Finding normal subgroups

Max Neunhöffer

University of St Andrews

Birmingham, 26.3.2009

The problem

Problem

Let $1 < N \triangleleft G = \langle g_1, \ldots, g_k \rangle$ be a finite group and N be a normal subgroup.

The problem

Problem

Let $1 < N \triangleleft G = \langle g_1, \ldots, g_k \rangle$ *be a finite group and* N *be a normal subgroup.*
Produce a non-trivial element of N *as a word in the* g_i

The problem

Problem

Let $1 < N \lhd G = \langle g_1, \ldots, g_k \rangle$ be a finite group and N be a normal subgroup.
Produce a non-trivial element of N as a word in the g_i

- Assume no more knowledge about G or N.

The problem

Problem

Let $1 < N \lhd G = \langle g_1, \ldots, g_k \rangle$ be a finite group and N be a normal subgroup.
Produce a non-trivial element of N as a word in the g_i

- Assume no more knowledge about G or N.
- I shall tell you soon why we want to do this.

The problem

Problem

Let $1 < N \lhd G = \langle g_1, \ldots, g_k \rangle$ be a finite group and N be a normal subgroup.
Produce a non-trivial element of N as a word in the g_i

- Assume no more knowledge about G or N.
- I shall tell you soon why we want to do this.
- We are looking for a randomised algorithm.

The problem

Problem

Let $1 < N \lhd G = \langle g_1, \ldots, g_k \rangle$ be a finite group and N be a normal subgroup.
Produce a non-trivial element of N as a word in the g_i

- Assume no more knowledge about G or N.
- I shall tell you soon why we want to do this.
- We are looking for a randomised algorithm.
- Assume we can generate uniformly distributed random elements in G.

The problem

Problem

Let $1 < N \lhd G = \langle g_1, \ldots, g_k \rangle$ be a finite group and N be a normal subgroup.
Produce a non-trivial element of N as a word in the g_i with "high probability".

- Assume no more knowledge about G or N.
- I shall tell you soon why we want to do this.
- We are looking for a randomised algorithm.
- Assume we can generate uniformly distributed random elements in G.

The problem

Problem

Let $1 < N \lhd G = \langle g_1, \ldots, g_k \rangle$ be a finite group and N be a normal subgroup.
Produce a non-trivial element of N as a word in the g_i with "high probability".

- Assume no more knowledge about G or N.
- I shall tell you soon why we want to do this.
- We are looking for a randomised algorithm.
- Assume we can generate uniformly distributed random elements in G.
- "High probability" means for the moment "higher than $1/[G : N]$".

Matrix groups . . .

Let $\mathbb{F}_q$ be the field with q elements and

$$\mathrm{GL}_n(\mathbb{F}_q) := \{M \in \mathbb{F}_q^{n \times n} \mid M \text{ invertible}\}$$

Matrix groups . . .

Let $\mathbb{F}_q$ be the field with q elements and

$$\mathrm{GL}_n(\mathbb{F}_q) := \{M \in \mathbb{F}_q^{n\times n} \mid M \text{ invertible}\}$$

Given: $M_1, \ldots, M_k \in \mathrm{GL}_n(\mathbb{F}_q)$

Matrix groups . . .

Let $\mathbb{F}_q$ be the field with q elements and

$$\mathrm{GL}_n(\mathbb{F}_q) := \{M \in \mathbb{F}_q^{n\times n} \mid M \text{ invertible}\}$$

Given: $M_1, \ldots, M_k \in \mathrm{GL}_n(\mathbb{F}_q)$

Then the M_i generate a group $G \leq \mathrm{GL}_n(\mathbb{F}_q)$.

Matrix groups . . .

Let $\mathbb{F}_q$ be the field with q elements and

$$\mathrm{GL}_n(\mathbb{F}_q) := \{M \in \mathbb{F}_q^{n \times n} \mid M \text{ invertible}\}$$

Given: $M_1, \ldots, M_k \in \mathrm{GL}_n(\mathbb{F}_q)$

Then the M_i generate a group $G \leq \mathrm{GL}_n(\mathbb{F}_q)$.

It is finite, we have $|\mathrm{GL}_n(\mathbb{F}_q)| = q^{n(n-1)/2} \prod_{i=1}^{n} (q^i - 1)$

Matrix groups . . .

Let $\mathbb{F}_q$ be the field with q elements and

$$\mathrm{GL}_n(\mathbb{F}_q) := \{M \in \mathbb{F}_q^{n\times n} \mid M \text{ invertible}\}$$

Given: $M_1, \ldots, M_k \in \mathrm{GL}_n(\mathbb{F}_q)$

Then the M_i generate a group $G \leq \mathrm{GL}_n(\mathbb{F}_q)$.

It is finite, we have $|\mathrm{GL}_n(\mathbb{F}_q)| = q^{n(n-1)/2} \prod_{i=1}^{n}(q^i - 1)$

What do we want to determine about G?

- The group order $|G|$

Matrix groups ...

