcal{D} Vi_N not homogeneous: \mathcal{D} a cases for N/2: \mathcal{D}_{-} - \mathcal{D}_9

Application

Classes D2 and D Class D7

A variant ...

Let $n \in \mathbb{N}$ and \mathbb{F}_q the field with $q = p^e$ elements. Let $V := \mathbb{F}_q^{1 \times n}$ be the \mathbb{F}_q -vector space of row vectors.

Theorem

Let $G \leq GL_n(\mathbb{F}_q)$ and $n \geq 2$. Then G lies in at least one of the classes \mathcal{D}_1 to \mathcal{D}_9 of subgroups of $GL_n(\mathbb{F}_q)$.

Max Neunhöffer

Background Constructive recognition Reductions Composition trees

Aschbacher's Theorem

A variant for GL and its proof The Satement Reducible: \mathcal{D}_1 Not abs. irred: \mathcal{D}_3 Subfield: \mathcal{D}_5 G/Z is simple: \mathcal{D}_6 or \mathcal{D}_9 Clifford theory W not abs. irred: \mathcal{D}_3 V_{I/N} not homogeneous: \mathcal{D} V_{I/N} not homogeneous: \mathcal{D}_4 cases for N/2: \mathcal{D}_2 - \mathcal{D}_9

Application

Classes D₂ and D Class D₇

A variant ...

Let $n \in \mathbb{N}$ and \mathbb{F}_q the field with $q = p^e$ elements. Let $V := \mathbb{F}_q^{1 \times n}$ be the \mathbb{F}_q -vector space of row vectors.

Theorem

Let $G \leq GL_n(\mathbb{F}_q)$ and $n \geq 2$. Then G lies in at least one of the classes \mathfrak{D}_1 to \mathfrak{D}_9 of subgroups of $GL_n(\mathbb{F}_q)$.

I will not tell you on this slide what the classes D₁ to D₉ are.

Max Neunhöffer

Background Constructive recognition Reductions Composition trees

Aschbacher's Theorem

A variant for GL and its proof The Statement Reducible: \mathcal{D}_1 Not abs. irred : \mathcal{D}_3 Subfiel: \mathcal{D}_2 G/Z is simple: \mathcal{D}_0 or \mathcal{D}_0 Clifford theory W not abs. irred : \mathcal{D}_3 V/w, not homogeneous: \mathcal{D}_2 V/w, homogeneous: \mathcal{D}_2 Clifford theory M as a cases for N/Z? \mathcal{D}_0 - \mathcal{D}_0

Application

Classes D₂ and D Class D₇

A variant ...

Let $n \in \mathbb{N}$ and \mathbb{F}_q the field with $q = p^e$ elements. Let $V := \mathbb{F}_q^{1 \times n}$ be the \mathbb{F}_q -vector space of row vectors.

Theorem

Let $G \leq GL_n(\mathbb{F}_q)$ and $n \geq 2$. Then G lies in at least one of the classes \mathfrak{D}_1 to \mathfrak{D}_9 of subgroups of $GL_n(\mathbb{F}_q)$.

- I will not tell you on this slide what the classes D₁ to D₉ are.
- I will show you a sketch of the proof of this statement.

Max Neunhöffer

Background Constructive recognition Reductions Composition trees

Aschbacher's Theorem

A variant for GL and its proof The Statement Reducible: \mathcal{D}_1 Not abs. tred : \mathcal{D}_3 Subfiel: \mathcal{D}_2 G/Z is simple: \mathcal{D}_0 or \mathcal{D}_0 Ciliford theory W not abs. tred : \mathcal{D}_3 V/ $_{W}$ not homogeneous: \mathcal{D}_4 V/ $_{W}$ homogeneous: \mathcal{D}_4 S cases for N/Z: \mathcal{D}_0 - \mathcal{D}_0

Application

Classes D2 and D Class D7

A variant ...

Let $n \in \mathbb{N}$ and \mathbb{F}_q the field with $q = p^e$ elements. Let $V := \mathbb{F}_q^{1 \times n}$ be the \mathbb{F}_q -vector space of row vectors.

Theorem

Let $G \leq GL_n(\mathbb{F}_q)$ and $n \geq 2$. Then G lies in at least one of the classes \mathfrak{D}_1 to \mathfrak{D}_9 of subgroups of $GL_n(\mathbb{F}_q)$.

- I will not tell you on this slide what the classes D₁ to D₉ are.
- I will show you a sketch of the proof of this statement.
- This is not new, lots of people have worked on this.

Max Neunhöffer

Background Constructive recognition Reductions Composition trees

Aschbacher's Theorem

A variant for GL and its proof The Statement Reducible: \mathfrak{P}_1 Not abs. irred:: \mathfrak{P}_3 Subfield: \mathfrak{P}_6 G/Z is simple: \mathfrak{P}_9 or \mathfrak{P}_9 Cilfford theory W not abs. irred:: \mathfrak{P}_0 VI_N not homogeneous: \mathfrak{P}_4 3 cases for N/Z:: \mathfrak{P}_6 - \mathfrak{P}_9

Application

Classes D2 and D Class D7

A variant ...

Let $n \in \mathbb{N}$ and \mathbb{F}_q the field with $q = p^e$ elements. Let $V := \mathbb{F}_q^{1 \times n}$ be the \mathbb{F}_q -vector space of row vectors.

Theorem

Let $G \leq GL_n(\mathbb{F}_q)$ and $n \geq 2$. Then G lies in at least one of the classes \mathfrak{D}_1 to \mathfrak{D}_9 of subgroups of $GL_n(\mathbb{F}_q)$.

- I will not tell you on this slide what the classes \mathcal{D}_1 to \mathcal{D}_9 are.
- I will show you a sketch of the proof of this statement.
- This is not new, lots of people have worked on this.
- Alongside the proof, we will
 - define \mathcal{D}_1 to \mathcal{D}_9 , and
 - keep an eye on how one can find reductions computationally.

Max Neunhöffer

Background Constructive recognitio Reductions Composition trees

Aschbacher's Theorem

A variant for GL and its proof The Statement Reductive: p_1 Not abs. irred: D_3 Subfiel: p_2 G/Z is simple: D_6 or D_9 Clifford theory W not abs. irred: D_3 V/i, not homogeneous: D_4 A cases for N/Z: p_2 - p_3

Application

Classes D2 and D Class D7

Reducible: \mathcal{D}_1

G could lie in \mathcal{D}_1 :

Definition of class \mathcal{D}_1

 $G \leq \operatorname{GL}_n(\mathbb{F}_q)$ lies in \mathcal{D}_1 if there is a subspace 0 < W < V with Wg = W for all $g \in G$.

Max Neunhöffer

Background Constructive recognition Reductions Composition trees

Aschbacher's Theorem

A variant for GL and its proof The Statement Reductive: \mathfrak{D}_1 Not abs. tred : \mathfrak{D}_3 Subfiel: \mathfrak{D}_2 Gifford theory W not abs. tred : \mathfrak{D}_3 or \mathfrak{D}_3 Clifford theory W not abs. tred : \mathfrak{D}_3 V $_{I_I}$ not homogeneous: \mathfrak{D}_4 V $_{I_I}$ not homogeneous: \mathfrak{D}_4 Cases for N/27: \mathfrak{D}_2 - \mathfrak{D}_3

Application

Classes D2 and Da Class D7

Reducible: \mathcal{D}_1

G could lie in \mathcal{D}_1 :

Definition of class \mathcal{D}_1

 $G \leq \operatorname{GL}_n(\mathbb{F}_q)$ lies in \mathcal{D}_1 if there is a subspace 0 < W < V with Wg = W for all $g \in G$.

We can decide computationally using the MeatAxe, whether such an invariant subspace W exists or not.

Max Neunhöffer

Background Constructive recognition Reductions Composition trees

Aschbacher's Theorem

A variant for GL and its proof The Statement **Reducible:** \mathcal{P}_1 Not abs. irred: \mathcal{P}_3 Subfield: \mathcal{P}_5 (*III*) is simple: \mathcal{P}_0 or \mathcal{P}_0 Clifford theory *W* not abs. irred: \mathcal{P}_3 V i_W not homogeneous: \mathcal{P}_4 V i_W not homogeneous: \mathcal{P}_4 S cases for *N*/2: \mathcal{P}_6 - \mathcal{P}_0

Application

Classes D₂ and D₄ Class D₇

Reducible: \mathcal{D}_1

G could lie in \mathcal{D}_1 :

Definition of class \mathcal{D}_1

 $G \leq \operatorname{GL}_n(\mathbb{F}_q)$ lies in \mathcal{D}_1 if there is a subspace 0 < W < V with Wg = W for all $g \in G$.

We can decide computationally using the MeatAxe, whether such an invariant subspace *W* exists or not.

Assumption

From now on we assume that G acts irreducibly on V.

