COMPUTING AUTOMORPHISMS OF SEMIGROUPS

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ABSTRACT. In this paper an algorithm is presented that can be used to calculate the automorphism group of a finite transformation semigroup. The general algorithm employs a special method to compute the automorphism group of a finite simple semigroup. As applications of the algorithm all the automorphism groups of semigroups of order at most 7 and of the multiplicative semigroups of some group rings are found. We also consider which groups occur as the automorphism groups of semigroups of several distinguished types.

1. INTRODUCTION

There is a tremendous amount of literature relating to automorphism groups of mathematical structures of every hue. An algorithm for computing the automorphism group of a finite group was first given in the 1960s and development of procedures with the same purpose continues to the present day; see [6], [8], and [9]. There are numerous papers concerning the automorphism groups of particular classes of semigroups, for example, Schreier [36] and Mal'cev [27] described all the automorphisms of the semigroup of all mappings from a set to itself. Similar results have been obtained for various other structures such as orders, equivalence relations, graphs, and hypergraphs; see the survey papers [31] and [32]. More examples are provided, among others, by Gluskĭn [13], Araújo and Konieczny [1, 2, 3], Fitzpatrick and Symons [10], Levi [22, 23], Liber [24], Magill [25], Schein [35], Sullivan [39], and Šutov [40]. However, there appears to have been no previous attempt to give an algorithm for computing the automorphisms of an arbitrary finite semigroup. The purpose of this paper is to give such an algorithm.

The most naïve approach to computing the automorphisms of a semigroup S would be to verify, one by one, whether each bijection ϕ from S to S satisfies $(x)\phi(y)\phi = (xy)\phi$ for all $x, y \in S$. To perform this calculation, except for extremely small examples, exceeds human patience. As the examples grow in size, it soon becomes impractical for computers to do the work for us. Our algorithm employs the following general strategy: a search is conducted through a relatively small set of bijections, which are tested to see if they are homomorphisms using the relations of a presentation that defines S.

The main algorithm for computing the automorphism group of a semigroup is given in Section 4. The main algorithm relies on another procedure for calculating the automorphisms of a special type of semigroup: Rees matrix semigroups. This procedure can be found in Section 2. In Section 3 we give an algorithm to compute the inner automorphisms of a transformation semigroup S. In Section 5 we apply the main algorithm to compute the automorphism groups of the semigroups of

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order at most 7. In Section 6 we compute the automorphism groups of the multiplicative semigroups of some group rings. Finally, in Section 7 we consider which groups can occur as the automorphism groups of semigroups belonging to various standard classes.

As part of the computation it is necessary to calculate the automorphisms of certain finite groups, partially ordered sets, and graphs associated with the semigroup. The efficiency of the well-developed algorithms used to perform these calculations is thus incorporated in the presented algorithm. The routines presented here have been implemented as part of the MONOID package [30] in the computational algebra system GAP [12].

In two of the three algorithms presented we make use of backtrack search. As applied here, backtrack search provides an efficient means of computing a subgroup of a group all of whose elements satisfy a certain property. Further details regarding backtrack algorithms in computational group theory can be found in [37, Chapter 9] or [19, Section 4.6].

Throughout we will write mappings on the right and compose them from left to right, and all sets, groups, and semigroups are assumed to be finite. The identity element of a semigroup with identity S will be denoted by 1_S.

2. Automorphisms of Rees matrix semigroups

In this section we describe how to compute the automorphism group of a special type of semigroup called Rees matrix semigroups, which are defined as follows. Let T be a semigroup, let I and J be disjoint index sets and let $P = (p_{j,i})_{j\in J, i\in I}$ be a $|J| \times |I|$ matrix with entries in $T \cup \{0\}$. Then the *Rees matrix semigroup* over T is the set $(I \times T \times J) \cup \{0\}$ with multiplication $(i, g, j)(k, h, l) = (i, gp_{j,k}h, l)$ and $0(i, g, j) = (i, g, j)0 = 0^2 = 0$; denoted by $\mathcal{M}^0[T; I, J; P]$.

An arbitrary finite semigroup can be partitioned into classes that correspond to Rees matrix semigroups with finite index sets over groups; for further details see Section 4. As such Rees matrix semigroups can be thought of as the building blocks of a finite semigroup.

The automorphism group of a semigroup with a zero adjoined is equal to the automorphism group of the original semigroup. Therefore we may assume without loss of generality that all the semigroups considered in this section contain a zero element.

The characterisation of all homomorphisms between two Rees matrix semigroups in the following theorem is taken from [33]; see also [20] and [21].

Theorem 2.1. *Let $M_1 = \mathcal{M}^0[G_1; I_1, J_1; P_1]$ and $M_2 = \mathcal{M}^0[G_2; I_2, J_2; P_2]$ be Rees matrix semigroups where $P_1 = (p^{(1)}_{j,i})_{j\in J_1, i\in I_1}$ and $P_2 = (p^{(2)}_{j,i})_{j\in J_2, i\in I_2}$, let $\lambda_I : I_1 \to I_2$ and $\lambda_J : J_1 \to J_2$ be arbitrary functions, let $\gamma : G_1 \to G_2$ be a homomorphism, and let $f : I_1 \cup J_1 \to G_2$. Then the mapping $(i, g, j) \mapsto (i\lambda_I, (if)(g\gamma)(jf)^{-1}, j\lambda_J)$ is a homomorphism if and only if*

(i) $p^{(1)}_{j,i} = 0$ *if and only if* $p^{(2)}_{j\lambda_J, i\lambda_I} = 0$;
(ii) $p^{(1)}_{j,i}\gamma = (jf)^{-1} \cdot (p^{(2)}_{j\lambda_J, i\lambda_I}) \cdot (if)$, *whenever* $p^{(1)}_{j,i} \neq 0$.

Furthermore, every homomorphism from M_1 to M_2 can be described in this way.

We require a reformulation of Theorem 2.1. Let $M = \mathcal{M}^0[G; I, J; P]$ be a Rees matrix semigroup over a group G, disjoint index sets I and J, and matrix $P = (p_{j,i})_{j \in J, i \in I}$. The automorphism group of M is denoted $\operatorname{Aut} M$.