Let $\mathbb{F}_q$ be the field with q elements and

$$\mathrm{GL}_n(\mathbb{F}_q) := \{M \in \mathbb{F}_q^{n\times n} \mid M \text{ invertible}\}$$

Given: $M_1, \ldots, M_k \in \mathrm{GL}_n(\mathbb{F}_q)$

Then the M_i generate a group $G \leq \mathrm{GL}_n(\mathbb{F}_q)$.

It is finite, we have $|\mathrm{GL}_n(\mathbb{F}_q)| = q^{n(n-1)/2} \prod_{i=1}^{n}(q^i - 1)$

What do we want to determine about G?

- The group order $|G|$
- Membership test: Is $M \in \mathrm{GL}_n(\mathbb{F}_q)$ in G?

Matrix groups ...

Let $\mathbb{F}_q$ be the field with q elements and

$$\mathrm{GL}_n(\mathbb{F}_q) := \{M \in \mathbb{F}_q^{n \times n} \mid M \text{ invertible}\}$$

Given: $M_1, \ldots, M_k \in \mathrm{GL}_n(\mathbb{F}_q)$

Then the M_i generate a group $G \leq \mathrm{GL}_n(\mathbb{F}_q)$.

It is finite, we have $|\mathrm{GL}_n(\mathbb{F}_q)| = q^{n(n-1)/2} \prod_{i=1}^{n} (q^i - 1)$

What do we want to determine about G?

- The group order $|G|$
- Membership test: Is $M \in \mathrm{GL}_n(\mathbb{F}_q)$ in G?
- Homomorphisms $\varphi : G \to H$?

Matrix groups . . .

Let $\mathbb{F}_q$ be the field with q elements and

$$\mathrm{GL}_n(\mathbb{F}_q) := \{M \in \mathbb{F}_q^{n\times n} \mid M \text{ invertible}\}$$

Given: $M_1, \ldots, M_k \in \mathrm{GL}_n(\mathbb{F}_q)$

Then the M_i generate a group $G \leq \mathrm{GL}_n(\mathbb{F}_q)$.

It is finite, we have $|\mathrm{GL}_n(\mathbb{F}_q)| = q^{n(n-1)/2} \prod_{i=1}^{n}(q^i - 1)$

What do we want to determine about G?

- The group order $|G|$
- Membership test: Is $M \in \mathrm{GL}_n(\mathbb{F}_q)$ in G?
- Homomorphisms $\varphi : G \to H$?
- Kernels of homomorphisms? Is G simple?

Matrix groups ...

Let $\mathbb{F}_q$ be the field with q elements and

$$\mathrm{GL}_n(\mathbb{F}_q) := \{M \in \mathbb{F}_q^{n\times n} \mid M \text{ invertible}\}$$

Given: $M_1, \ldots, M_k \in \mathrm{GL}_n(\mathbb{F}_q)$

Then the M_i generate a group $G \leq \mathrm{GL}_n(\mathbb{F}_q)$.

It is finite, we have $|\mathrm{GL}_n(\mathbb{F}_q)| = q^{n(n-1)/2} \prod_{i=1}^{n}(q^i - 1)$

What do we want to determine about G?

- The group order $|G|$
- Membership test: Is $M \in \mathrm{GL}_n(\mathbb{F}_q)$ in G?
- Homomorphisms $\varphi : G \to H$?
- Kernels of homomorphisms? Is G simple?
- Comparison with known groups

Matrix groups . . .

Let $\mathbb{F}_q$ be the field with q elements and

$$\mathrm{GL}_n(\mathbb{F}_q) := \{M \in \mathbb{F}_q^{n\times n} \mid M \text{ invertible}\}$$

Given: $M_1, \ldots, M_k \in \mathrm{GL}_n(\mathbb{F}_q)$

Then the M_i generate a group $G \leq \mathrm{GL}_n(\mathbb{F}_q)$.

It is finite, we have $|\mathrm{GL}_n(\mathbb{F}_q)| = q^{n(n-1)/2} \prod_{i=1}^{n}(q^i - 1)$

What do we want to determine about G?

- The group order $|G|$
- Membership test: Is $M \in \mathrm{GL}_n(\mathbb{F}_q)$ in G?
- Homomorphisms $\varphi : G \to H$?
- Kernels of homomorphisms? Is G simple?
- Comparison with known groups
- (Maximal) subgroups?

Matrix groups . . .

Let $\mathbb{F}_q$ be the field with q elements and

$$\mathrm{GL}_n(\mathbb{F}_q) := \{M \in \mathbb{F}_q^{n\times n} \mid M \text{ invertible}\}$$

Given: $M_1, \ldots, M_k \in \mathrm{GL}_n(\mathbb{F}_q)$

Then the M_i generate a group $G \leq \mathrm{GL}_n(\mathbb{F}_q)$.

It is finite, we have $|\mathrm{GL}_n(\mathbb{F}_q)| = q^{n(n-1)/2} \prod_{i=1}^{n}(q^i - 1)$

What do we want to determine about G?

- The group order $|G|$
- Membership test: Is $M \in \mathrm{GL}_n(\mathbb{F}_q)$ in G?
- Homomorphisms $\varphi : G \to H$?
- Kernels of homomorphisms? Is G simple?
- Comparison with known groups
- (Maximal) subgroups?
- . . .