Max Neunhöffer

Background Constructive recognition Reductions Composition trees

Aschbacher's Theorem

A variant for GL and its proof The Statement Reducible: \mathcal{D}_1 Not abs. irred: \mathcal{D}_3 Subfiel: \mathcal{D}_2 Gilford theory V not abs. irred: \mathcal{D}_3 or \mathcal{D}_3 Clifford theory V not abs. irred: \mathcal{D}_3 V $_{IN}$ not homogeneous: \mathcal{D}_2 V $_{IN}$ homogeneous: \mathcal{D}_2 S cases for N/2: \mathcal{D}_2 - \mathcal{D}_3

Application

Classes D2 and I Class D7

Not absolutely irreducible: \mathcal{D}_3

G could act irreducibly but not absolutely irreducibly.

Max Neunhöffer

Background Constructive recognition Reductions Composition trees

Aschbacher's Theorem

A variant for GL and its proof The Statement Reducible: \mathcal{D}_1 Not abs. irred:: \mathcal{D}_0 Subfiel: \mathcal{D}_0 Gilford theory V not abs. irred:: \mathcal{D}_0 or \mathcal{D}_0 Clifford theory V not abs. irred:: \mathcal{D}_1 V no desne irred:: \mathcal{D}_1 V n_0 not homogeneous: \mathcal{D}_1 d cases for N/2: \mathcal{D}_0 - \mathcal{D}_0

Application

Classes D2 and I Class D7

Not absolutely irreducible: \mathcal{D}_3

G could act irreducibly but not absolutely irreducibly. (*G* acts absolutely irreducibly iff $C_{GL_n(\mathbb{F}_q)}(G) = \{c \cdot 1\}$.)

Max Neunhöffer

Background Constructive recognition Reductions Composition trees

Aschbacher's Theorem

A variant for GL and its proof The Statement Reducible: \mathcal{D}_1 Not abs. irred: \mathcal{D}_0 Subfiel: \mathcal{D}_0 G/Z is simple: \mathcal{D}_0 or \mathcal{D}_0 Clifford theory W not abs. irred: \mathcal{D}_0 V $|_N$ not homogeneous: \mathcal{D} V $|_N$ nonogeneous: \mathcal{D}_1 cases for N/2: \mathcal{D}_0 - \mathcal{D}_0

Application

Classes D2 and D Class D7

Not absolutely irreducible: \mathcal{D}_3

G could act irreducibly but not absolutely irreducibly. (*G* acts absolutely irreducibly iff $C_{GL_n(\mathbb{F}_q)}(G) = \{c \cdot 1\}$.)

Lemma

If $G \leq GL_n(\mathbb{F}_q)$ acts irreducibly but not absolutely irreducibly on the natural module V, then G lies in \mathcal{D}_3 .

Max Neunhöffer

Background Constructive recognition Reductions Composition trees

Aschbacher's Theorem

A variant for GL and its proof The Statement Reducible: \mathfrak{D}_1 Not abs. ired: \mathfrak{D}_3 Subfield: \mathfrak{D}_2 G/Z is simple: \mathfrak{D}_3 or \mathfrak{D}_3 Clifford theory W not abs. ired: \mathfrak{D}_3 V $_{IN}$ not homogeneous: \mathfrak{D}_2 V $_{IN}$ not mongeneous: \mathfrak{D}_2 V $_{IN}$ not mongeneous: \mathfrak{D}_2

Application

Classes D2 and D Class D7

Not absolutely irreducible: \mathcal{D}_3

G could act irreducibly but not absolutely irreducibly. (*G* acts absolutely irreducibly iff $C_{GL_n(\mathbb{F}_q)}(G) = \{c \cdot 1\}$.)

Lemma

If $G \leq GL_n(\mathbb{F}_q)$ acts irreducibly but not absolutely irreducibly on the natural module V, then G lies in \mathcal{D}_3 .

We can decide computationally using the MeatAxe, whether G acts absolutely irreducibly on V.

Max Neunhöffer

Background Constructive recognition Reductions Composition trees

Aschbacher's Theorem

A variant for GL and its proof The Statement Reducible: \mathfrak{D}_1 Not abs. irred: \mathfrak{D}_3 Subfield: \mathfrak{D}_2 G/Z is simple: \mathfrak{D}_3 or \mathfrak{D}_3 Clifford theory W not abs. irred: \mathfrak{D}_3 V i_N not homogeneous: \mathfrak{D}_4 V i_N not homogeneous: \mathfrak{D}_4 S cases for N/Z: \mathfrak{D}_8 - \mathfrak{D}_9 - \mathfrak{D}_9

Application

Classes D₂ and *L* Class D₇

Not absolutely irreducible: \mathcal{D}_3

G could act irreducibly but not absolutely irreducibly. (*G* acts absolutely irreducibly iff $C_{GL_n(\mathbb{F}_q)}(G) = \{c \cdot 1\}$.)

Lemma

If $G \leq GL_n(\mathbb{F}_q)$ acts irreducibly but not absolutely irreducibly on the natural module V, then G lies in \mathcal{D}_3 .

We can decide computationally using the MeatAxe, whether G acts absolutely irreducibly on V.

Assumption

From now on we assume that G acts absolutely irreducibly on V.

Max Neunhöffer

Background Constructive recognition Reductions Composition trees

Aschbacher's Theorem

A variant for GL and its proof The Statement Reducible: \mathfrak{D}_1 Not abs. irred: \mathfrak{D}_3 Solutied: \mathfrak{D}_2 G/Z is simple: \mathfrak{D}_3 or \mathfrak{D}_5 Clifford theory W not abs. irred: \mathfrak{D}_3 Vi_N not homogeneous: \mathfrak{D}_4 3 cases for N/Z: \mathfrak{D}_6 - \mathfrak{D}_5

Application

Classes D2 and D4 Class D7

Semilinear: \mathcal{D}_3

Definition of class \mathcal{D}_3

- $G \leq \operatorname{GL}_n(\mathbb{F}_q)$ lies in \mathcal{D}_3 if
 - the natural module V is irreducible and
 - there is a finite field 𝔽_{q^s}, for which we can extend the 𝔽_q-vector space structure of *V* to an 𝔽_{q^s}-vector space structure of dimension *n*/*s*, such that:

 $\forall g \in G \ \exists \alpha_g \in Aut(\mathbb{F}_{q^s})$ with:

 $(\mathbf{v} + \lambda \mathbf{w}) \cdot \mathbf{g} = \mathbf{v} \cdot \mathbf{g} + \lambda^{\alpha_g} \cdot \mathbf{w} \cdot \mathbf{g}$ for all $\mathbf{v}, \mathbf{w} \in V$ and all $\lambda \in \mathbb{F}_{q^s}$.

(i.e. the action of *G* on *V* is \mathbb{F}_{q^s} -semilinear)

Max Neunhöffer

Background Constructive recognition Reductions Composition trees

Aschbacher's Theorem

A variant for GL and its proof The Statement Reducible: \mathfrak{D}_1 Not abs. irred: \mathfrak{D}_3 Solutied: \mathfrak{D}_2 G/Z is simple: \mathfrak{D}_3 or \mathfrak{D}_5 Clifford theory W not abs. irred: \mathfrak{D}_3 Vi_N not homogeneous: \mathfrak{D}_4 3 cases for N/Z: \mathfrak{D}_6 - \mathfrak{D}_5

Application

Classes D2 and D4 Class D7

Semilinear: \mathcal{D}_3

Definition of class \mathcal{D}_3

- $G \leq \operatorname{GL}_n(\mathbb{F}_q)$ lies in \mathcal{D}_3 if
 - the natural module V is irreducible and
 - there is a finite field 𝔽_{q^s}, for which we can extend the 𝔽_q-vector space structure of *V* to an 𝔽_{q^s}-vector space structure of dimension *n*/*s*, such that:

 $\forall g \in G \ \exists \alpha_g \in Aut(\mathbb{F}_{q^s})$ with:

 $(\mathbf{v} + \lambda \mathbf{w}) \cdot \mathbf{g} = \mathbf{v} \cdot \mathbf{g} + \lambda^{\alpha_g} \cdot \mathbf{w} \cdot \mathbf{g}$ for all $\mathbf{v}, \mathbf{w} \in V$ and all $\lambda \in \mathbb{F}_{q^s}$.

(i.e. the action of *G* on *V* is \mathbb{F}_{q^s} -semilinear)

Non-absolutely irred. case: all automorphisms are trivial!

Max Neunhöffer

Background Constructive recognition Reductions Composition trees

Aschbacher's Theorem

A variant for GL and its proof The Statement Reducible: \mathcal{D}_1 Not abs. irred: \mathcal{D}_3 Subfiel: \mathcal{D}_2 G/Z is simple: \mathcal{D}_3 or \mathcal{D}_3 Clifford theory W not abs. irred: \mathcal{D}_3 V /_N not homogeneous: \mathcal{D}_2 V /_N not homogeneous: \mathcal{D}_2 Casaes for N/2: \mathcal{D}_3 - \mathcal{D}_3 -

Application

Classes D2 and Da Class D7 Subfield: \mathcal{D}_5 G could lie in \mathcal{D}_5 :

Definition of class \mathcal{D}_5

- $G \leq \operatorname{GL}_n(\mathbb{F}_q)$ lies in \mathcal{D}_5 if
 - the natural module V is absolutely irreducible and
 - there is a proper subfield \mathbb{F}_{q_0} of \mathbb{F}_q and $T \in GL_n(\mathbb{F}_q)$ and $(\beta_g)_{g \in G}$ with $\beta_g \in \mathbb{F}_q$ such that

 $\beta_g \cdot T^{-1}gT \in \operatorname{GL}_n(\mathbb{F}_{q_0})$ for all $g \in G$.