Let $\Gamma(M)$ be the bipartite graph with vertices $I \cup J$ and edges $(i,j) \in I \times J$ whenever $p_{j,i} \neq 0$. The automorphism group $\operatorname{Aut}\Gamma(M)$ of $\Gamma(M)$ is defined as the group of all bijections $\alpha : \Gamma(M) \to \Gamma(M)$ such that $(i\alpha, j\alpha) \in I \times J$ is an edge in $\Gamma(M)$ if and only if $(i,j) \in I \times J$ is an edge in $\Gamma(M)$. It is obvious that pairs of bijections λ_I and λ_J satisfying Theorem 2.1(i) are equivalent to elements of $\operatorname{Aut}\Gamma(M)$. So, the problem of finding mappings λ_I and λ_J satisfying Theorem 2.1(i) is exchanged for the problem of computing $\operatorname{Aut}\Gamma(M)$. The latter problem has been well studied; the implementation in GAP [30] of the algorithm in this section uses the GRAPE package [38] to compute $\operatorname{Aut}\Gamma(M)$.

Every automorphism of M can be represented as a triple of mappings $\lambda \in \operatorname{Aut}\Gamma(M)$, $\gamma \in \operatorname{Aut} G$, and $f : I \cup J \to G$; a more precise formulation of this is given in the next theorem. Let $G^{I \cup J}$ denote the set of all functions from $I \cup J$ to G and let M^M denote the monoid of all mappings from M to M under composition. The following theorem is an immediate consequence of Theorem 2.1.

Theorem 2.2. *Let $\alpha \in M^M$ and let $\Psi : Aut\,\Gamma(M) \times Aut\,G \times G^{I \cup J} \to M^M$ be defined by*

$$(i, g, j)([\lambda, \gamma, f]\Psi) = (i\lambda, (if)(g\gamma)(jf)^{-1}, j\lambda).$$

Then $\alpha \in Aut\,M$ if and only if $\alpha = [\lambda, \gamma, f]\Psi$ for some $[\lambda, \gamma, f] \in Aut\,\Gamma(M) \times Aut\,G \times G^{I \cup J}$ satisfying

$$p_{j,i}\gamma = (jf)^{-1} \cdot (p_{j\lambda, i\lambda}) \cdot (if) \tag{1}$$

for all $p_{j,i} \neq 0$.

It is straightforward to verify that $\operatorname{Aut}\Gamma(M) \times \operatorname{Aut} G \times G^{I \cup J}$ with multiplication $\diamond$ defined by

$$[\lambda_1, \gamma_1, f_1] \diamond [\lambda_2, \gamma_2, f_2] = [\lambda_1\lambda_2, \gamma_1\gamma_2, \lambda_1 f_2 \star f_1\gamma_2],$$

is a group, where $f \star g : x \mapsto (xf)(xg)$; the identity is $[1_{\operatorname{Aut}\Gamma(M)}, 1_{\operatorname{Aut} G}, x \mapsto 1_G]$ and $[\lambda^{-1}, \gamma^{-1}, x \mapsto (x\lambda^{-1}f\gamma^{-1})^{-1}]$ is the inverse of $[\lambda, \gamma, f]$. We note that it is not practical to compute with $\operatorname{Aut}\Gamma(M) \times \operatorname{Aut} G \times G^{I \cup J}$ directly as $G^{I \cup J}$ is prohibitively large even for relatively small G, I, and J.

Lemma 2.3. *The mapping $\Psi : Aut\,\Gamma(M) \times Aut\,G \times G^{I \cup J} \to M^M$ defined in Theorem 2.2 is a homomorphism of monoids.*

Proof. From the definition of Ψ,

$$[\lambda_1, \gamma_1, f_1][\lambda_2, \gamma_2, f_2]\Psi = [\lambda_1\lambda_2, \gamma_1\gamma_2, \lambda_1 f_2 \star f_1\gamma_2]\Psi$$

is the mapping in M^M given by

$$(i, g, j) \mapsto (i\lambda_1\lambda_2, i\lambda_1 f_2 \cdot if_1\gamma_2 \cdot g\gamma_1\gamma_2 \cdot (jf_1\gamma_2)^{-1} \cdot (j\lambda_1 f_2)^{-1}, j\lambda_1\lambda_2).$$

On the other hand, if $\alpha = [\lambda_1, \gamma_1, f_1]\Psi$ and $\beta = [\lambda_2, \gamma_2, f_2]\Psi$, then

$$\begin{aligned}
(i, g, j)\alpha\beta &= (i\lambda_1, if_1 \cdot g\gamma_1 \cdot (jf_1)^{-1}, j\lambda_1)\beta \\
&= (i\lambda_1\lambda_2, i\lambda_1 f_2 \cdot [if_1 \cdot g\gamma_1 \cdot (jf_1)^{-1}]\gamma_2 \cdot (j\lambda_1 f_2)^{-1}, j\lambda_1\lambda_2) \\
&= (i\lambda_1\lambda_2, i\lambda_1 f_2 \cdot if_1\gamma_2 \cdot g\gamma_1\gamma_2 \cdot (jf_1\gamma_2)^{-1} \cdot (j\lambda_1 f_2)^{-1}, j\lambda_1\lambda_2),
\end{aligned}$$

as required. □

Since $\mathrm{Aut}\, M$ is a subgroup of M^M, it follows from Lemma 2.3 that $(\mathrm{Aut}\, M)\Psi^{-1}$ is a subgroup of $\mathrm{Aut}\,\Gamma(M) \times \mathrm{Aut}\, G \times G^{I\cup J}$.

Lemma 2.4. *Let* $[\lambda, \gamma, f] \in \mathit{Aut}\,\Gamma(M) \times \mathit{Aut}\, G \times G^{I\cup J}$. *Then* $[\lambda, \gamma, f] \in \mathit{ker}(\Psi)$ *if and only if* $\lambda = 1_{\mathit{Aut}\,\Gamma(M)}$ *and there exists* $h \in G$ *such that* $\gamma : g \mapsto hgh^{-1}$ *and* $f : x \mapsto h^{-1}$.

Proof. ($\Rightarrow$) Since $[\lambda, \gamma, f]\Psi = 1_{M^M}$, we have that $(i, g, j) = (i\lambda, if \cdot g\gamma \cdot (jf)^{-1}, j\lambda)$ for all $(i, g, j) \in M$. It follows that $\lambda = 1_{\mathrm{Aut}\,\Gamma(M)}$ and $if \cdot g\gamma \cdot (jf)^{-1} = g$ for all $g \in G$. In particular, if $g = 1_G$, then we deduce that $if = jf$ for all $i \in I$ and $j \in J$. Thus f is constant with value h^{-1}, for some $h \in G$. Finally, rearrange $if \cdot g\gamma \cdot (jf)^{-1} = g$ to obtain $g\gamma = (if)^{-1} \cdot g \cdot jf = hgh^{-1}$.