Constructive recognition

Problem

Let $\mathbb{F}_q$ be the field with q elements and

$$M_1, \ldots, M_k \in \mathrm{GL}_n(\mathbb{F}_q).$$

Find for $G := \langle M_1, \ldots, M_k \rangle$:

- The group order $|G|$ and
- an algorithm that, given $M \in \mathrm{GL}_n(\mathbb{F}_q)$,
  - decides, whether or not $M \in G$, and,
  - if so, expresses M as word in the M_i.

Constructive recognition

Problem

Let $\mathbb{F}_q$ be the field with q elements and

$$M_1, \ldots, M_k \in \mathrm{GL}_n(\mathbb{F}_q).$$

Find for $G := \langle M_1, \ldots, M_k \rangle$:

- The group order $|G|$ and
- an algorithm that, given $M \in \mathrm{GL}_n(\mathbb{F}_q)$,
  - decides, whether or not $M \in G$, and,
  - if so, expresses M as word in the M_i.
- The runtime should be bounded from above by a polynomial in n, k and $\log q$.

Constructive recognition

Problem

Let $\mathbb{F}_q$ be the field with q elements and

$$M_1, \ldots, M_k \in \mathrm{GL}_n(\mathbb{F}_q).$$

Find for $G := \langle M_1, \ldots, M_k \rangle$:

- The group order $|G|$ and
- an algorithm that, given $M \in \mathrm{GL}_n(\mathbb{F}_q)$,
  - decides, whether or not $M \in G$, and,
  - if so, expresses M as word in the M_i.
- The runtime should be bounded from above by a polynomial in n, k and $\log q$.
- A Monte Carlo Algorithm is enough.

Constructive recognition

Problem

Let $\mathbb{F}_q$ be the field with q elements and

$$M_1, \ldots, M_k \in \mathrm{GL}_n(\mathbb{F}_q).$$

Find for $G := \langle M_1, \ldots, M_k \rangle$:

- The group order $|G|$ and
- an algorithm that, given $M \in \mathrm{GL}_n(\mathbb{F}_q)$,
  - decides, whether or not $M \in G$, and,
  - if so, expresses M as word in the M_i.
- The runtime should be bounded from above by a polynomial in n, k and $\log q$.
- A Monte Carlo Algorithm is enough. (Verification!)

Constructive recognition

Problem

Let $\mathbb{F}_q$ be the field with q elements and

$$M_1, \ldots, M_k \in \mathrm{GL}_n(\mathbb{F}_q).$$

Find for $G := \langle M_1, \ldots, M_k \rangle$:

- The group order $|G|$ and
- an algorithm that, given $M \in \mathrm{GL}_n(\mathbb{F}_q)$,
  - decides, whether or not $M \in G$, and,
  - if so, expresses M as word in the M_i.
- The runtime should be bounded from above by a polynomial in n, k and $\log q$.
- A Monte Carlo Algorithm is enough. (Verification!)

If this problem is solved, we call

$\langle M_1, \ldots, M_k \rangle$ recognised constructively.

What is a reduction?

Let $G := \langle M_1, \ldots, M_k \rangle \leq \mathrm{GL}_n(\mathbb{F}_q)$.

What is a reduction?

Let $G := \langle M_1, \ldots, M_k \rangle \leq \mathrm{GL}_n(\mathbb{F}_q)$.

A reduction is a group homomorphism

$$\begin{array}{cccc} \varphi : & G & \to & H \\ & M_i & \mapsto & P_i \end{array} \qquad \text{for all } i$$

with the following properties:

What is a reduction?

Let $G := \langle M_1, \ldots, M_k \rangle \leq \mathrm{GL}_n(\mathbb{F}_q)$.

A reduction is a group homomorphism

$$\begin{array}{cccc} \varphi : & G & \to & H \\ & M_i & \mapsto & P_i \end{array} \qquad \text{for all } i$$

with the following properties:

- $\varphi(M)$ is explicitly computable for all $M \in G$

What is a reduction?

Let $G := \langle M_1, \ldots, M_k \rangle \leq \mathrm{GL}_n(\mathbb{F}_q)$.

A reduction is a group homomorphism

$$\begin{array}{rcccl} \varphi & : & G & \to & H \\ & & M_i & \mapsto & P_i \qquad \text{for all } i \end{array}$$

with the following properties:

- $\varphi(M)$ is explicitly computable for all $M \in G$
- φ is surjective: $H = \langle P_1, \ldots, P_k \rangle$

What is a reduction?

Let $G := \langle M_1, \ldots, M_k \rangle \leq \mathrm{GL}_n(\mathbb{F}_q)$.

A reduction is a group homomorphism

$$\begin{array}{rcccl} \varphi & : & G & \to & H \\ & & M_i & \mapsto & P_i \qquad \text{for all } i \end{array}$$

with the following properties:

- $\varphi(M)$ is explicitly computable for all $M \in G$
- φ is surjective: $H = \langle P_1, \ldots, P_k \rangle$
- H is in some sense "smaller"
- or at least "easier to recognise constructively"

What is a reduction?