Max Neunhöffer

Background Constructive recognition Reductions Composition trees

Aschbacher's Theorem

A variant for GL and its proof The Statement Reducible: \mathcal{D}_1 Not abs. irred: \mathcal{D}_2 Subfield: \mathcal{D}_3 Ciliford theory Wind tabs. irred: \mathcal{D}_2 Vily not homogeneous: \mathcal{D}_4 Vily not homogeneous: \mathcal{D}_4 a cases for N/Z: \mathcal{D}_6 - \mathcal{D}_9

Application

Classes D2 and D Class D7 Subfield: \mathcal{D}_5 G could lie in \mathcal{D}_5 :

Definition of class \mathcal{D}_5

- $G \leq \operatorname{GL}_n(\mathbb{F}_q)$ lies in \mathcal{D}_5 if
 - the natural module V is absolutely irreducible and
 - there is a proper subfield \mathbb{F}_{q_0} of \mathbb{F}_q and $T \in GL_n(\mathbb{F}_q)$ and $(\beta_g)_{g \in G}$ with $\beta_g \in \mathbb{F}_q$ such that

 $\beta_g \cdot T^{-1}gT \in \operatorname{GL}_n(\mathbb{F}_{q_0})$ for all $g \in G$.

We can decide computationally whether *G* lies in \mathcal{D}_5 (see Glasby, Leedham-Green, and O'Brien (2006) and Carlson, N. and Roney-Dougal (submitted)).

Max Neunhöffer

Background Constructive recognition Reductions Composition trees

Aschbacher's Theorem

A variant for GL and its proof The Statement Reducible: \mathfrak{D}_1 Not abs. irred: \mathfrak{D}_2 Subfiel: \mathfrak{D}_2 Clifford theory W not abs. irred: \mathfrak{D}_2 or \mathfrak{D}_3 Clifford theory W not abs. irred: \mathfrak{D}_2 Vi_N not homogeneous: \mathfrak{D}_4 S cases for N/2: \mathfrak{D}_6 – \mathfrak{D}_3

Application

Classes D₂ and D Class D₇ Subfield: \mathcal{D}_5

G could lie in \mathcal{D}_5 :

Definition of class \mathcal{D}_5

- $G \leq \operatorname{GL}_n(\mathbb{F}_q)$ lies in \mathcal{D}_5 if
 - the natural module V is absolutely irreducible and
 - there is a proper subfield \mathbb{F}_{q_0} of \mathbb{F}_q and $T \in GL_n(\mathbb{F}_q)$ and $(\beta_g)_{g \in G}$ with $\beta_g \in \mathbb{F}_q$ such that

 $\beta_g \cdot T^{-1}gT \in \operatorname{GL}_n(\mathbb{F}_{q_0})$ for all $g \in G$.

We can decide computationally whether *G* lies in \mathcal{D}_5 (see Glasby, Leedham-Green, and O'Brien (2006) and Carlson, N. and Roney-Dougal (submitted)).

Assumption

From now on we assume that G does not lie in \mathcal{D}_5 .

Max Neunhöffer

Background Constructive recognition Reductions Composition trees

Aschbacher's Theorem

A variant for GL and its proof The Statement Reducible: D₁ Not abs. Irred: D₂ Subfield: D₂ **G/Z is simple: D₂ or D₃** Cillford theory W not abs. Irred: D₃ V_{1/N} noncogeneous: D₂ V_{1/N} homogeneous: D₂ Carses for W/2: D₂-D₂

Application

Classes D₂ and a Class D₇

G/Z is simple: \mathcal{D}_8 or \mathcal{D}_9

From now on denote $Z := Z(G) = G \cap Z(GL_n(\mathbb{F}_q))$.

Max Neunhöffer

Background Constructive recognition Reductions Composition trees

Aschbacher's Theorem

A variant for GL and its proof The Statement Reducible: D₁ Not abs. Irred:: D₃ Subfiel: D₅ **G/Z is simple: D₆ or D₅** Cillford theory W not abs. Irred:: D₃ V_{/N} not morgeneous: D₄ V_{/N} not morgeneous: D₄ Composite V(2: D₆-D₇)

Application

Classes D2 and I Class D7

G/Z is simple: \mathcal{D}_8 or \mathcal{D}_9

From now on denote $Z := Z(G) = G \cap Z(GL_n(\mathbb{F}_q))$.

The group G/Z could be simple.

Max Neunhöffer

Background Constructive recognition Reductions Composition trees

Aschbacher's Theorem

A variant for GL and its proof The Statement Reducible: D₁ Not abs. Irred: D₃ Subfield: D₆ **G/Z is simple: D₈ or D₉** Cillford theory W not abs. Irred: D₃ V_{1/N} noncogeneous: D₂ V_{1/N} homogeneous: D₄ Carses for N/27: D₂-D₁

Application

Classes D2 and D Class D7

G/Z is simple: \mathcal{D}_8 or \mathcal{D}_9

From now on denote $Z := Z(G) = G \cap Z(GL_n(\mathbb{F}_q))$.

The group G/Z could be simple.

If G/Z were cyclic, then G would be abelian and V not absolutely irreducible.

Max Neunhöffer

Background Constructive recognition Reductions Composition trees

Aschbacher's Theorem

A variant for GL and its proof The Statement Reducible: D_1 Not abs. irred: D_3 Subfiel: D_2 **6**/2 is simple: D_0 or D_0 Clifford theory W not abs. irred: D_3 V_{IN} not homogeneous: D_2 V_{IN} nongeneous: D_2 S cases for W/2; D_2 - D_0 -

Application

Classes D2 and D Class D7

G/Z is simple: \mathcal{D}_8 or \mathcal{D}_9

From now on denote $Z := Z(G) = G \cap Z(GL_n(\mathbb{F}_q))$.

The group G/Z could be simple.

If G/Z were cyclic, then G would be abelian and V not absolutely irreducible.

Then G/Z is either a classical simple group in its natural representation (then *G* lies in \mathcal{D}_8), or *G* lies in \mathcal{D}_9 .

Max Neunhöffer

Background Constructive recognition Reductions Composition trees

Aschbacher's Theorem

A variant for GL and its proof The Statement Reductive: \mathfrak{D}_1 Not abs. irred: \mathfrak{D}_2 Subfield: \mathfrak{D}_2 G/Z is simple: \mathfrak{D}_0 or \mathfrak{D}_0 Clifford theory Wint abs. irred: \mathfrak{D}_2 Vi_N not homogeneous: \mathfrak{D}_2 Vi_N not homogeneous: \mathfrak{D}_2 Scasse for N/Z: \mathfrak{D}_0 - \mathfrak{D}_0

Application

Classes D2 and D Class D7

G/Z is simple: \mathcal{D}_8 or \mathcal{D}_9

From now on denote $Z := Z(G) = G \cap Z(GL_n(\mathbb{F}_q))$.

The group G/Z could be simple.

If G/Z were cyclic, then G would be abelian and V not absolutely irreducible.

Then G/Z is either a classical simple group in its natural representation (then *G* lies in \mathcal{D}_8), or *G* lies in \mathcal{D}_9 .

We cannot find a reduction in this case.

Max Neunhöffer

Background Constructive recognition Reductions Composition trees

Aschbacher's Theorem

A variant for GL and its proof The Statement Reducible: \mathcal{D}_1 Not abs. irred: \mathcal{D}_3 Subfield: \mathcal{D}_5 **G/Z is simple:** \mathcal{D}_6 or \mathcal{D}_6 Cilling theory W not abs. irred: \mathcal{D}_3 V_{1/N} not homogeneous: \mathcal{D}_2 V_{1/N} hot homogeneous: \mathcal{D}_4 3 cases for N/Z: \mathcal{D}_6 - \mathcal{D}_6

Application

Classes D2 and D Class D7

G/Z is simple: \mathcal{D}_8 or \mathcal{D}_9

From now on denote $Z := Z(G) = G \cap Z(GL_n(\mathbb{F}_q))$.

The group G/Z could be simple.

If G/Z were cyclic, then G would be abelian and V not absolutely irreducible.

Then G/Z is either a classical simple group in its natural representation (then *G* lies in \mathcal{D}_8), or *G* lies in \mathcal{D}_9 .

We cannot find a reduction in this case. Thus we have to recognise *G* constructively in some other way!

