Finding normal subgroups of even order

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The problem

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- "High probability" means for the moment "higher than $1/[G : N]$".

Matrix groups . . .

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Constructive recognition

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$$M_1, \ldots, M_k \in \mathrm{GL}_n(\mathbb{F}_q).$$

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    - decides, whether or not $M \in G$, and,
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If this problem is solved, we call

$\langle M_1, \ldots, M_k \rangle$ recognised constructively.

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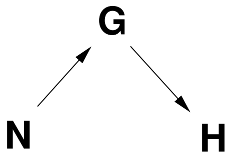
- or at least "easier to recognise constructively"
- e.g. $H \leq S_m$ or $H \leq \mathrm{GL}_{n'}(\mathbb{F}_{q'})$ with $n' \log q' < n \log q$

Recursion: composition trees

We get a tree:



Up arrows: inclusions
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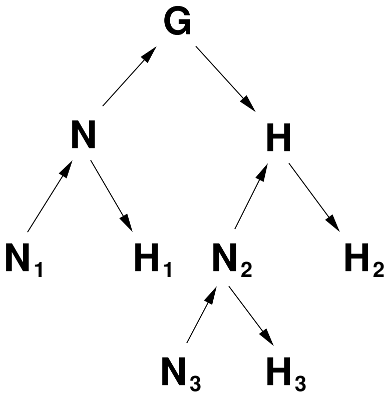

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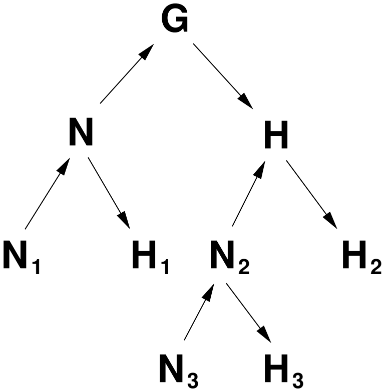

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Old idea, substantial improvements are still being made

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$$V|_N = W_1 \oplus W_2 \oplus \cdots \oplus W_k,$$

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Since we can compute normal closures, our initial problem is exactly, what we need to do.

Finding even order normal subgroups

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That is, we can replace (N, G) with $(C \cap N, C)$ and use the statement again, provided we find another non-central involution.

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We find involutions by powering up random elements.

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The following method by John Bray does the job:

Algorithm 2: InvolutionCentraliser

Input: $G = \langle g_1, \ldots, g_k \rangle$ and an involution $x \in G$.
initialise *gens* $:= [x]$
repeat
$y := \text{RandomElement}(G)$
$c := x^{-1}y^{-1}xy$ **and** $o := \text{Order}(c)$

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  $c := x^{-1}y^{-1}xy$ **and** $o :=$ ORDER(c)
  if o is even **then**
    append $c^{o/2}$ and $(x^{-1}yxy^{-1})^{o/2}$ to *gens*
  else
    append $z := y \cdot c^{(o-1)/2}$ to *gens*

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Input: $G = \langle g_1, \ldots, g_k \rangle$ and an involution $x \in G$.
initialise *gens* $:= [x]$
repeat
  $y := \textsc{RandomElement}(G)$
  $c := x^{-1}y^{-1}xy$ **and** $o := \textsc{Order}(c)$
  if o is even **then**
    append $c^{o/2}$ and $(x^{-1}yxy^{-1})^{o/2}$ to *gens*
  else
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And: If o is odd, then z is uniformly distributed in $C_G(x)$.

Finding $N \lhd G$

We want to find an N with $1 < N \trianglelefteq G$ and $2 \mid |N|$, or conclude that there is none.

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Initialise $H := G$. Then

1. Find a non-central involution $x \in H$. If none found, goto 4.
2. Compute its involution centraliser $C := C_H(x)$.
3. Replace H with C and goto 1.
4. Let D be the group generated by all central involutions we found.
5. For all $1 \neq x \in D$: Test if $\langle x^G \rangle \neq G$.
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What if D is large?

Blind descent (Babai, Beals)

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3. Compute generators $\{y_i\}$ for $Y := \langle y^G \rangle$.
   - If some $c_i := [x, y_i] \neq 1$, then take $z := c_i$ as in 1.
   - Otherwise $x \in C_G(Y)$ but $x \notin Z(G)$, thus $Y \neq G$, we take $z := y$.

Combining Algorithms 1 and 3

Algorithm 4: FINDELMOFEVENNORMALSUBGROUP

Let $G = \langle g_1, \ldots, g_k \rangle \leq \mathrm{GL}(d, q)$.

1. Use Algorithm INVOLUTIONDESCENT to produce candidate elements.
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- One non-trivial group element is returned.
- The algorithm is Monte Carlo and could return a wrong result.

✓

Examples

This approach works well in many important cases:

G	N	time
$A_{20} \wr A_{30}$	$A_5^{\times 30}$	120
$\mathrm{SL}(3,3) \wr A_{10} < \mathrm{GL}(30,3)$	$\mathrm{SL}(3,3)^{\times 10}$	724
$\mathrm{Sp}(6,3) \otimes 2.\mathrm{O}(7,3) < \mathrm{GL}(48,3)$ (computing projectively)	$\mathrm{Sp}(6,3) \otimes 1$ or $1 \otimes 2.\mathrm{O}(7,3)$	645
$6.\mathrm{Suz} < \mathrm{GL}(12,25)$	central 2	227
S_{100}	A_{100}	165
A_{100}	—	148
$\mathrm{PSL}(10,5)$	—	1248
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(here we have averaged over 10 runs, times in ms)

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The success rate was 100% in all cases (using 200 runs).

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Reductions for imprimitive matrix groups

Situation

Let $G \leq \mathrm{GL}_n(\mathbb{F}_q)$ acting linearly on $V := \mathbb{F}_q^{1\times n}$, such that V is irreducible. Assume there is N with $Z(G) < N \lhd G$ such that

$$V|_N = W_1 \oplus W_2 \oplus \cdots \oplus W_k,$$

all W_i are invariant under N, and G permutes the W_i transitively. Then there is a reduction $\varphi : G \to S_k$.

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all W_i are invariant under N, and G permutes the W_i transitively. Then there is a reduction $\varphi : G \to S_k$.

We use Algorithm FindElmOfEvenNormalSubgroup, for the result x, do:

- compute the normal closure $M := \langle x^G \rangle$,
- use the MeatAxe to check whether $V|_M$ is reducible,
- if $x \in N$, we find a reduction.

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- G might not have an even order normal subgroup.

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