Let $G := \langle M_1, \ldots, M_k \rangle \leq \mathrm{GL}_n(\mathbb{F}_q)$.

A reduction is a group homomorphism

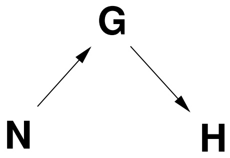
$$\begin{array}{rcccc} \varphi & : & G & \to & H \\ & & M_i & \mapsto & P_i \end{array} \qquad \text{for all } i$$

with the following properties:

- $\varphi(M)$ is explicitly computable for all $M \in G$
- φ is surjective: $H = \langle P_1, \ldots, P_k \rangle$
- H is in some sense "smaller"
- or at least "easier to recognise constructively"
- e.g. $H \leq S_m$ or $H \leq \mathrm{GL}_{n'}(\mathbb{F}_{q'})$ with $n' \log q' < n \log q$

Recursion: composition trees

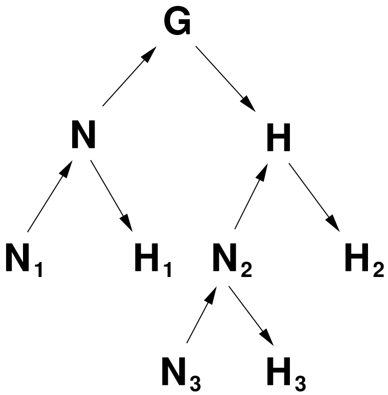
We get a tree:



Up arrows: inclusions
Down arrows: homomorphisms

Recursion: composition trees

We get a tree:

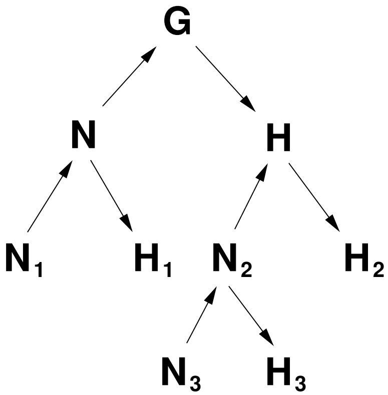


Up arrows: inclusions

Down arrows: homomorphisms

Recursion: composition trees

We get a tree:



Up arrows: inclusions
Down arrows: homomorphisms

Old idea, substantial improvements are still being made

Reduction in the imprimitive case

One case, in which we want to find a reduction, is:

Reduction in the imprimitive case

One case, in which we want to find a reduction, is:

Situation

Let $G \leq \mathrm{GL}_n(\mathbb{F}_q)$ acting linearly on $V := \mathbb{F}_q^{1\times n}$, such that V is irreducible.

Reduction in the imprimitive case

One case, in which we want to find a reduction, is:

Situation

Let $G \leq \mathrm{GL}_n(\mathbb{F}_q)$ acting linearly on $V := \mathbb{F}_q^{1\times n}$, such that V is irreducible. Assume there is N with $Z(G) < N \lhd G$ such that

$$V|_N = W_1 \oplus W_2 \oplus \cdots \oplus W_k,$$

all W_i are invariant under N, and

Reduction in the imprimitive case

One case, in which we want to find a reduction, is:

Situation

Let $G \leq \mathrm{GL}_n(\mathbb{F}_q)$ acting linearly on $V := \mathbb{F}_q^{1\times n}$, such that V is irreducible. Assume there is N with $Z(G) < N \lhd G$ such that

$$V|_N = W_1 \oplus W_2 \oplus \cdots \oplus W_k,$$

all W_i are invariant under N, and G permutes the W_i transitively. Then there is a reduction $\varphi : G \to S_k$.

Reduction in the imprimitive case

One case, in which we want to find a reduction, is:

Situation

Let $G \leq \mathrm{GL}_n(\mathbb{F}_q)$ acting linearly on $V := \mathbb{F}_q^{1\times n}$, such that V is irreducible. Assume there is N with $Z(G) < N \lhd G$ such that

$$V|_N = W_1 \oplus W_2 \oplus \cdots \oplus W_k,$$

all W_i are invariant under N, and G permutes the W_i transitively. Then there is a reduction $\varphi : G \to S_k$.

We can compute the reduction once N is found.

Reduction in the imprimitive case

One case, in which we want to find a reduction, is:

Situation

Let $G \leq \mathrm{GL}_n(\mathbb{F}_q)$ acting linearly on $V := \mathbb{F}_q^{1\times n}$, such that V is irreducible. Assume there is N with $Z(G) < N \lhd G$ such that

$$V|_N = W_1 \oplus W_2 \oplus \cdots \oplus W_k,$$

all W_i are invariant under N, and G permutes the W_i transitively. Then there is a reduction $\varphi : G \to S_k$.

We can compute the reduction once N is found.

Since we can compute normal closures, our initial problem is exactly, what we need to do.