Max Neunhöffer

Background Constructive recognition Reductions Composition trees

Aschbacher's Theorem

A variant for GL and its proof The Statement Reducible: \mathcal{D}_1 Not abs. irred: \mathcal{D}_2 Subletic: \mathcal{D}_2 G/Z is simple: \mathcal{D}_0 or \mathcal{D}_0 Clifford theory W not abs. irred: \mathcal{D}_3 Vi_{II} not theoregeneous: \mathcal{D}_2 Vi_{II} not theoregeneous: \mathcal{D}_2 A cases for N/Z: \mathcal{D}_0 - \mathcal{D}_0

Application Classes ற2 and ற4

G/Z is simple: \mathcal{D}_8 or \mathcal{D}_9

From now on denote $Z := Z(G) = G \cap Z(GL_n(\mathbb{F}_q))$.

The group G/Z could be simple.

If G/Z were cyclic, then G would be abelian and V not absolutely irreducible.

Then G/Z is either a classical simple group in its natural representation (then *G* lies in \mathcal{D}_8), or *G* lies in \mathcal{D}_9 .

We cannot find a reduction in this case. Thus we have to recognise *G* constructively in some other way!

Assumption

Assume from now on that G/Z is not simple.

Max Neunhöffer

Background Constructive recognition Reductions Composition trees

Aschbacher's Theorem

A variant for GL and its proof The Statement Reducible: D₁ Not abs. Irred: D₂ Subfield: D₅ **G/2 is simple: D₈ or D₉** Cillford theory W not abs. Irred:: D₉ V_{|N} not morgeneous: D₂ V_{|N} not morgeneous: D₂

Application

Classes D_2 and IClass D_7

Classical in natural representation: \mathcal{D}_8

Definition of class \mathcal{D}_8

 $G \leq GL_n(\mathbb{F}_q)$ lies in \mathcal{D}_8 if G/Z contains a classical simple group in its natural representation

Max Neunhöffer

Background Constructive recognition Reductions Composition trees

Aschbacher's Theorem

A variant for GL and its proof The Statement Reducible: \mathcal{D}_1 Not abs. irred: \mathcal{D}_2 Subfield: \mathcal{D}_2 Gifford theory W not abs. irred: \mathcal{D}_3 or \mathcal{D}_3 Clifford theory W not abs. irred: \mathcal{D}_3 V $|_{V_1}$ non-domogeneous: \mathcal{D}_2 V $|_{V_1}$ non-domogeneous: \mathcal{D}_2 S cases for $V(2; 2)_{2}$ - \mathcal{D}_3 - \mathcal{D}_3

Application

Classes D2 and Da Class D7

Classical in natural representation: \mathcal{D}_8

Definition of class \mathcal{D}_8

 $G \leq GL_n(\mathbb{F}_q)$ lies in \mathcal{D}_8 if G/Z contains a classical simple group in its natural representation in one of the following ways:

• G/Z contains $PSL_n(\mathbb{F}_q)$ and $(n, q) \notin \{(2, 2), (2, 3)\},\$

- *n* is even, *G* is contained in N_{GLn(𝔽q)}(Sp_n(𝔽q)) for some non-singular symplectic form, *G*/*Z* contains PSp_n(𝔽q) and (*n*, *q*) ∉ {(2, 2), (2, 3), (4, 2)},
- q is a square, G is contained in $N_{\operatorname{GL}_n(\mathbb{F}_q)}(\operatorname{SU}_n(\mathbb{F}_{q^{1/2}}))$ for some non-singular Hermitian form, G/Z contains $\operatorname{PSU}_n(\mathbb{F}_{q^{1/2}})$ and $(n, q^{1/2}) \notin \{(2, 2), (2, 3), (3, 2)\},$

• *G* is contained in $N_{\operatorname{GL}_n(\mathbb{F}_q)}(\Omega_n^{\epsilon}(\mathbb{F}_q))$, the corresponding $P\Omega_n^{\epsilon}(\mathbb{F}_q)$ is simple and contained in G/Z.

Max Neunhöffer

Background Constructive recognition Reductions Composition trees

Aschbacher's Theorem

A variant for GL and its proof The Statement Reducible: \mathcal{D}_1 Not abs. irred: \mathcal{D}_2 Subfield: \mathcal{D}_2 Gifford theory W not abs. irred: \mathcal{D}_3 or \mathcal{D}_3 Clifford theory W not abs. irred: \mathcal{D}_3 V $|_{V_1}$ non-domogeneous: \mathcal{D}_2 V $|_{V_1}$ non-domogeneous: \mathcal{D}_2 S cases for $V(2; 2)_{2}$ - \mathcal{D}_3 - \mathcal{D}_3

Application

Classes D2 and D Class D7

Classical in natural representation: \mathcal{D}_8

Definition of class \mathcal{D}_8

 $G \leq GL_n(\mathbb{F}_q)$ lies in \mathcal{D}_8 if G/Z contains a classical simple group in its natural representation in one of the following ways:

• G/Z contains $PSL_n(\mathbb{F}_q)$ and $(n, q) \notin \{(2, 2), (2, 3)\},\$

- *n* is even, *G* is contained in N_{GLn(𝔽q)}(Sp_n(𝔽q)) for some non-singular symplectic form, *G*/*Z* contains PSp_n(𝔽q) and (*n*, *q*) ∉ {(2, 2), (2, 3), (4, 2)},
- q is a square, G is contained in $N_{\operatorname{GL}_n(\mathbb{F}_q)}(\operatorname{SU}_n(\mathbb{F}_{q^{1/2}}))$ for some non-singular Hermitian form, G/Z contains $\operatorname{PSU}_n(\mathbb{F}_{q^{1/2}})$ and $(n, q^{1/2}) \notin \{(2, 2), (2, 3), (3, 2)\},$

• *G* is contained in $N_{\operatorname{GL}_n(\mathbb{F}_q)}(\Omega_n^{\epsilon}(\mathbb{F}_q))$, the corresponding $P\Omega_n^{\epsilon}(\mathbb{F}_q)$ is simple and contained in *G*/*Z*. The group $P\Omega_n^{\epsilon}(\mathbb{F}_q)$ is simple if and only if

- * $n \ge 3$, and
- * *q* is odd if *n* is odd, and
- * ϵ is if n = 4, and
- * $(n, q) \notin \{(3, 3), (4, 2)\}.$

Max Neunhöffer

Background Constructive recognitio Reductions Composition trees

Aschbacher's Theorem

A variant for GL and its proof The Statement Reducible: \mathcal{D}_1 Not abs. irred: \mathcal{D}_2 Subfiel: \mathcal{D}_2 Gilfard theory W not abs. irred: \mathcal{D}_3 or \mathcal{D}_3 Cilifard theory W not abs. irred: \mathcal{D}_3 V $|_{V_1}$ noncogeneous: \mathcal{D}_2 V $|_{V_1}$ noncogeneous: \mathcal{D}_2 d cases for W/2; \mathcal{D}_2 - \mathcal{D}_3

Application

Classes D₂ and I Class D₇

G/Z almost simple: \mathcal{D}_9

Definition of class \mathcal{D}_9

 $G \leq \operatorname{GL}_n(\mathbb{F}_q)$ lies in \mathcal{D}_9 , if

• it is not in \mathcal{D}_8 and

Max Neunhöffer

Background Constructive recognition Reductions Composition trees

Aschbacher's Theorem

A variant for GL and its proof The Statement Reducible: \mathcal{D}_1 Not abs. irred: \mathcal{D}_3 Subfiel: \mathcal{D}_2 Gilford theory W not abs. irred: \mathcal{D}_3 or \mathcal{D}_3 Clifford theory W not abs. irred: \mathcal{D}_3 V $_{IN}$ nonogeneous: \mathcal{D}_4 V $_{IN}$ nonogeneous: \mathcal{D}_2 a cases for N/2: \mathcal{D}_2 - \mathcal{D}_3

Application

Classes D₂ and D Class D₇

G/Z almost simple: \mathcal{D}_9

Definition of class \mathcal{D}_9

- $G \leq \operatorname{GL}_n(\mathbb{F}_q)$ lies in \mathcal{D}_9 , if
 - it is not in \mathcal{D}_8 and
 - there is a non-abelian simple group N and a group T with N ≤ T ≤ Aut(N) such that

•
$$G/Z \cong T$$
 and

Max Neunhöffer

Background Constructive recognition Reductions Composition trees

Aschbacher's Theorem

A variant for GL and its proof The Statement Reducible: \mathfrak{D}_1 Not abs. irred: \mathfrak{D}_2 G/Z is simple: \mathfrak{D}_0 or \mathfrak{D}_0 Clifford theory W not abs. irred: \mathfrak{D}_2 V_{1N} not homogeneous: \mathfrak{D}_2 V_{1N} not homogeneous: \mathfrak{D}_2 V_{1N} ob a cases for N/Z: \mathfrak{D}_0 - \mathfrak{D}_0

Application

Classes D2 and D Class D7

G/Z almost simple: \mathcal{D}_9

Definition of class \mathcal{D}_9

- $G \leq \operatorname{GL}_n(\mathbb{F}_q)$ lies in \mathcal{D}_9 , if
 - it is not in \mathcal{D}_8 and
 - there is a non-abelian simple group N and a group T with N ≤ T ≤ Aut(N) such that
 - $G/Z \cong T$ and
 - *V* gives rise to an absolutely irreducible projective representation for *T*,

Max Neunhöffer

Background Constructive recognition Reductions Composition trees

Aschbacher's Theorem

A variant for GL and its proof The Statement Reducible: \mathcal{D}_1 Not abs. irred: \mathcal{D}_3 Subfiel: \mathcal{D}_2 Gilford theory W not abs. irred: \mathcal{D}_3 or \mathcal{D}_3 Clifford theory W not abs. irred: \mathcal{D}_3 V $_{IN}$ nonogeneous: \mathcal{D}_4 V $_{IN}$ nonogeneous: \mathcal{D}_2 a cases for N/2: \mathcal{D}_2 - \mathcal{D}_3

Application

Classes D₂ and *L* Class D₇

G/Z almost simple: \mathcal{D}_9

Definition of class \mathcal{D}_9

- $G \leq \operatorname{GL}_n(\mathbb{F}_q)$ lies in \mathcal{D}_9 , if
 - it is not in \mathcal{D}_8 and
 - there is a non-abelian simple group N and a group T with N ≤ T ≤ Aut(N) such that
 - $G/Z \cong T$ and
 - *V* gives rise to an absolutely irreducible projective representation for *T*,

which is not realisable over a proper subfield of \mathbb{F}_q .