($\Leftarrow$) Let $(i, g, j) \in M$ be arbitrary. Then

$$(i, g, j)([\lambda, \gamma, f]\Psi) = (i\lambda, if \cdot g\gamma \cdot (jf)^{-1}, j\lambda) = (i, h^{-1} \cdot hgh^{-1} \cdot h, j) = (i, g, j),$$

and so $[\lambda, \gamma, f]\Psi \in \ker(\Psi)$. □

It follows from the previous two lemmas that $\mathrm{Aut}\, M$ is isomorphic to the quotient of the subgroup $(\mathrm{Aut}\, M)\Psi^{-1}$ (consisting of elements in $\mathrm{Aut}\,\Gamma(M) \times \mathrm{Aut}\, G \times G^{I\cup J}$ satisfying (1)) by the normal subgroup of $\mathrm{Aut}\,\Gamma(M) \times \mathrm{Aut}\, G \times G^{I\cup J}$ with elements of the form

$$[1_{\mathrm{Aut}\,\Gamma(M)}, g \mapsto hgh^{-1}, x \mapsto h^{-1}],$$

for some $h \in G$.

Roughly speaking, a preliminary version of the algorithm to compute $\mathrm{Aut}\, M$ is now clear; search through a transversal of cosets of $\ker(\Psi)$ in $\mathrm{Aut}\,\Gamma(M) \times \mathrm{Aut}\, G \times G^{I\cup J}$ and test if every element satisfies (1). The size of the search space in this case is

$$|\mathrm{Aut}\,\Gamma(M)| \cdot |\mathrm{Aut}\, G| \cdot |G|^{|I|+|J|-1}.$$

With a little more thought we can reduce the size of the search space considerably.

We start by considering how to find triples in $\mathrm{Aut}\,\Gamma(M) \times \mathrm{Aut}\, G \times G^{I\cup J}$ that satisfy (1). We give a method of constructing all the functions $f \in G^{I\cup J}$ such that $[\lambda, \gamma, f]$ satisfies (1) for fixed $\lambda \in \mathrm{Aut}\,\Gamma(M)$ and $\gamma \in \mathrm{Aut}\, G$.

Let $K_1, K_2, \ldots, K_t$ be the connected components of $\Gamma(M)$, for every i let T_i be a fixed spanning tree for K_i and let r_i be a fixed vertex in K_i.

If $\lambda \in \mathrm{Aut}\,\Gamma(M)$, $\gamma \in \mathrm{Aut}\, G$, and $g_i \in G$ are arbitrary, then we will define a binary relation $\rho_i = \rho_{K_i}(\lambda, \gamma, g_i) \subseteq K_i \times G$ using a function $\rho' : K_i \to G$ in the three steps below. The idea is to define ρ' to equal g_i on the representative r_i and to propagate this value to the other vertices using the tree T_i. The edges of $K_i \setminus T_i$ are then used to obtain the full relation ρ_i.

Step 1: the definition of ρ' is initiated by letting $r_i\rho' = g_i$ for $1 \leqslant i \leqslant t$.

Step 2: if (x, y) is an edge in T_i with $y\rho'$ defined but $x\rho'$ undefined, then assign

$$x\rho' = \begin{cases} p_{y\lambda,x\lambda}^{-1} \cdot y\rho' \cdot p_{y,x}\gamma & \text{if } x \in I \\ p_{x\lambda,y\lambda} \cdot y\rho' \cdot (p_{x,y}\gamma)^{-1} & \text{if } x \in J. \end{cases}$$

Step 2 is repeated until $x\rho'$ is defined for all vertices x in K_i. Since T_i is a tree, ρ' is a function.

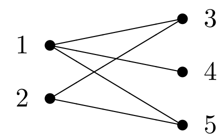


FIGURE 1. The graph $\Gamma(M)$ from Example 2.5.

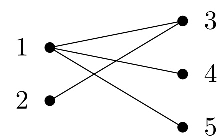


FIGURE 2. The spanning tree T of $\Gamma(M)$ from Example 2.5.

Step 3: if $x \in I$, then define $x\rho_i$ to be the union of $\{x\rho'\}$ and

$$\{\ p^{-1}_{y\lambda,x\lambda} \cdot y\rho' \cdot p_{y,x}\gamma\ :\ (x,y) \in K_i \setminus T_i\ \}.$$

Otherwise, $x \in J$ and $x\rho_i$ is defined to be the union of $\{x\rho'\}$ and

$$\{\ p_{x\lambda,y\lambda} \cdot y\rho' \cdot (p_{x,y}\gamma)^{-1}\ :\ (x,y) \in K_i \setminus T_i\ \}.$$

The following is an example of the above procedure.

Example 2.5. Let M denote the Rees matrix semigroup $\mathcal{M}^0[C_3; \{1,2\}, \{3,4,5\}; P]$ where $C_3 = \{1, x, x^2\}$ is the cyclic group of order 3 and

$$P = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 1 & x \end{pmatrix}.$$

A diagram of the graph $\Gamma(M)$ is shown in Figure 1 and by inspection $\operatorname{Aut}\Gamma(M) = \langle\ (3\,5)\ \rangle$. The automorphism group of C_3 is $\operatorname{Aut} C_3 = \langle\ x \mapsto x^2\ \rangle$. Let $r = 1$ be the fixed vertex in the unique connected component K of $\Gamma(M)$ and let T be the spanning tree for $\Gamma(M)$ with edges $\{(1,3),(1,4),(1,5),(2,3)\}$ as shown in Figure 2.

Now, let $\lambda = (3\,5)$, $\gamma : x \mapsto x$ and $g = 1 \in C_3$. Then from Steps 1 and 2 we obtain

$$\rho' = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & x^2 & 1 & 1 & 1 \end{pmatrix}.$$

From Step 3, $1\rho = \{1\rho'\} = \{1\} = 3\rho = 4\rho$,

$$2\rho = \{2\rho', p^{-1}_{5\lambda,2\lambda} \cdot 5\rho' \cdot p_{5,2}\gamma\} = \{x, x^2\}$$

and

$$5\rho = \{5\rho', p_{5\lambda,2\lambda} \cdot 2\rho' \cdot (p_{5,2}\gamma)^{-1}\} = \{1, x\},$$

and the example is complete.

Throughout the remainder of the paper we will denote the relation

$$\bigcup_{i=1}^{t} \rho_{K_i}(\lambda, \gamma, g_i) \subseteq (I \cup J) \times G$$

by $\rho(\lambda, \gamma, \vec{g})$ where $\vec{g} = (g_1, g_2, \ldots, g_t) \in G^t$ (the direct product of t copies of G).