Things we can do in matrix groups

We can efficiently:

- store and compare elements

Things we can do in matrix groups

We can efficiently:

- store and compare elements
- form products and inverses,

Things we can do in matrix groups

We can efficiently:

- store and compare elements
- form products and inverses,
- act on vectors, subspaces and matrices,

Things we can do in matrix groups

We can efficiently:

- store and compare elements
- form products and inverses,
- act on vectors, subspaces and matrices,
- compute element orders

Things we can do in matrix groups

We can efficiently:

- store and compare elements
- form products and inverses,
- act on vectors, subspaces and matrices,
- compute element orders
- produce uniformly distributed random elements

Things we can do in matrix groups

We can efficiently:

- store and compare elements
- form products and inverses,
- act on vectors, subspaces and matrices,
- compute element orders
- produce uniformly distributed random elements
- use previously assembled data about groups and representations

Things we can do in matrix groups

We can efficiently:

- store and compare elements
- form products and inverses,
- act on vectors, subspaces and matrices,
- compute element orders
- produce uniformly distributed random elements
- use previously assembled data about groups and representations
- compute normal closures (at least Monte Carlo).

Things we can do in matrix groups

We can efficiently:

- store and compare elements
- form products and inverses,
- act on vectors, subspaces and matrices,
- compute element orders
- produce uniformly distributed random elements
- use previously assembled data about groups and representations
- compute normal closures (at least Monte Carlo).

The latter means that for $H < G$, we can compute some elements that generate with high probability the smallest normal subgroup of G containing H.

Blind descent (Babai, Beals)

Let $1 \neq x, y \in G$ and G non-abelian.

Blind descent (Babai, Beals)

Let $1 \neq x, y \in G$ and G non-abelian.

Assume at least one of x, y is contained in a non-trivial proper normal subgroup.

Blind descent (Babai, Beals)

Let $1 \neq x, y \in G$ and G non-abelian.

Assume at least one of x, y is contained in a non-trivial proper normal subgroup.

We do not know which!

Blind descent (Babai, Beals)

Let $1 \neq x, y \in G$ and G non-abelian.

Assume at least one of x, y is contained in a non-trivial proper normal subgroup.

We do not know which!

Aim: Produce $1 \neq z \in G$ that is contained in a non-trivial proper normal subgroup.

Blind descent (Babai, Beals)

Let $1 \neq x, y \in G$ and G non-abelian.

Assume at least one of x, y is contained in a non-trivial proper normal subgroup.

We do not know which!

Aim: Produce $1 \neq z \in G$ that is contained in a non-trivial proper normal subgroup.

1. Consider $c := [x, y] := x^{-1}y^{-1}xy$,
   if $c \neq 1$, we take $z := c$.

Blind descent (Babai, Beals)

Let $1 \neq x, y \in G$ and G non-abelian.

Assume at least one of x, y is contained in a non-trivial proper normal subgroup.

We do not know which!

Aim: Produce $1 \neq z \in G$ that is contained in a non-trivial proper normal subgroup.

1. Consider $c := [x, y] := x^{-1}y^{-1}xy$, if $c \neq 1$, we take $z := c$.
2. If $c = 1$, the elements x and y commute. If $x \in Z(G)$, take $z := x$.

Blind descent (Babai, Beals)

Let $1 \neq x, y \in G$ and G non-abelian.

Assume at least one of x, y is contained in a non-trivial proper normal subgroup.

We do not know which!

Aim: Produce $1 \neq z \in G$ that is contained in a non-trivial proper normal subgroup.

1. Consider $c := [x, y] := x^{-1}y^{-1}xy$,
   if $c \neq 1$, we take $z := c$.
2. If $c = 1$, the elements x and y commute.
   If $x \in Z(G)$, take $z := x$.
3. Compute generators $\{y_i\}$ for $Y := \langle y^G \rangle$.
   - If some $c_i := [x, y_i] \neq 1$, then take $z := c_i$ as in 1.

Blind descent (Babai, Beals)

Let $1 \neq x, y \in G$ and G non-abelian.

Assume at least one of x, y is contained in a non-trivial proper normal subgroup.

We do not know which!

Aim: Produce $1 \neq z \in G$ that is contained in a non-trivial proper normal subgroup.

1. Consider $c := [x, y] := x^{-1}y^{-1}xy$, if $c \neq 1$, we take $z := c$.
2. If $c = 1$, the elements x and y commute. If $x \in Z(G)$, take $z := x$.
3. Compute generators $\{y_i\}$ for $Y := \langle y^G \rangle$.
   - If some $c_i := [x, y_i] \neq 1$, then take $z := c_i$ as in 1.
   - Otherwise $g \in C_G(Y)$ but $g \notin Z(G)$, thus $Y \neq G$, we take $z := y$.