Max Neunhöffer

Background Constructive recognition Reductions Composition trees

Aschbacher's Theorem

A variant for GL and its proof The Statement Reducible: \mathcal{D}_1 Not abs. irred: \mathcal{D}_3 Subfield: \mathcal{D}_5 G/Z is simple: \mathcal{D}_6 or \mathcal{D}_9 Clifford theory W not abs. irred.: \mathcal{D}_3

 $V|_N$ not homogeneous: \mathcal{D}_4 $V|_N$ homogeneous: \mathcal{D}_4 3 cases for N/Z: $\mathcal{D}_6-\mathcal{D}_9$

Application

Classes D2 and D Class D7

Clifford theory

Let now \overline{N} be a minimal normal subgroup of G/Z and let $Z < N \triangleleft G$ be the full preimage.

Max Neunhöffer

Background Constructive recognition Reductions Composition trees

Aschbacher's Theorem

A variant for GL and its proof The Statement Reducible: \mathcal{D}_1 Not abs. irred.: \mathcal{D}_3 Subfield: \mathcal{D}_5 G/Z is simple: \mathcal{D}_6 or \mathcal{D}_9 Clifford theory W not abs. irred.: \mathcal{D}_3

 $V|_N$ not nomogeneous: \mathcal{D}_4 $V|_N$ homogeneous: \mathcal{D}_4 3 cases for N/Z: $\mathcal{D}_6-\mathcal{D}_9$

Application

Classes D₂ and D Class D₇

Clifford theory

Let now \overline{N} be a minimal normal subgroup of G/Z and let $Z < N \triangleleft G$ be the full preimage.

Theorem (Clifford)

The restriction $V|_N$ of the natural module to the normal subgroup N is a direct sum

$$V|_N = \bigoplus_{i=1}^k W_i$$

of irreducible N-modules W_i which are all G-conjugates of a single submodule $W \leq V|_N$, i.e. $W_i = Wg_i$ for some $g_i \in G$.

Max Neunhöffer

Background Constructive recognition Reductions Composition trees

Aschbacher's Theorem

A variant for GL and its proof The Statement Reducible: D₁ Not abs. irred.: D₃ Subfield: D₅ G/Z is simple: D₆ or D₉ **Clifford theory** W not abs. irred.: D₃

 $V|_N$ homogeneous: \mathcal{D}_4 3 cases for N/Z: $\mathcal{D}_6-\mathcal{D}_9$

Application

Classes D2 and D Class D7

Clifford theory

Let now \overline{N} be a minimal normal subgroup of G/Z and let $Z < N \triangleleft G$ be the full preimage.

Theorem (Clifford)

The restriction $V|_N$ of the natural module to the normal subgroup N is a direct sum

$$V|_N = \bigoplus_{i=1}^k W_i$$

of irreducible N-modules W_i which are all G-conjugates of a single submodule $W \leq V|_N$, i.e. $W_i = Wg_i$ for some $g_i \in G$.

Now we distinguish cases for this decomposition.

Max Neunhöffer

Background Constructive recognition Reductions Composition trees

Aschbacher's Theorem

A variant for GL and its proof The Statement Reducible: \mathcal{D}_1 Not abs. irred: \mathcal{D}_3 Subfield: \mathcal{D}_5 G/Z is simple: \mathcal{D}_8 or \mathcal{D}_5 Ciliford theory

W not abs. irred.: \mathcal{D}_3 $V|_N$ not homogeneous: \mathcal{D}_2 $V|_N$ homogeneous: \mathcal{D}_4 3 cases for N/Z: $\mathcal{D}_8-\mathcal{D}_9$

Application

Classes D2 and Class D7

W not absolutely irreducible: \mathcal{D}_3

Remember: $Z < N \triangleleft G$ such that N/Z is minimal normal.

Lemma

Let W be an irreducible submodule of $V|_N$. If W is not absolutely irreducible, then G lies in \mathcal{D}_3 .

Max Neunhöffer

Background Constructive recognition Reductions Composition trees

Aschbacher's Theorem

A variant for GL and its proof The Statement Reducible: D₁ Not abs. irred:: D₃ Subfield: D₅ G/Z is simple: D₆ or D₉ Clifford theory W not abs. irred:: D₃

 $V|_N$ not homogeneous: \mathcal{D}_2 $V|_N$ homogeneous: \mathcal{D}_4 3 cases for N/Z: $\mathcal{D}_6-\mathcal{D}_9$

Application

Classes D2 and D Class D7

W not absolutely irreducible: \mathcal{D}_3

Remember: $Z < N \triangleleft G$ such that N/Z is minimal normal.

Lemma

Let W be an irreducible submodule of $V|_N$. If W is not absolutely irreducible, then G lies in \mathcal{D}_3 .

This is computationally under control, see "SMASH": Holt, Leedham-Green, O'Brien and Rees (1996) or Carlson, N., Roney-Dougal (submitted).

Max Neunhöffer

Background Constructive recognition Reductions Composition trees

Aschbacher's Theorem

A variant for GL and its proof The Statement Reducible: \mathcal{D}_1 Not abs. irred.: \mathcal{D}_3 Subfield: \mathcal{D}_5 G/Z is simple: \mathcal{D}_6 or \mathcal{D}_9 Clifford theory

W not abs. irred.: \mathcal{D}_3 $V|_N$ not homogeneous: \mathcal{D}_2 $V|_N$ homogeneous: \mathcal{D}_4 3 cases for N/Z: $\mathcal{D}_6-\mathcal{D}_9$

Application Classes D₂ and D₄

W not absolutely irreducible: \mathcal{D}_3

Remember: $Z < N \triangleleft G$ such that N/Z is minimal normal.

Lemma

Let W be an irreducible submodule of $V|_N$. If W is not absolutely irreducible, then G lies in \mathcal{D}_3 .

This is computationally under control, see "SMASH": Holt, Leedham-Green, O'Brien and Rees (1996) or Carlson, N., Roney-Dougal (submitted).

Assumption

From now on we assume that *W* is absolutely irreducible.

Max Neunhöffer

Background Constructive recognition Reductions Composition trees

Aschbacher's Theorem

A variant for GL and its proof The Statement Reducible: \mathcal{D}_1 Not abs. irred: \mathcal{D}_3 Subfield: \mathcal{D}_5 Grifford theory W not abs. irred: \mathcal{D}_3 Vijn on tomogeneous: \mathcal{D}_4 Vijn to mogeneous: \mathcal{D}_6

Application

Classes D2 and Da Class D7

$V|_N$ not homogeneous: \mathcal{D}_2

Assume that not all W_i are isomorphic to W.

Max Neunhöffer

Background Constructive recognition Reductions Composition trees

Aschbacher's Theorem

A variant for GL and its proof The Statement Reductive: \mathfrak{I}_1 Not abs. irred: \mathfrak{D}_3 Subfield: \mathfrak{D}_3 Gilford theory W not abs. irred: \mathfrak{D}_3 W in thomogeneous: \mathfrak{D}_2 Vi_l homogeneous: \mathfrak{D}_4

Application

Classes D₂ and D Class D₇

$V|_N$ not homogeneous: \mathcal{D}_2

Assume that not all W_i are isomorphic to W.

Then *G* permutes the homogeneous components and lies in \mathcal{D}_2 :

$$V|_N = \bigoplus_{i=1}^k W_i = \bigoplus_j \left(\bigoplus_a W_a^{(j)}\right)$$

where $W_a^{(j)} \cong W_b^{(l)}$ iff j = l.

Max Neunhöffer

Background Constructive recognition Reductions Composition trees

Aschbacher's Theorem

A variant for GL and its proof The Statement Reducible: \mathcal{D}_1 Not abs. ined: \mathcal{D}_3 Sublied: \mathcal{D}_3 Sublied: \mathcal{D}_3 Cilliford theory Wing tabs. ined: \mathcal{D}_3 Vin not thomogeneous: \mathcal{D}_4 Vin homogeneous: \mathcal{D}_4 S cases for N/2: \mathcal{D}_9 - \mathcal{D}_9

١

Application

Classes D₂ and D Class D₇

$V|_N$ not homogeneous: \mathcal{D}_2

Assume that not all W_i are isomorphic to W.