Lemma 2.6. *Let $[\lambda, \gamma, f] \in \mathit{Aut}\,\Gamma(M) \times \mathit{Aut}\,G \times G^{I\cup J}$. Then $[\lambda, \gamma, f]\Psi \in \mathit{Aut}\,M$ if and only if the relation $\rho(\lambda, \gamma, \vec{g})$ where $\vec{g} = (r_1 f, r_2 f, \ldots, r_t f) \in G^t$ equals f.*

Proof. Throughout the proof we will denote $\rho(\lambda, \gamma, \vec{g})$ by ρ.

($\Rightarrow$) We start by proving that f equals the function ρ', given in the above procedure, by a finite induction on the least length $d(x)$ of a path from any $x \in T_i$ to the fixed vertex $r_i \in K_i$. Starting the induction with $x \in I \cup J$ where $d(x) = 0$, we get $x = r_i$ and so $x\rho' = r_i\rho' = r_i f = xf$.

Assume that $y\rho' = yf$ for all $y \in I \cup J$ such that $d(y) \leqslant m-1$. Then let $x \in J$ where $d(x) = m$. It follows, from the construction of ρ', that $x\rho' = p_{x\lambda, y\lambda} \cdot y\rho' \cdot (p_{x,y}\gamma)^{-1}$ for some $y \in I$ with $d(y) = m-1$. Thus $x\rho' = p_{x\lambda,y\lambda} \cdot yf \cdot (p_{x,y}\gamma)^{-1} = xf$ since f satisfies (1). The proof in the case that $x \in I$ follows analogously.

Now, if $x \in I$, then

$$x\rho = \{x\rho'\} \cup \{\, p_{y\lambda,x\lambda}^{-1} \cdot y\rho' \cdot p_{y,x}\gamma \;:\; (x, y) \in K_i \setminus T_i \,\}$$

But $p_{y\lambda,x\lambda}^{-1} \cdot y\rho' \cdot p_{y,x}\gamma = p_{y\lambda,x\lambda}^{-1} \cdot yf \cdot p_{y,x}\gamma = xf$, by (1), and $x\rho' = xf$. Thus $x\rho = xf$, as required. The proof in the case that $x \in J$ follows analogously.

($\Leftarrow$) By the construction of ρ and the fact that ρ is a function, we have that $i\rho = p_{j\lambda,i\lambda}^{-1} \cdot j\rho \cdot p_{j,i}\gamma$ for all $i \in I$ and $j \in J$ with $p_{j,i} \neq 0$. Hence ρ satisfies (1) and so, by Theorem 2.2,

$$[\lambda, \gamma, \rho]\Psi = [\lambda, \gamma, f]\Psi \in \mathrm{Aut}\, M,$$

as required. □

So, to find the functions in $G^{I\cup J}$ that satisfy (1) it suffices, by Lemma 2.6, to find which of the relations $\rho(\lambda, \gamma, \vec{g})$ are functions. More precisely, let $G_i = \{\, g \in G \;:\; \rho_{K_i}(\lambda, \gamma, g) \text{ is a function} \,\}$. Then $\rho(\lambda, \gamma, \vec{g})$ is a function for some $\vec{g} \in G^t$ if and only if $\vec{g} \in G_1 \times G_2 \times \cdots \times G_t$. In other words, the relation $\rho(\lambda, \gamma, \vec{g})$ can be defined on each connected component K_i independently of the other connected components. Therefore the size of the search space is reduced from $|\mathrm{Aut}\,\Gamma(M)| \cdot |\mathrm{Aut}\, G| \cdot |G|^{|I|+|J|-1}$ to

$$|\mathrm{Aut}\,\Gamma(M)| \cdot |\mathrm{Aut}\, G| \cdot t|G|, \tag{2}$$

where t is the number of connected components of the graph $\Gamma(M)$. Note that if $\Gamma(M)$ is a tree, then $\rho(\lambda, \gamma, \vec{g})$ is a function for all $\vec{g} \in G^t$.

The *centre* of G is denoted $Z(G)$. The following lemma allows us to reduce the size of the search space given in (2) further still.

Lemma 2.7. *Let $[\lambda, \gamma, f] \in \mathit{Aut}\,\Gamma(M) \times \mathit{Aut}\,G \times G^{I\cup J}$ such that $[\lambda, \gamma, f]\Psi \in \mathit{Aut}\,M$ and let $\delta \in \gamma\mathrm{Inn}\,G$. Then there exist $\vec{g} = (g_1, \ldots, g_t) \in G^t$ such that $[\lambda, \gamma, f]\Psi = [\lambda, \delta, \rho(\lambda, \delta, \vec{g})]\Psi$.*

Moreover, if $h_1 \in g_1 Z(G)$, then there exists $h_2, \ldots, h_t \in G$ such that $[\lambda, \gamma, f]\Psi = [\lambda, \delta, \rho(\lambda, \delta, \vec{h})]\Psi$ where $\vec{h} = (h_1, h_2, \ldots, h_t)$.

Proof. We start by proving a related claim. Let $\lambda \in \operatorname{Aut}\Gamma(M)$, $\gamma \in \operatorname{Aut} G$, let $\vec{g} = (g_1, g_2, \ldots, g_t) \in G^t$, and let $k \in G$ such that $[\lambda, \gamma, \rho(\lambda, \gamma, \vec{g}k)]\Psi \in \operatorname{Aut} M$. Then we will prove that

$$[\lambda, \gamma, \rho(\lambda, \gamma, \vec{g}k)]\Psi = [\lambda, \gamma\phi_k, \rho(\lambda, \gamma\phi_k, \vec{g})]\Psi, \tag{3}$$

where $\phi_k : g \mapsto kgk^{-1} \in \operatorname{Inn} G$.

Let $\rho = \rho(\lambda, \gamma, \vec{g}k)$. Then it suffices to prove that $[\lambda, \gamma\phi_k, \rho(\lambda, \gamma\phi_k, \vec{g})]$ and $[\lambda, \gamma, \rho]$ are in the same coset of $\ker(\Psi)$ in $\operatorname{Aut}\Gamma(M) \times \operatorname{Aut} G \times G^{I\cup J}$. Consider the product

$$[\lambda, \gamma, \rho] \diamond [1_{\operatorname{Aut}\Gamma(M)}, \phi_k, c_{k^{-1}} : x \mapsto k^{-1}] = [\lambda, \gamma\phi_k, \lambda c_{k^{-1}} \star \rho\phi_k].$$

By Lemma 2.4, $[1_{\operatorname{Aut}\Gamma(M)}, \phi_k, c_{k^{-1}}] \in \ker(\Psi)$ and so

$$[\lambda, \gamma, \rho]\Psi = [\lambda, \gamma\phi_k, \lambda c_{k^{-1}} \star \rho\phi_k]\Psi \in \operatorname{Aut} M.$$

If $y = \lambda c_{k^{-1}} \star \rho\phi_k : I \cup J \to G$, then $(x)y = x\rho \cdot k^{-1}$. In particular, if $x = r_i$ for some $1 \leqslant i \leqslant t$, then $(x)y = x\rho \cdot k^{-1} = r_i\rho \cdot k^{-1} = g_i$. Hence, by Lemma 2.6, $y = \rho(\lambda, \gamma\phi_k, \vec{g})$, as required.