A first try

Algorithm 1 (Babai, Beals)

Initialize $1 \neq x := \textsc{RandomElement}(G)$

A first try

Algorithm 1 (Babai, Beals)

Initialize $1 \neq x := \textsc{RandomElement}(G)$

Repeat K times:

1. $y := \textsc{RandomElement}(G)$

A first try

Algorithm 1 (Babai, Beals)

Initialize $1 \neq x := \textsc{RandomElement}(G)$

Repeat K times:

1. $y := \textsc{RandomElement}(G)$
2. $o := \textsc{Order}(y)$

A first try

Algorithm 1 (Babai, Beals)

Initialize $1 \neq x := \textsc{RandomElement}(G)$

Repeat K times:

1. $y := \textsc{RandomElement}(G)$
2. $o := \textsc{Order}(y)$
3. $p :=$ some prime divisor of o
4. $y' := y^{o/p}$ has order p

A first try

Algorithm 1 (Babai, Beals)

Initialize $1 \neq x := \text{RANDOMELEMENT}(G)$

Repeat K times:

1. $y := \text{RANDOMELEMENT}(G)$
2. $o := \text{ORDER}(y)$
3. $p :=$ some prime divisor of o
4. $y' := y^{o/p}$ has order p
5. $x := \text{BLINDDESCENT}(x, y')$

Return x

✓

What is the Involution Jumper?

What is the Involution Jumper?

Input: $G = \langle g_1, \ldots, g_k \rangle$ and an involution $x \in G$.
repeat
$y := \textsc{RandomElement}(G)$
$c := x^{-1}y^{-1}xy$ **and** $o := \textsc{Order}(c)$
if o is even **then**
return $c^{o/2}$
else
$z := y \cdot c^{(o-1)/2}$ **and** $o' := \textsc{Order}(z)$
if o' is even **then**
return $z^{o'/2}$
until patience lost
return $\textsc{Fail}$

What is the Involution Jumper?

Input: $G = \langle g_1, \ldots, g_k \rangle$ and an involution $x \in G$.
repeat
  $y := \text{RandomElement}(G)$
  $c := x^{-1}y^{-1}xy$ **and** $o := \text{Order}(c)$
  if o is even **then**
    return $c^{o/2}$
  else
    $z := y \cdot c^{(o-1)/2}$ **and** $o' := \text{Order}(z)$
    if o' is even **then**
      return $z^{o'/2}$
until patience lost
return FAIL

Note: If $xy = yx$ then $c = 1_G$ and $o = 1$ and $z = y$.

What is the Involution Jumper?

Input: $G = \langle g_1, \ldots, g_k \rangle$ and an involution $x \in G$.
repeat
    $y := \textsc{RandomElement}(G)$
    $c := x^{-1}y^{-1}xy$ **and** $o := \textsc{Order}(c)$
    if o is even **then**
        return $c^{o/2}$
    else
        $z := y \cdot c^{(o-1)/2}$ **and** $o' := \textsc{Order}(z)$
        if o' is even **then**
            return $z^{o'/2}$
until patience lost
return $\textsc{Fail}$

Note: If $xy = yx$ then $c = 1_G$ and $o = 1$ and $z = y$.
But this happens rarely.

What does the Involution Jumper do?

Input: $G = \langle g_1, \ldots, g_k \rangle$ and an involution $x \in G$.

- If it does not fail, it returns an involution $\tilde{x} \in G$.

What does the Involution Jumper do?

Input: $G = \langle g_1, \ldots, g_k \rangle$ and an involution $x \in G$.

- If it does not fail, it returns an involution $\tilde{x} \in G$.
- $x\tilde{x} = \tilde{x}x$

What does the Involution Jumper do?

Input: $G = \langle g_1, \ldots, g_k \rangle$ and an involution $x \in G$.

- If it does not fail, it returns an involution $\tilde{x} \in G$.
- $x\tilde{x} = \tilde{x}x$
- Every involution of $C_G(x)$ occurs with probability > 0.

What does the Involution Jumper do?

Input: $G = \langle g_1, \ldots, g_k \rangle$ and an involution $x \in G$.

- If it does not fail, it returns an involution $\tilde{x} \in G$.
- $x\tilde{x} = \tilde{x}x$
- Every involution of $C_G(x)$ occurs with probability > 0.
- Using product replacement to produce random elements, this is a practical method for
  - permutation groups,
  - matrix groups and
  - projective groups,

What does the Involution Jumper do?

Input: $G = \langle g_1, \ldots, g_k \rangle$ and an involution $x \in G$.

- If it does not fail, it returns an involution $\tilde{x} \in G$.
- $x\tilde{x} = \tilde{x}x$
- Every involution of $C_G(x)$ occurs with probability > 0.
- Using product replacement to produce random elements, this is a practical method for
  - permutation groups,
  - matrix groups and
  - projective groups,

  if nothing goes wrong.

What does the Involution Jumper do?

Input: $G = \langle g_1, \ldots, g_k \rangle$ and an involution $x \in G$.

- If it does not fail, it returns an involution $\tilde{x} \in G$.
- $x\tilde{x} = \tilde{x}x$
- Every involution of $C_G(x)$ occurs with probability > 0.
- Using product replacement to produce random elements, this is a practical method for
  - permutation groups,
  - matrix groups and
  - projective groups,

  if nothing goes wrong.
- It needs an involution to start with.

Jumping between classes

Notation: Let x^G denote the conjugacy class of x in G.