Then *G* permutes the homogeneous components and lies in \mathcal{D}_2 :

$$V|_N = \bigoplus_{i=1}^k W_i = \bigoplus_j \left(\bigoplus_a W_a^{(j)}\right)$$

where
$$W_a^{(j)} \cong W_b^{(l)}$$
 iff $j = l$.

Definition of class \mathcal{D}_2

- $G \leq \operatorname{GL}_n(\mathbb{F}_q)$ lies in \mathcal{D}_2 if
 - the natural module V is absolutely irreducible and
 - there is Z < N ⊲ G such that V|_N = ⊕^k_{i=1} W_i and the W_i are absolutely irreducible F_qN-modules and not all isomorphic.

Max Neunhöffer

Background Constructive recognition Reductions Composition trees

Aschbacher's Theorem

A variant for GL and its proof The Statement Reducible: \mathcal{D}_1 Not abs. irred: \mathcal{D}_3 Subfiel: \mathcal{D}_2 G/Z is simple: \mathcal{D}_3 or \mathcal{D}_3 Clifford theory W not abs. irred: \mathcal{D}_3 V I_N not homogeneous: \mathcal{D}_2 VI_N nongeneous: \mathcal{D}_2 d cases for N/2: \mathcal{D}_2 - \mathcal{D}_3

Application Classes D2 and D4 Class D7

$V|_N$ homogeneous: \mathcal{D}_4

Assume that all W_i are isomorphic to W and k > 1.

Max Neunhöffer

Background Constructive recognition Reductions Composition trees

Aschbacher's Theorem

A variant for GL and its proof The Statement Reducible: \mathcal{D}_1 Not abs. irred: \mathcal{D}_3 Subfiel: \mathcal{D}_2 G/Z is simple: \mathcal{D}_3 or \mathcal{D}_3 Clifford theory W not abs. irred: \mathcal{D}_3 V I_N not homogeneous: \mathcal{D}_2 VI_N nongeneous: \mathcal{D}_2 d cases for N/2: \mathcal{D}_2 - \mathcal{D}_3

Application Classes D2 and D4

$V|_N$ homogeneous: \mathcal{D}_4

Assume that all W_i are isomorphic to W and k > 1.

If dim_{\mathbb{F}_q}(*W*) = 1 then *N* would be scalar.

Max Neunhöffer

Background Constructive recognition Reductions Composition trees

Aschbacher's Theorem

A variant for GL and its proof The Statement Reducible: \mathfrak{D}_1 Not abs. irred: \mathfrak{D}_3 Subfiel: \mathfrak{D}_2 G/Z is simple: \mathfrak{D}_0 or \mathfrak{D}_3 Clifford theory W not abs. irred: \mathfrak{D}_3 or \mathfrak{D}_3 Vi_N not homogeneous: \mathfrak{D}_4 Vi_N not homogeneous: \mathfrak{D}_4 3 cases for N/Z: \mathfrak{D}_9 - \mathfrak{D}_9

Application Classes D2 and D4 Class D7

$V|_N$ homogeneous: \mathcal{D}_4

Assume that all W_i are isomorphic to W and k > 1.

If dim_{\mathbb{F}_q}(*W*) = 1 then *N* would be scalar.

Definition of class D₄

- $G \leq GL_n(\mathbb{F}_q)$ lies in class \mathcal{D}_4 if
 - the natural module V is absolutely irreducible and
 - there is $N \triangleleft G$ such that $V|_N = \bigoplus_{i=1}^k W_i$ with $k \ge 2$ and $W_i \cong W$ for all *i*, where *W* is an absolutely irreducible $\mathbb{F}_q N$ -module with $\dim_{\mathbb{F}_q}(W) > 1$.

Max Neunhöffer

Background Constructive recognition Reductions Composition trees

Aschbacher's Theorem

A variant for GL and its proof The Statement Reducible: \mathcal{D}_1 Not abs. irred: \mathcal{D}_3 Subfield: \mathcal{D}_5 G/Z is simple: \mathcal{D}_6 or \mathcal{D}_6 Clifford theory W not abs. irred: \mathcal{D}_3 V_{1N} not homogeneous: \mathcal{D}_4 V_{1N} not homogeneous: \mathcal{D}_4 S cases for N/Z: \mathcal{D}_6 - \mathcal{D}_9

Application Classes D₂ and D₄ Class D₇

$V|_N$ homogeneous: \mathcal{D}_4

Assume that all W_i are isomorphic to W and k > 1.

If dim_{\mathbb{F}_q}(*W*) = 1 then *N* would be scalar.

Definition of class D₄

- $G \leq GL_n(\mathbb{F}_q)$ lies in class \mathcal{D}_4 if
 - the natural module V is absolutely irreducible and
 - there is $N \triangleleft G$ such that $V|_N = \bigoplus_{i=1}^k W_i$ with $k \ge 2$ and $W_i \cong W$ for all *i*, where *W* is absolutely irreducible $\mathbb{F}_q N$ -module with $\dim_{\mathbb{F}_q}(W) > 1$.

Assumption

We assume from now on that $W = V|_N$.

Max Neunhöffer

Background Constructive recognition Reductions Composition trees

Aschbacher's Theorem

A variant for GL and its proof The Statement Reducible: \mathcal{D}_1 Not abs. irred: \mathcal{D}_3 Subfiel: \mathcal{D}_2 G/Z is simple: \mathcal{D}_3 or \mathcal{D}_3 Clifford theory W not abs. irred: \mathcal{D}_3 V $_{IN}$ not homogeneous: \mathcal{D}_2 V $_{IN}$ homogeneous: \mathcal{D}_2 S cases for N/2: \mathcal{D}_2 - \mathcal{D}_3

Application

Classes D2 and A Class D7

Minimal normal subgroups

Now look at the group structure of N/Z:

Max Neunhöffer

Background Constructive recognition Reductions Composition trees

Aschbacher's Theorem

A variant for GL and its proof The Statement Reducible: \mathfrak{D}_1 Not abs. irred: \mathfrak{D}_2 Subfield: \mathfrak{D}_2 G/Z is simple: \mathfrak{D}_3 or \mathfrak{D}_3 Cilified theory W not abs. irred: \mathfrak{D}_3 Vi_N not thomogeneous: \mathfrak{D}_2 Vi_N not monogeneous: \mathfrak{D}_2 Scasses for N/Z: \mathfrak{D}_2 - \mathfrak{D}_3

Application

Classes D₂ and D Class D₇

Minimal normal subgroups

Now look at the group structure of N/Z:

Lemma (Minimal normal subgroups)

Let $1 < K \triangleleft H$ be a minimal normal subgroup. Then

 $K \cong T_1 \times T_2 \times \cdots \times T_k$

and the T_i are copies of a simple group which are all conjugate under H.

Therefore,

$$N/Z \cong T_1 \times T_2 \times \cdots \times T_k,$$

the T_i are pairwise isomorphic simple groups which are all conjugate under G/Z and thus G.

Max Neunhöffer

Background Constructive recognition Reductions Composition trees

Aschbacher's Theorem

A variant for GL and its proof The Statement Reducible: \mathcal{D}_1 Not abs. irred: \mathcal{D}_3 Subfiel: \mathcal{D}_2 G/Z is simple: \mathcal{D}_0 or \mathcal{D}_0 Clifford theory W not abs. irred: \mathcal{D}_3 V I_N not homogeneous: \mathcal{D}_2 V I_N not homogeneous: \mathcal{D}_2 S cases for V/Z; \mathcal{D}_2 - \mathcal{D}_0

Application Classes D2 and D

Class D7

Minimal normal subgroups

Now look at the group structure of N/Z:

Lemma (Minimal normal subgroups)

Let $1 < K \triangleleft H$ be a minimal normal subgroup. Then

 $K \cong T_1 \times T_2 \times \cdots \times T_k$

and the T_i are copies of a simple group which are all conjugate under H.

Therefore,

$$N/Z \cong T_1 \times T_2 \times \cdots \times T_k,$$

the T_i are pairwise isomorphic simple groups which are all conjugate under G/Z and thus G.