We will now use (3) to prove the lemma. Let $[\lambda, \gamma, f] \in \operatorname{Aut}\Gamma(M) \times \operatorname{Aut} G \times G^{I\cup J}$ such that $[\lambda, \gamma, f]\Psi \in \operatorname{Aut} M$ and let $\delta \in \gamma\operatorname{Inn} G$ be arbitrary. Then there exists $k \in G$ such that $\gamma = \delta\phi_k$ where $\phi_k : g \mapsto kgk^{-1} \in \operatorname{Inn} G$. By Lemma 2.6, $f = \rho(\lambda, \gamma, \vec{g}k^{-1})$ for some $\vec{g} = (g_1, \ldots, g_t) \in G^t$. Hence by (3) we have that

$$[\lambda, \gamma, f]\Psi = [\lambda, \delta\phi_k, \rho(\lambda, \delta\phi_k, \vec{g}k^{-1})]\Psi = [\lambda, \delta, \rho(\lambda, \delta, \vec{g})]\Psi,$$

and the proof of the first part of the lemma is complete.

Let $h_1 = g_1 z \in g_1 Z(G)$ be arbitrary, let $h_i = g_i g_1^{-1} h_1$ for all $1 < i \leqslant t$, and let $\vec{h} = (h_1, \ldots, h_t)$. Then $h_1^{-1} g_1 \in Z(G)$ and so $\phi_{h_1^{-1} g_1} = 1_{\operatorname{Aut} G}$. Thus again using (3) we obtain

$$\begin{aligned}[\lambda, \delta, \rho(\lambda, \delta, \vec{h})]\Psi &= [\lambda, \delta\phi_{h_1^{-1}g_1}, \rho(\lambda, \delta\phi_{h_1^{-1}g_1}, \vec{h})]\Psi = \\ &\qquad [\lambda, \delta, \rho(\lambda, \delta, \vec{h}h_1^{-1}g_1)]\Psi = [\lambda, \delta, \rho(\lambda, \delta, \vec{g})]\Psi = [\lambda, \gamma, f]\Psi,\end{aligned}$$

as required. $\square$

From Lemma 2.7, the size of the search space becomes

$$|\operatorname{Aut}\Gamma(M)| \cdot |\operatorname{Aut} G / \operatorname{Inn} G| \cdot (|G/Z(G)| + (t-1)|G|), \tag{4}$$

where t is the number of connected components of $\Gamma(M)$ and the number of automorphism is at most

$$|\operatorname{Aut}\Gamma(M)| \cdot |\operatorname{Aut} G / \operatorname{Inn} G| \cdot |G/Z(G)| \cdot |G|^{t-1}. \tag{5}$$

Note that there are some small values where (5) is smaller than (4), as can be seen in Example 2.11.

It is routine to verify that

$$U = \{\, (\lambda, \gamma) \in \operatorname{Aut}\Gamma(M) \times \operatorname{Aut} G \ :\ (\exists f \in G^{I\cup J})\, ([\lambda, \gamma, f]\Psi \in \operatorname{Aut} M) \,\}$$

is a subgroup of $\operatorname{Aut}\Gamma(M) \times \operatorname{Aut} G$. As such, in our algorithm to compute $\operatorname{Aut} M$, we can use backtrack search in $\operatorname{Aut}\Gamma(M) \times \operatorname{Aut} G$ to determine U. The worst case complexity of such a search is $|\operatorname{Aut}\Gamma(M)| \cdot |\operatorname{Aut} G|$ but in many examples we have better complexity. We prune the search tree using the fact that $(\lambda, \gamma) \in U$ if and only if $(\lambda, \delta) \in U$ for all $\delta \in \gamma\operatorname{Inn} G$ (by Lemma 2.7).

If $(\lambda, \gamma) \in U$, then choose $f_{\lambda,\gamma} \in G^{I\cup J}$ such that $[\lambda, \gamma, f_{\lambda,\gamma}]\Psi \in \operatorname{Aut} M$. Let T be a fixed transversal of $\operatorname{Inn} G$ in $\operatorname{Aut} G$, let

$$A = \{ [\lambda, \gamma, f_{\lambda,\gamma}] \ : \ (\lambda, \gamma) \in U \text{ and } \gamma \in T \},$$

and let

$$B = \{ [1_{\operatorname{Aut}\Gamma(M)}, 1_{\operatorname{Aut} G}, f] \in (\operatorname{Aut} M)\Psi^{-1} \ : \ f \in G^{I\cup J} \}.$$

Lemma 2.8. $\langle A\Psi, B\Psi \rangle = \mathit{Aut}\, M$.

Proof. We start by proving that if $[\lambda, \gamma, f], [\lambda, \gamma, g] \in (\operatorname{Aut} M)\Psi^{-1}$ are arbitrary, then there exists $[1_{\operatorname{Aut}\Gamma(M)}, 1_{\operatorname{Aut} G}, h] \in (\operatorname{Aut} M)\Psi^{-1}$ such that

$$[\lambda, \gamma, f] \diamond [1_{\operatorname{Aut}\Gamma(M)}, 1_{\operatorname{Aut} G}, h] = [\lambda, \gamma, g]. \tag{6}$$

If $h = \lambda^{-1}g \star \lambda^{-1}f^{-1}$, where $xf^{-1} = (xf)^{-1}$, then

$$[\lambda, \gamma, f] \diamond [1_{\operatorname{Aut}\Gamma(M)}, 1_{\operatorname{Aut} G}, h] = [\lambda, \gamma, \lambda \circ (\lambda^{-1}g \star \lambda^{-1}f^{-1}) \star f].$$

and $(x)\lambda \circ (\lambda^{-1}g \star \lambda^{-1}f^{-1}) \star f = xg \cdot (xf)^{-1} \cdot xf = xg$. Hence it suffices to prove that $[1_{\operatorname{Aut}\Gamma(M)}, 1_{\operatorname{Aut} G}, h]\Psi \in \operatorname{Aut} M$.