Jumping between classes

Notation: Let x^G denote the conjugacy class of x in G.

Lemma

Let $x, a \in G$ be involutions and $g \in G$. Then

$$Prob(IJ(x) \in a^G) = Prob(IJ(x^g) \in a^G).$$

Jumping between classes

Notation: Let x^G denote the conjugacy class of x in G.

Lemma

Let $x, a \in G$ be involutions and $g \in G$. Then

$$Prob(IJ(x) \in a^G) = Prob(IJ(x^g) \in a^G).$$

or equivalently

Lemma

Let $x \in G$ be an involution. Then the distribution of $IJ(x)^G$ only depends on x^G and not on the choice of x in x^G.

Jumping between classes

Notation: Let x^G denote the conjugacy class of x in G.

Lemma

Let $x, a \in G$ be involutions and $g \in G$. Then

$$Prob(IJ(x) \in a^G) = Prob(IJ(x^g) \in a^G).$$

or equivalently

Lemma

Let $x \in G$ be an involution. Then the distribution of $IJ(x)^G$ only depends on x^G and not on the choice of x in x^G.

Proof: $f(x, y) :=$

$$\begin{cases} [x,y]^k & \text{if } \mathrm{ORDER}([x,y]) = 2k \\ (y[x,y]^k)^l & \text{if } \mathrm{ORDER}([x,y]) = 2k+1 > 1 \text{ and} \\ & \quad \mathrm{ORDER}([y[x,y]^k]) = 2l \\ y^k & \text{if } xy = yx \text{ and } \mathrm{ORDER}(y) = 2k \end{cases}$$

and we have $f(x^g, y^g) = f(x,y)^g$ whenever f is defined. ✓

A Markov chain $\mathcal{M}$

The states are the conjugacy classes of involutions in G.

A Markov chain $\mathcal{M}$

The states are the conjugacy classes of involutions in G.

The transition is done as follows: At a class a^G:

- Pick an arbitrary involution $x \in a^G$.
- Compute $\tilde{x} := IJ(x)$ until $\tilde{x} \neq$ FAIL.
- Next state is $\tilde{x}^G$.

A Markov chain $\mathcal{M}$

The states are the conjugacy classes of involutions in G.

The transition is done as follows: At a class a^G:

- Pick an arbitrary involution $x \in a^G$.
- Compute $\tilde{x} := IJ(x)$ until $\tilde{x} \neq$ FAIL.
- Next state is $\tilde{x}^G$.

By the lemma, the distribution of the class $\tilde{x}^G$ does not depend on the choice of x.

A Markov chain $\mathcal{M}$

The states are the conjugacy classes of involutions in G.

The transition is done as follows: At a class a^G:

- Pick an arbitrary involution $x \in a^G$.
- Compute $\tilde{x} := IJ(x)$ until $\tilde{x} \neq \textsc{Fail}$.
- Next state is $\tilde{x}^G$.

By the lemma, the distribution of the class $\tilde{x}^G$ does not depend on the choice of x.

Theorem

The above Markov chain $\mathcal{M}$ is irreducible and aperiodic and thus has a stationary distribution in which every state has non-zero probability.

✓

Back to the original question

Problem

Let $1 < N \lhd G = \langle g_1, \ldots, g_k \rangle$ be a finite group and N be a normal subgroup.
Produce a non-trivial element of N as a word in the g_i with "high probability".

Back to the original question

Problem

Let $1 < N \lhd G = \langle g_1, \ldots, g_k \rangle$ be a finite group and N be a normal subgroup.
Produce a non-trivial element of N as a word in the g_i with "high probability".

- If we find an involution in G to start with
- and N contains at least one involution class,

Back to the original question

Problem

Let $1 < N \lhd G = \langle g_1, \ldots, g_k \rangle$ be a finite group and N be a normal subgroup.
Produce a non-trivial element of N as a word in the g_i with "high probability".

- If we find an involution in G to start with
- and N contains at least one involution class,

the IJ will eventually jump onto an involution class in N.

✓

A better try

Algorithm 2

Initialize $1 \neq x := \textsc{RandomElement}(G)$ and
$z := \textsc{RandomInvolution}(G)$

A better try

Algorithm 2

Initialize $1 \neq x := \text{RANDOMELEMENT}(G)$ and
$z := \text{RANDOMINVOLUTION}(G)$

Repeat K times:

1. $y := \text{RANDOMELEMENT}(G)$
2. $o := \text{ORDER}(y)$
3. For a few prime divisors p of o do:
   - $y' := y^{o/p}$ has order p
   - $x := \text{BLINDDESCENT}(x, y')$

A better try

Algorithm 2

Initialize $1 \neq x := \text{RandomElement}(G)$ and
$z := \text{RandomInvolution}(G)$

Repeat K times:

1. $y := \text{RandomElement}(G)$
2. $o := \text{Order}(y)$
3. For a few prime divisors p of o do:
   - $y' := y^{o/p}$ has order p
   - $x := \text{BlindDescent}(x, y')$
4. $z := \text{InvolutionJumper}(G, z)$
5. $x := \text{BlindDescent}(x, z)$

Return x

Examples

In practice, the IJ works extremely well in many cases:

G	N	# hops*
$S_5 \wr S_{10}$	$S_5^{\times 10}$	1.91
$\mathrm{GL}(3,3) \wr S_6 < \mathrm{GL}(18,3)$	$\mathrm{GL}(3,3)^{\times 6}$	1.17
$\mathrm{Sp}(6,3) \otimes 2.\mathrm{O}(7,3) < \mathrm{GL}(48,3)$	$\mathrm{Sp}(6,3) \otimes 1$	1.83

* average number of IJ hops needed to reach N.