We distinguish 3 cases:

- the T_i are cyclic groups of prime order $r(\mathcal{D}_6)$
- 2 the T_i are non-abelian simple and $k \ge 2$ (\mathcal{D}_7)
- (3) k = 1 and T_1 is non-abelian simple (\mathcal{D}_8 or \mathcal{D}_9)

Max Neunhöffer

Background Constructive recognition Reductions Composition trees

Aschbacher's Theorem

A variant for GL and its proof The Statement Reducible: \mathcal{D}_1 Not abs. irred: \mathcal{D}_3 Subfiel: \mathcal{D}_2 G/Z is simple: \mathcal{D}_0 or \mathcal{D}_0 Clifford theory W not abs. irred: \mathcal{D}_3 or \mathcal{D}_1 V_{IN} not homogeneous: \mathcal{D}_2 V_{IN} no homogeneous: \mathcal{D}_2 S cases for N/2: \mathcal{D}_0 - \mathcal{D}_0

Application

Classes D2 and D Class D7

Extraspecial: \mathcal{D}_6

Definition of class \mathcal{D}_6

- $G \leq \operatorname{GL}_n(\mathbb{F}_q)$ lies in \mathcal{D}_6 if
 - the natural module V is absolutely irreducible,

Max Neunhöffer

Background Constructive recognition Reductions Composition trees

Aschbacher's Theorem

A variant for GL and its proof The Statement Reducible: \mathcal{D}_1 Not abs. irred: \mathcal{D}_3 Subfiel: \mathcal{D}_2 G/Z is simple: \mathcal{D}_3 or \mathcal{D}_3 Clifford theory W not abs. irred: \mathcal{D}_3 V /_N not homogeneous: \mathcal{D}_4 V /_N not homogeneous: \mathcal{D}_4 Cases for N/27: \mathcal{D}_8 - \mathcal{D}_9

Application

Classes D2 and D Class D7

Extraspecial: \mathcal{D}_6

Definition of class \mathcal{D}_6

- $G \leq \operatorname{GL}_n(\mathbb{F}_q)$ lies in \mathcal{D}_6 if
 - the natural module V is absolutely irreducible,
 - $n = r^m$ for a prime r and

Max Neunhöffer

- Background Constructive recognition Reductions Composition trees
- Aschbacher's Theorem
- A variant for GL and its proof The Statement Reductible: 3:1 Not abs. irred:: 3:3 Subfield: 3:5 G/Z is simple: 3:0 or 3:0 Cliffort theory W not abs. irred:: 3:0 VIN not thomogeneous: 3:2 3 cases for N/Z: 3:0-5;

Application

Classes D2 and D Class D7

Extraspecial: D₆

Definition of class \mathcal{D}_6

- $G \leq \operatorname{GL}_n(\mathbb{F}_q)$ lies in \mathcal{D}_6 if
 - the natural module V is absolutely irreducible,
 - $n = r^m$ for a prime r and
 - either *r* is odd and *G* has a normal subgroup *E* that is an extraspecial *r*-group of order r^{1+2m} and exponent *r*,
 - or r = 2 and G has a normal subgroup E that is either extraspecial of order 2^{1+2m} or a central product of a cyclic group of order 4 with an extraspecial group of order 2^{1+2m} ,

Max Neunhöffer

- Background Constructive recognition Reductions Composition trees
- Aschbacher's Theorem
- A variant for GL and its proof The Statement Reductile: 3, Not abs. irred:: 3, Subfiel: 3, G/Z is simple: 3, or 3, Cilliard theory W not abs. irred:: 3, V I/w homogeneous: 3, 3 cases for N/Z: 3, -3, -3,
- Application
- Classes D2 and D Class D7

Extraspecial: D₆

Definition of class \mathcal{D}_6

- $G \leq \operatorname{GL}_n(\mathbb{F}_q)$ lies in \mathcal{D}_6 if
 - the natural module V is absolutely irreducible,
 - $n = r^m$ for a prime r and
 - either *r* is odd and *G* has a normal subgroup *E* that is an extraspecial *r*-group of order r^{1+2m} and exponent *r*,
 - or r = 2 and G has a normal subgroup E that is either extraspecial of order 2^{1+2m} or a central product of a cyclic group of order 4 with an extraspecial group of order 2^{1+2m} ,
 - and in both cases the linear action of *G* on the \mathbb{F}_r -vector space E/Z(E) of dimension 2m is irreducible.

Max Neunhöffer

- Background Constructive recognition Reductions Composition trees
- Aschbacher's Theorem
- A variant for GL and its proof The Statement Reductile: 3, Not abs. irred.: 3, Subfiel: 3, G/Z is simple: 3, or 3, Cilford theory W not abs. irred.: 3, V I/v not homogeneous: 3, 3 cases for N/Z: 3, -3, -3,
- Application
- Classes D2 and I Class D7

Extraspecial: D₆

Definition of class \mathcal{D}_6

- $G \leq \operatorname{GL}_n(\mathbb{F}_q)$ lies in \mathcal{D}_6 if
 - the natural module V is absolutely irreducible,
 - $n = r^m$ for a prime r and
 - either *r* is odd and *G* has a normal subgroup *E* that is an extraspecial *r*-group of order r^{1+2m} and exponent *r*,
 - or r = 2 and G has a normal subgroup E that is either extraspecial of order 2^{1+2m} or a central product of a cyclic group of order 4 with an extraspecial group of order 2^{1+2m} ,
 - and in both cases the linear action of *G* on the \mathbb{F}_r -vector space E/Z(E) of dimension 2m is irreducible.

This class is in practice computationally under control.

Max Neunhöffer

Background Constructive recognition Reductions Composition trees

Aschbacher's Theorem

A variant for GL and its proof The Statement Reducible: \mathcal{D}_1 Not abs. irred: \mathcal{D}_3 Subfiel: \mathcal{D}_2 G/Z is simple: \mathcal{D}_0 or \mathcal{D}_0 Clifford theory W not abs. irred: \mathcal{D}_3 or \mathcal{D}_3 V_{IN} not homogeneous: \mathcal{D}_2 V_{IN} homogeneous: \mathcal{D}_2 a cases for N/2: \mathcal{D}_0 - \mathcal{D}_0

Application

Classes D₂ and D Class D₇

Tensor-induced: \mathcal{D}_7

Definition of class \mathcal{D}_7

 $G \leq \operatorname{GL}_n(\mathbb{F}_q)$ lies in \mathcal{D}_7 if

• the natural module V is absolutely irreducible and,

Max Neunhöffer

Background Constructive recognition Reductions Composition trees

Aschbacher's Theorem

A variant for GL and its proof The Statement Reductive: \mathcal{D}_1 Not abs. irred: \mathcal{D}_2 Subfield: \mathcal{D}_2 Gilford theory Wint abs. irred: \mathcal{D}_2 or \mathcal{D}_3 Ciliford theory Wint abs. irred: \mathcal{D}_2 Vi_N not homogeneous: \mathcal{D}_2 Vi_N not homogeneous: \mathcal{D}_3 a cases for N/2: B₂-m₂-M

Application

Classes D2 and D Class D7

Tensor-induced: \mathcal{D}_7

Definition of class \mathcal{D}_7

- $G \leq \operatorname{GL}_n(\mathbb{F}_q)$ lies in \mathcal{D}_7 if
 - the natural module V is absolutely irreducible and,
 - there is $Z < N \triangleleft G$ such that for some k > 1,

$$N \cong \underbrace{T \circ \cdots \circ T}_{k \text{ factors}} \quad \text{(central product)},$$

where T/Z is a non-abelian simple group, such that:

Max Neunhöffer

Background Constructive recognition Reductions Composition trees

Aschbacher's Theorem

A variant for GL and its proof The Statement Reducible: \mathcal{D}_1 Not abs. irred: \mathcal{D}_3 Subfield: \mathcal{D}_5 G/Z is simple: \mathcal{D}_6 or \mathcal{D}_6 Clifford theory W not abs. irred: \mathcal{D}_2 Vi_N not theoregeneous: \mathcal{D}_4 Vi_N homogeneous: \mathcal{D}_4 3 cases for N/Z: \mathcal{D}_6 - \mathcal{D}_9

Application

Classes D2 and D Class D7

Tensor-induced: \mathcal{D}_7

Definition of class \mathcal{D}_7

- $G \leq \operatorname{GL}_n(\mathbb{F}_q)$ lies in \mathcal{D}_7 if
 - the natural module V is absolutely irreducible and,
 - there is $Z < N \triangleleft G$ such that for some k > 1,

$$N \cong \underbrace{T \circ \cdots \circ T}_{k \text{ factors}} \quad \text{(central product)},$$

where T/Z is a non-abelian simple group, such that:

V|_N ≅ W₁ ⊗_{F_q} ··· ⊗_{F_q} W_k where the W_i are absolutely irreducible F_qT-modules of the same dimension on which Z acts as scalars,

Max Neunhöffer

Background Constructive recognition Reductions Composition trees

Aschbacher's Theorem

A variant for GL and its proof The Statement Reducible: \mathcal{D}_1 Not abs. irred: \mathcal{D}_2 Subfield: \mathcal{D}_3 Subfield: \mathcal{D}_5 G/Z is simple: \mathcal{D}_6 or \mathcal{D}_6 Clifford theory W not abs. irred: \mathcal{D}_2 V_{1/1} not homogeneous: \mathcal{D}_4 3 cases for N/Z: \mathcal{D}_6 - \mathcal{D}_9

Application

Classes D2 and D Class D7

Tensor-induced: \mathcal{D}_7

Definition of class \mathcal{D}_7

- $G \leq \operatorname{GL}_n(\mathbb{F}_q)$ lies in \mathcal{D}_7 if
 - the natural module V is absolutely irreducible and,
 - there is $Z < N \triangleleft G$ such that for some k > 1,

$$N \cong \underbrace{T \circ \cdots \circ T}_{k \text{ factors}} \quad \text{(central product)},$$

where T/Z is a non-abelian simple group, such that:

- V|_N ≅ W₁ ⊗_{F_q} ··· ⊗_{F_q} W_k where the W_i are absolutely irreducible F_qT-modules of the same dimension on which Z acts as scalars,
- and *G*/*N* permutes the tensor factors transitively.