Let $i \in I$ and $j \in J$ such that $p_{j,i} \neq 0$. Then, since $[\lambda, \gamma, f]$ and $[\lambda, \gamma, g]$ satisfy (1), we have that

$$\begin{aligned}
(jh)^{-1} \cdot p_{j,i} \cdot ih &= [(j)\lambda^{-1}g \star \lambda^{-1}f^{-1}]^{-1} \cdot p_{j,i} \cdot (i)\lambda^{-1}g \star \lambda^{-1}f^{-1} \\
&= (j\lambda^{-1}f^{-1})^{-1} \cdot (j\lambda^{-1}g)^{-1} \cdot p_{j,i} \cdot i\lambda^{-1}g \cdot (i)\lambda^{-1}f^{-1} \\
&= (j\lambda^{-1}f^{-1})^{-1} \cdot p_{j\lambda^{-1}, i\lambda^{-1}}\gamma \cdot (i)\lambda^{-1}f^{-1} \\
&= j\lambda^{-1}f \cdot p_{j\lambda^{-1}, i\lambda^{-1}}\gamma \cdot (i\lambda^{-1}f)^{-1} = p_{j,i}.
\end{aligned}$$

Hence $[1_{\operatorname{Aut}\Gamma(M)}, 1_{\operatorname{Aut} G}, h]$ satisfies (1) and so is in $(\operatorname{Aut} M)\Psi^{-1}$.

To conclude, let $[\lambda, \gamma, f] \in \operatorname{Aut}\Gamma(M) \times \operatorname{Aut} G \times G^{I\cup J}$ such that $[\lambda, \gamma, f]\Psi \in \operatorname{Aut} M$ be arbitrary and let $\delta \in T$ such that $\gamma \in \delta \operatorname{Inn} G$. Then, by Lemma 2.7, there exists $g \in G^{I\cup J}$ such that $[\lambda, \gamma, f]\Psi = [\lambda, \delta, g]\Psi$. From (6), there exists $[1_{\operatorname{Aut}\Gamma(M)}, 1_{\operatorname{Aut} G}, h] \in (\operatorname{Aut} M)\Psi^{-1}$ such that

$$[\lambda, \delta, f_{\lambda,\delta}] \diamond [1_{\operatorname{Aut}\Gamma(M)}, 1_{\operatorname{Aut} G}, h] = [\lambda, \delta, g].$$

Thus, since Ψ is a homomorphism, it follows that

$$[\lambda, \delta, f_{\lambda,\delta}]\Psi \cdot [1_{\operatorname{Aut}\Gamma(M)}, 1_{\operatorname{Aut} G}, h]\Psi = [\lambda, \delta, g]\Psi = [\lambda, \gamma, f]\Psi.$$

But $[\lambda, \delta, f_{\lambda,\delta}] \in A$ and $[1_{\operatorname{Aut}\Gamma(M)}, 1_{\operatorname{Aut} G}, h] \in B$ and the proof is complete. □

To improve matters further, it is useful to have an *a priori* known subgroup of the group U. The following lemma provides such a subgroup.

The group $\operatorname{Aut}\Gamma(M)$ acts on the set of $|J| \times |I|$ matrices with entries in $G \cup \{0\}$ by permuting its rows and columns. More precisely, if $\lambda \in \operatorname{Aut}\Gamma(M)$, then define

$$(p_{j,i})^{\lambda}_{j\in J, i\in I} = (p_{j\lambda, i\lambda})_{j\in J, i\in I}.$$

Hence we can consider the pointwise stabilizer $\operatorname{Aut}\Gamma(M)_{(P)}$ of (the point) P under the action of $\operatorname{Aut}\Gamma(M)$. Moreover, as $\operatorname{Aut} G$ acts on G, we can consider the pointwise stabilizer $\operatorname{Aut} G_{(P)}$ of the entries in P in $\operatorname{Aut} G$.

Lemma 2.9. *$\mathit{Aut}\,\Gamma(M)_{(P)} \times \mathit{Aut}\, G_{(P)}$ is a subgroup of U.*

Proof. Let $\lambda \in \operatorname{Aut}\Gamma(M)_{(P)}$, let $\gamma \in \operatorname{Aut} G_{(P)}$, and let $f \in G^{I\cup J}$ be defined by $xf = 1_G$. Then

$$p_{j,i}\gamma = p_{j,i} = 1_G \cdot p_{j,i} \cdot 1_G = (jf)^{-1} \cdot p_{j\lambda,i\lambda} \cdot (if)$$

for all $p_{j,i} \neq 0$. Hence $[\lambda, \gamma, f]\Psi \in \operatorname{Aut} M$ and so $\operatorname{Aut}\Gamma(M)_{(P)} \times \operatorname{Aut} G_{(P)}$ is a subgroup of U. □

The algorithm used to compute the automorphisms of an arbitrary finite Rees matrix semigroup is given in Algorithm 1.

Algorithm 1 - The automorphism group of a Rees matrix semigroup.

$S \leftarrow$ stabilizer chain for $\operatorname{Aut}\Gamma(M) \times \operatorname{Aut} G$
find $\operatorname{Aut}\Gamma(M)_{(P)} \times \operatorname{Aut} G_{(P)}$
backtrack in S to find $U \leqslant \operatorname{Aut}\Gamma(M) \times \operatorname{Aut} G$ and simultaneously the set A
$T \leftarrow$ a transversal of $Z(G)$ in G
$G_1, G_2, \ldots, G_t \leftarrow \emptyset$
for i in $\{1, 2, \ldots, t\}$ **do**
    for $g \in T$ if $i = 1$ or $g \in G$ if $i \neq 1$ **do**
        find $\rho = \rho_{K_i}(1_{\operatorname{Aut}\Gamma(M)}, 1_{\operatorname{Aut} G}, g)$
        if ρ is a function **then**
            $G_i \leftarrow G_i \cup \{\rho\}$
        end if
    end for
end for
$B \leftarrow \{\, [1_{\operatorname{Aut}\Gamma(M)}, 1_{\operatorname{Aut} G}, f] \;:\; f|_{K_i} = \rho \in G_i \text{ for all } i \,\}$
return $\langle\, A\Psi, B\Psi \,\rangle$

2.1. **Examples.** To conclude the section we give several examples.