Examples

In practice, the IJ works extremely well in many cases:

G	N	# hops*
$S_5 \wr S_{10}$	$S_5^{\times 10}$	1.91
$\mathrm{GL}(3,3) \wr S_6 < \mathrm{GL}(18,3)$	$\mathrm{GL}(3,3)^{\times 6}$	1.17
$\mathrm{Sp}(6,3) \otimes 2.\mathrm{O}(7,3) < \mathrm{GL}(48,3)$	$\mathrm{Sp}(6,3) \otimes 1$	1.83

* average number of IJ hops needed to reach N.

Running Algorithm 2 (with $K = 5$) also works nicely:

G	N	succ.
$S_5 \wr S_{10}$	$S_5^{\times 10}$	100%
$\mathrm{GL}(3,3) \wr S_6 < \mathrm{GL}(18,3)$	$\mathrm{GL}(3,3)^{\times 6}$	100%
$\mathrm{Sp}(6,3) \otimes 2.\mathrm{O}(7,3) < \mathrm{GL}(48,3)$	$\mathrm{Sp}(6,3) \otimes 1$	100%

(here we have done 100 runs)

Reductions for imprimitive matrix groups

Situation

Let $G \leq \mathrm{GL}_n(\mathbb{F}_q)$ acting linearly on $V := \mathbb{F}_q^{1\times n}$, such that V is irreducible. Assume there is N with $Z(G) < N \lhd G$ such that

$$V|_N = W_1 \oplus W_2 \oplus \cdots \oplus W_k,$$

all W_i are invariant under N, and G permutes the W_i transitively. Then there is a reduction $\varphi : G \to S_k$.

Reductions for imprimitive matrix groups

Situation

Let $G \leq \mathrm{GL}_n(\mathbb{F}_q)$ acting linearly on $V := \mathbb{F}_q^{1\times n}$, such that V is irreducible. Assume there is N with $Z(G) < N \lhd G$ such that

$$V|_N = W_1 \oplus W_2 \oplus \cdots \oplus W_k,$$

all W_i are invariant under N, and G permutes the W_i transitively. Then there is a reduction $\varphi : G \to S_k$.

We use Algorithm 2, for the result x, do:

- compute the normal closure $M := \langle x^G \rangle$,
- use the MeatAxe to check whether $V|_M$ is reducible,
- if $x \in N$, we find a reduction.

Possible problems

The InvolutionJumper is in trouble, if at least one of the following happens:

- we do not easily find an involution in G (like for example in $\mathrm{SL}(2, 2^n)$ for big n),

Possible problems

The InvolutionJumper is in trouble, if at least one of the following happens:

- we do not easily find an involution in G (like for example in $\mathrm{SL}(2, 2^n)$ for big n),
- the involution classes of N have a small probability in the limit distribution (when does this happen?),

Possible problems

The InvolutionJumper is in trouble, if at least one of the following happens:

- we do not easily find an involution in G (like for example in $\mathrm{SL}(2, 2^n)$ for big n),
- the involution classes of N have a small probability in the limit distribution (when does this happen?),
- the Markov chain does not converge quick enough to its limiting distribution (how quick does it converge?),

Possible problems

The InvolutionJumper is in trouble, if at least one of the following happens:

- we do not easily find an involution in G (like for example in $\mathrm{SL}(2, 2^n)$ for big n),
- the involution classes of N have a small probability in the limit distribution (when does this happen?),
- the Markov chain does not converge quick enough to its limiting distribution (how quick does it converge?),
- the Involution Jumper returns FAIL too often (when does this happen?),

Possible problems

The InvolutionJumper is in trouble, if at least one of the following happens:

- we do not easily find an involution in G (like for example in $\mathrm{SL}(2, 2^n)$ for big n),
- the involution classes of N have a small probability in the limit distribution (when does this happen?),
- the Markov chain does not converge quick enough to its limiting distribution (how quick does it converge?),
- the Involution Jumper returns FAIL too often (when does this happen?),
- N has odd order.

Possible problems

The InvolutionJumper is in trouble, if at least one of the following happens:

- we do not easily find an involution in G (like for example in $\mathrm{SL}(2, 2^n)$ for big n),
- the involution classes of N have a small probability in the limit distribution (when does this happen?),
- the Markov chain does not converge quick enough to its limiting distribution (how quick does it converge?),
- the Involution Jumper returns FAIL too often (when does this happen?),
- N has odd order.

Fortunately: Centralisers of involutions seem to contain enough involutions.