Max Neunhöffer

Background Constructive recognition Reductions Composition trees

Aschbacher's Theorem

A variant for GL and its proof The Statement Reducible: \mathcal{D}_1 Not abs. irred: \mathcal{D}_3 Subfiel: \mathcal{D}_2 G/Z is simple: \mathcal{D}_3 or \mathcal{D}_3 Clifford theory W not abs. irred: \mathcal{D}_3 V /_N not homogeneous: \mathcal{D}_2 V /_N no homogeneous: \mathcal{D}_2 a cases for N/2: \mathcal{D}_2 - \mathcal{D}_3

Application Classes D₂ and D₄ Class D₇

Finding reductions for groups in \mathcal{D}_2 and \mathcal{D}_4

 \mathcal{D}_2 and \mathcal{D}_4 in this formulation have in common:

Max Neunhöffer

Background Constructive recognition Reductions Composition trees

Aschbacher's Theorem

A variant for GL and its proof The Statement Reducible: \mathcal{D}_1 Not abs. irred: \mathcal{D}_3 Subfiel: \mathcal{D}_2 G/Z is simple: \mathcal{D}_0 or \mathcal{D}_0 Clifford theory W not abs. irred: \mathcal{D}_3 V I_N not homogeneous: \mathcal{D}_2 V I_N not homogeneous: \mathcal{D}_2 S cases for W/Z: \mathcal{D}_2 - \mathcal{D}_3

Application Classes D₂ and D₄ Class D₇

Finding reductions for groups in \mathcal{D}_2 and \mathcal{D}_4

 \mathcal{D}_2 and \mathcal{D}_4 in this formulation have in common:

• In both cases there is an N with $Z < N \triangleleft G$.

Max Neunhöffer

Background Constructive recognition Reductions Composition trees

Aschbacher's Theorem

A variant for GL and its proof The Statement Reducible: \mathcal{D}_1 Not abs. irred: \mathcal{D}_3 Subfiel: \mathcal{D}_2 G/Z is simple: \mathcal{D}_0 or \mathcal{D}_0 Clifford theory W not abs. irred: \mathcal{D}_3 or \mathcal{D}_3 V_{IN} not homogeneous: \mathcal{D}_2 V_{IN} nongeneous: \mathcal{D}_2 a cases for N/2: \mathcal{D}_2 - \mathcal{D}_2 - \mathcal{D}_3

Application Classes D₂ and D₄ Class D₇

Finding reductions for groups in \mathcal{D}_2 and \mathcal{D}_4

 \mathcal{D}_2 and \mathcal{D}_4 in this formulation have in common:

In both cases there is an N with Z < N ⊲ G.
V|_N is reducible such that the MeatAxe can:

Max Neunhöffer

Background Constructive recognition Reductions Composition trees

Aschbacher's Theorem

A variant for GL and its proof The Statement Reducible: \mathcal{D}_1 Not abs. irred: \mathcal{D}_3 Subfiel: \mathcal{D}_2 G/Z is simple: \mathcal{D}_0 or \mathcal{D}_0 Clifford theory W not abs. irred: \mathcal{D}_3 or \mathcal{D}_3 V_{IN} not homogeneous: \mathcal{D}_2 V_{IN} nongeneous: \mathcal{D}_2 a cases for N/2: \mathcal{D}_2 - \mathcal{D}_2 - \mathcal{D}_3

Application Classes D₂ and D₄ Class D₇

Finding reductions for groups in \mathcal{D}_2 and \mathcal{D}_4

 \mathcal{D}_2 and \mathcal{D}_4 in this formulation have in common:

- In both cases there is an N with Z < N ⊲ G.
 V|_N is reducible such that the MeatAxe can:
 - determine whether $H \leq N$ for some $H \triangleleft G$ and

Max Neunhöffer

Background Constructive recognition Reductions Composition trees

Aschbacher's Theorem

A variant for GL and its proof The Statement Reducible: \mathcal{D}_1 Not abs. irred: \mathcal{D}_3 Subfiel: \mathcal{D}_2 G/Z is simple: \mathcal{D}_3 or \mathcal{D}_3 Clifford theory W not abs. irred: \mathcal{D}_3 V /_N not homogeneous: \mathcal{D}_4 V /_N not homogeneous: \mathcal{D}_4 Casaes for N/27: \mathcal{D}_2 - \mathcal{D}_3

Application Classes D₂ and D₄ Class D₇

Finding reductions for groups in \mathcal{D}_2 and \mathcal{D}_4

 \mathcal{D}_2 and \mathcal{D}_4 in this formulation have in common:

- In both cases there is an N with $Z < N \triangleleft G$.
- $V|_N$ is reducible such that the MeatAxe can:
 - determine whether $H \leq N$ for some $H \triangleleft G$ and
 - find a reduction in that case.

Max Neunhöffer

Background Constructive recognition Reductions Composition trees

Aschbacher's Theorem

A variant for GL and its proof The Statement Reducible: \mathfrak{D}_1 Not abs. irred: \mathfrak{D}_3 Subfield: \mathfrak{D}_2 Gilfiord theory Win ot abs. irred: \mathfrak{D}_3 Vi_W not thomogeneous: \mathfrak{D}_4 Vi_W not thomogeneous: \mathfrak{D}_4 3 cases for NZ: \mathfrak{D}_6 - \mathfrak{D}_9

Application Classes D₂ and D₄ Class D₇ Finding reductions for groups in \mathcal{D}_2 and \mathcal{D}_4

 \mathcal{D}_2 and \mathcal{D}_4 in this formulation have in common:

- In both cases there is an N with $Z < N \triangleleft G$.
- $V|_N$ is reducible such that the MeatAxe can:
 - determine whether $H \leq N$ for some $H \triangleleft G$ and
 - find a reduction in that case.

Since we can compute normal closures in *G*, all we need is to solve:

Problem

Find one element $n \in N \setminus Z$ with high probability.

Max Neunhöffer

Background Constructive recognition Reductions Composition trees

Aschbacher's Theorem

A variant for GL and its proof The Statement Reducible: \mathcal{D}_1 Not abs. irred: \mathcal{D}_2 Subfiel: \mathcal{D}_2 Gilford theory V not abs. irred: \mathcal{D}_2 or \mathcal{D}_3 Clifford theory V not abs. irred: \mathcal{D}_2 V $_{IN}$ nonogeneous: \mathcal{D}_2 V $_{IN}$ nonogeneous: \mathcal{D}_2 S cases for N/2: \mathcal{D}_2 - \mathcal{D}_3

Application Classes D2 and D4 Class D7

Finding a reduction for groups in \mathcal{D}_7

Also the definition of \mathcal{D}_7 involves *N* with $Z < N \triangleleft G$.

However, this time $V|_N$ is irreducible, so we do not notice, whether some $H \le N!$

Max Neunhöffer

Background Constructive recognition Reductions Composition trees

Aschbacher's Theorem

A variant for GL and its proof The Statement Reducible: \mathcal{D}_1 Not abs. irred: \mathcal{D}_3 Subfiel: \mathcal{D}_2 G/Z is simple: \mathcal{D}_0 or \mathcal{D}_0 Clifford theory W not abs. irred: \mathcal{D}_3 or \mathcal{D}_1 V I_N not homogeneous: \mathcal{D}_2 V I_N no homogeneous: \mathcal{D}_2 S cases for N/2: \mathcal{D}_0 - \mathcal{D}_0

Application Classes D2 and D4 Class D7

Finding a reduction for groups in \mathcal{D}_7

Also the definition of \mathcal{D}_7 involves *N* with $Z < N \triangleleft G$.

However, this time $V|_N$ is irreducible, so we do not notice, whether some $H \le N!$

But: *N* in \mathcal{D}_7 lies itself in \mathcal{D}_4 !

Max Neunhöffer

Background Constructive recognition Reductions Composition trees

Aschbacher's Theorem

A variant for GL and its proof The Statement Reducible: D_1 Not abs. irred.: D_3 Subfield: D_2 G/Z is simple: D_3 or D_5 Cillford theory W not abs. irred.: D_3 Vi_N not homogeneous: D_4 2 access for N/Z: D_6 — D_9

Application Classes D₂ and D₄ Class D₇

Finding a reduction for groups in \mathcal{D}_7

Also the definition of \mathcal{D}_7 involves *N* with $Z < N \triangleleft G$.

However, this time $V|_N$ is irreducible, so we do not notice, whether some $H \le N!$

But: N in \mathcal{D}_7 lies itself in \mathcal{D}_4 !

Idea

If we had a provably nice way to produce elements in a normal subgroup, then we could use the trick twice.