Example 2.10. Let M denote the Rees matrix semigroup given in Example 2.5, where $G = \{1, x, x^2\}$, $\operatorname{Aut}\Gamma(M) = \langle\, (3\,5) \,\rangle$, $\operatorname{Aut} G = \langle\, x \mapsto x^2 \,\rangle$, $G/Z(G)$ is trivial, and the number of connected components in $\Gamma(M)$ is 1.

The orbit of P under $\operatorname{Aut}\Gamma(M)$ in the set of 3×3 matrices with entries in $G \cup \{0\}$ is

$$\left\{ P = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 1 & x \end{pmatrix}, \begin{pmatrix} 1 & x \\ 1 & 0 \\ 1 & 1 \end{pmatrix} \right\}$$

and so $\operatorname{Aut}\Gamma(M)_{(P)}$ is trivial. Likewise, the orbit of $\operatorname{Aut} G$ on the set of entries $\{1, x\}$ in P is

$$\{\{1, x\}, \{1, x^2\}\}$$

and so $\operatorname{Aut} G_{(P)}$ is trivial.

The only non-identity element of the subgroup U is

$$((3\,5), x \mapsto x^2) = (\lambda, \gamma)$$

and the corresponding function is

$$f_{\lambda,\gamma} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & x^2 & 1 & 1 & 1 \end{pmatrix}.$$

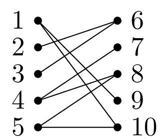
Since $G/Z(G)$ is trivial, B consists of a single element represented by the triple $[1_{\text{Aut}\,\Gamma(M)}, 1_{\text{Aut}\,G}, (1)]$.

It follows that $\text{Aut}\,M = \langle\, [1_{\text{Aut}\,\Gamma(M)}, 1_{\text{Aut}\,G}, (1)], [(3\,5), x \mapsto x^2, (1)]\,\rangle \cong C_2$.

Example 2.11. Let M denote the Rees matrix semigroup $\mathcal{M}^0[C_6; 5, 5; P]$ where $C_6 = \{1, x, \ldots, x^5\}$ is the cyclic group of order 6 and

$$P = \begin{pmatrix} 0 & x^4 & x & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & x^4 & x^5 \\ x^4 & 0 & 0 & 0 & 0 \\ x^4 & 0 & 0 & 0 & x \end{pmatrix}.$$

The graph $\Gamma(M)$ is



and so $\text{Aut}\,\Gamma(M) = \langle\, (1\,4)(7\,9)(8\,10), (2\,3)\,\rangle$. The automorphism group of C_6 is $\text{Aut}\,C_6 = \langle\, x \mapsto x^5\,\rangle$ and $\text{Inn}\,C_6$ is trivial. Since C_6 is abelian, $Z(C_6) = C_6$. Thus there are at most

$$|\text{Aut}\,\Gamma(M)| \cdot |\text{Aut}\,G/\text{Inn}\,G| \cdot |G/Z(G)| \cdot |G| = 4 \cdot 2 \cdot 6 = 48$$

automorphisms of M.

The stabilizer of P under $\text{Aut}\,\Gamma(M)$ in the set of 5×5 matrices with entries in $G \cup \{0\}$ is trivial. Likewise, the stabilizer under $\text{Aut}\,G$ of the entries $\{1, x, x^4, x^5\}$ in P is trivial. The generators of U found during the backtrack search are:

$$(1_{\text{Aut}\,\Gamma(M)}, x \mapsto x^5), ((2,3), 1_{\text{Aut}\,G}), ((1\,4)(7\,9)(8\,10), 1_{\text{Aut}\,G})$$

with the corresponding functions

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 1 & 1 & 1 & x^2 & 1 & x^2 & x^2 & x^4 & x^2 & x^2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 1 & 1 & 1 & 1 & 1 & x^3 & 1 & 1 & 1 & 1 \end{pmatrix},$$
$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 1 & 1 & 1 & x^4 & x^2 & 1 & x^2 & x^4 & x^2 & 1 \end{pmatrix},$$

respectively. The set B consists of 6 elements represented by the triples

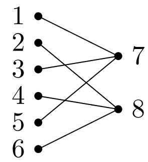
$$[1_{\text{Aut}\,\Gamma(M)}, 1_{\text{Aut}\,G}, (1, x^i)]$$

with $1 \leqslant i \leqslant 6$. The group generated by $A\Psi$ and $B\Psi$ has 48 elements and it can be shown that $\text{Aut}\,M \cong C_2 \times C_2 \times C_2 \times \mathcal{S}_3$.

Example 2.12. Let M denote the Rees matrix semigroup $\mathcal{M}^0[\mathcal{S}_4; 6, 2; P]$ where $\mathcal{S}_4$ is the symmetric group of degree 4, let 1 denote the identity of $\mathcal{S}_4$, let $x = (1\,2\,3)$, and let

$$P = \begin{pmatrix} 1 & 0 & x^{-1} & 0 & x & 0 \\ 0 & 1 & 0 & x^{-1} & 0 & x \end{pmatrix}.$$

The graph $\Gamma(M)$ is



and it can be shown that $\operatorname{Aut}\Gamma(M) = \langle (3\,5), (4\,6), (2\,4), (1\,2)(3\,4)(5\,6)(7\,8)\rangle (\cong (\mathcal{S}_3 \times \mathcal{S}_3) \wr C_2)$. The group $\operatorname{Aut}\mathcal{S}_4$ equals $\operatorname{Inn}\mathcal{S}_4$ and is isomorphic to $\mathcal{S}_4$. The centre $Z(\mathcal{S}_4)$ of $\mathcal{S}_4$ is trivial. Thus there are at most

$$|\operatorname{Aut}\Gamma(M)| \cdot |\operatorname{Aut}\mathcal{S}_4/\operatorname{Inn}\mathcal{S}_4| \cdot |\mathcal{S}_4/Z(\mathcal{S}_4)| \cdot |\mathcal{S}_4| = 72 \cdot 1 \cdot 24^2 = 41472$$

automorphisms of M.

