Matrix group recognition

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Matrix groups . . .

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We can determine about G algorithmically (e.g.):

- The group order $|G|$
- Membership test: Is $M \in S_n$ in G?
- Homomorphisms $\varphi : G \to H$?
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Constructive recognition — first formulation

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If this problem is solved, we call

$\langle M_1, \ldots, M_k \rangle$ recognised constructively.

Complexity of algorithms

To measure the efficiency of an algorithm, we consider a class $\mathcal{P}$ of problems, that the algorithm can solve.

We assign to each $P \in \mathcal{P}$ its size $g(P)$,

and prove an upper bound for the runtime $L(P)$ of the algorithm for P:

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Example (Constructive matrix group recognition)

- Problem given by $M_1, \ldots, M_k \in \mathrm{GL}_n(\mathbb{F}_q)$.
- Size determined by n, k and $\log q$.
- Runtime should be $\leq$ a polynomial in n, k and $\log q$.

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A Monte Carlo algorithm with error probability ϵ is an algorithm, that is guaranteed to terminate after a finite time, such that the probability that it returns a wrong result is at most ϵ.

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deterministic alg.: 112s   Monte Carlo $\epsilon = 1\%$: 6s

Saving: 95% of runtime

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Troubles

The discrete logarithm problem

If $M_1 = [z] \in \mathbb{F}_q^{1\times 1}$ with z a primitive root of $\mathbb{F}_q$. Then:

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In practice q is small $\Rightarrow$ no problem.

We ignore both!

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- or at least "easier to recognise constructively"
- e.g. $H \leq S_m$ or $H \leq \mathrm{GL}_{n'}(\mathbb{F}_{q'})$ with $n' \log q' < n \log q$

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6. If M is uniformly distributed in G
then $M \cdot M'^{-1}$ is uniformly distributed in N

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$\to$ Monte Carlo algorithm to compute N

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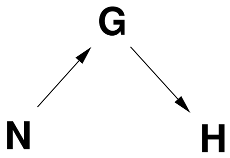
7. If $M \notin G$, then at least one step does not work.

Recursion: composition trees

We get a tree:



Up arrows: inclusions
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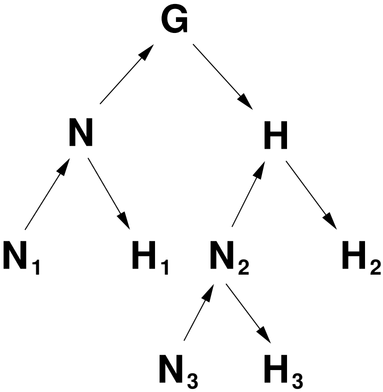

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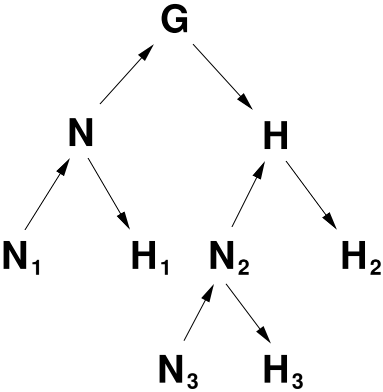

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Old idea, substantial improvements are still being made

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Choose basis $(w_1, \ldots, w_d)$ of W and extend to a basis

$$(w_1, \ldots, w_d, w_{d+1}, \ldots, w_n)$$

of V. After a base change the matrices in G look like this:

$$\left[\begin{array}{c|c} A & B \\ \hline \mathbf{0} & D \end{array}\right] \qquad \text{with } A \in \mathbb{F}_q^{d \times d}, B \in \mathbb{F}_q^{d \times (n-d)}, D \in \mathbb{F}_q^{(n-d) \times (n-d)}$$

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Together with a reduction additional information is gained!

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Let $G \leq \mathrm{GL}_n(\mathbb{F}_q)$ and $Z := G \cap Z(\mathrm{GL}_n(\mathbb{F}_q))$ the subgroup of scalar matrices. Then G lies in at least one of the classes C1 to C8 or we have:

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- the modular representation theory of simple groups

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This uses:

- the classification of finite simple groups
- information about their automorphism groups
- information about element orders
- information about conjugacy classes
- classifications of the irreducible representations
- information about the subgroup structure

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Usually this leads to Monte Carlo algorithms.

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- Check the relations and thus prove the result.