The stabilizer of P under $\operatorname{Aut}\Gamma(M)$ in the set of 2×6 matrices with entries in $\mathcal{S}_4 \cup \{0\}$ is $\langle\, (1\,2)(3\,4)(5\,6)(7\,8)\,\rangle (\cong C_2)$. The stabilizer under $\operatorname{Aut}\mathcal{S}_4$ of the entries $\{1, x, x^{-1}\}$ in P is $\langle\, y \mapsto y^x \,\rangle (\cong C_3)$ where y^x denotes conjugation by $x = (1\,2\,3)$. The generators of U found during the backtrack search are:

$$(1_{\operatorname{Aut}\Gamma(M)}, x \mapsto x^{(1\,2)}), ((4\,6), x \mapsto x^{(1\,4\,2)}), ((3\,5), x \mapsto x^{(1\,4\,2)}),$$
$$((2\,6\,4), x \mapsto x^{(1\,3\,4)}), ((1\,2)(3\,4)(5\,6)(7\,8), 1_{\operatorname{Aut}\mathcal{S}_4})$$

with the corresponding functions arising in every case from the pair $(1_{\mathcal{S}_4}, 1_{\mathcal{S}_4})$. The set B consists of $576 = (4!)^2$ elements represented by the triples

$$[1_{\operatorname{Aut}\Gamma(M)}, 1_{\operatorname{Aut}\mathcal{S}_4}, (f, g)]$$

with $f, g \in \mathcal{S}_4$.

3. Inner automorphisms

Let S be a semigroup of transformations of the n-element set $\{1, 2, \dots, n\}$ and let g be an element of $\mathcal{S}_n$, the symmetric group on $\{1, 2, \dots, n\}$. If the mapping $\phi_g : s \mapsto gsg^{-1}$ is an automorphism of S, then it is called an *inner automorphism*. Note that the notion of an inner automorphism of a semigroup differs from the notion of the same name for groups. The group of all inner automorphisms of S is denoted by $\operatorname{Inn} S$. The purpose of this section is to give an algorithm to compute the inner automorphisms of S.

In what follows $\operatorname{Ims}(S)$ denotes the set of images that elements of S admit. If $f \in S$, then the *kernel* of f is the equivalence relation $\ker(f) = \{\, (x, y) \in S \;:\; xf = yf \,\}$. We let $\operatorname{Kers}(S)$ denote the set of kernels that the elements of S admit. Both $\operatorname{Ims}(S)$ and $\operatorname{Kers}(S)$ can be found by using a simple orbit algorithm without computing the elements of S. As usual, if G is a subgroup of $\mathcal{S}_n$ and N is a subset of $\{1, 2, \dots, n\}$, set of subsets of $\{1, 2, \dots, n\}$, or subset of S, then $G_{\{N\}}$ denotes the setwise stabilizer of N in G.

Algorithm 2 makes use of the following straightforward lemma to compute the inner automorphism group of S.

Lemma 3.1. *Let S be a semigroup of transformations on $\{1, 2, \dots, n\}$, let X be a generating set for S, and let $I = \mathcal{S}_n$. Then* $\operatorname{Inn} S = \{\, \phi_f : s \mapsto f^{-1}sf \mid f \in I_{\{\operatorname{Ims}(S)\}} \cap I_{\{\operatorname{Kers}(S)\}}$ *and* $X\phi_f \subseteq S \,\}$.

Proof. Let $\phi_f \in \operatorname{Inn} S$ where $f \in \mathcal{S}_n$. Then ϕ_f is bijection from S to S and so $\operatorname{Ims}(S\phi_f) = \operatorname{Ims}(S)f = \operatorname{Ims}(S)$, $\operatorname{Kers}(S\phi_f) = \operatorname{Kers}(S)f = \operatorname{Kers}(S)$, and $X\phi_f \subseteq S$. In particular, $f \in I_{\{\operatorname{Ims}(S)\}} \cap I_{\{\operatorname{Kers}(S)\}}$.

For the converse, let $f \in I_{\{\operatorname{Ims}(S)\}} \cap I_{\{\operatorname{Kers}(S)\}}$ such that $X\phi_f \subseteq S$. The latter condition implies that ϕ_f is a homomorphism from S into S. Since $f \in \mathcal{S}_n$, it follows that ϕ_f is injective. Hence $\phi_f \in \operatorname{Inn} S$. □

The sets $\operatorname{Ims}(S)$ and $\operatorname{Kers}(S)$ are a fundamental part of almost every computation involving transformation semigroups, so much so, that if we cannot compute these sets, then we are unlikely to be able to compute anything else of interest. There are sophisticated methods for determining stabilizers of sets efficiently in permutation groups using partition backtrack search. In particular, such methods apply to the computation of $I_{\{\operatorname{Ims}(S)\}} \cap I_{\{\operatorname{Kers}(S)\}}$ when $I = \mathcal{S}_n$. Such methods are implemented in GAP and are utilized in our implementation in [30].

To find the inner automorphisms of a semigroup S on $\{1, 2, \ldots, n\}$ generated by a set X, we perform a backtrack search in $G = I_{\{\operatorname{Ims}(S)\}} \cap I_{\{\operatorname{Kers}(S)\}}$ for elements f such that $X\phi_f \subseteq S$. We can improve the backtrack search in the following three ways. Firstly, we take the elements $x_1, \ldots, x_m$ of X as the base points for our stabilizer chain for G. In this way, we can prune the search tree by never considering elements $f \in G$ such that $f^{-1}x_i f \notin S$. Secondly, if $f, g \in G$ such that $x_i^f = x_i^g$ for all $1 \leqslant i \leqslant m$, then we do not distinguish between f and g. This improves the search as we can ignore any remaining base points after x_m. Thirdly, the setwise stabilizer $G_{\{X\}}$ of the generators of S in G is a subgroup of $\operatorname{Inn} S$ that can be easily computed.

Algorithm 2 - Inner automorphisms of a transformation semigroup $S = \langle\, X \,\rangle$.

```
compute Ims(S) and Kers(S)
I ← S_n
I ← I_{Ims(S)}
if I is not trivial then
    I ← I_{Kers(S)}
    if I is not trivial then
        compute I_{X} ⩽ Inn S
        backtrack in I to find A = { f ∈ I : Xφ_f ⊆ S }
    end if
end if
return { φ_f : f ∈ A }
```

Example 3.2. Let R denote the group ring of the cyclic group C_4 of order 4 over the field with 2 elements. Then using the semigroup theoretic analogue of Cayley's theorem we can find a transformation semigroup S with generating set

$$X = \left\{ x = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\ 1 & 3 & 7 & 9 & 1 & 15 & 5 & 11 & 13 & 11 & 13 & 3 & 5 & 15 & 7 & 9 \end{pmatrix}, \right.$$

$$y = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\ 1 & 16 & 9 & 6 & 5 & 8 & 13 & 12 & 15 & 2 & 3 & 4 & 7 & 10 & 11 & 14 \end{pmatrix},$$

$$z = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\ 1 & 8 & 11 & 2 & 5 & 16 & 13 & 14 & 3 & 6 & 15 & 10 & 7 & 4 & 9 & 12 \end{pmatrix} \Big\}$$

that is isomorphic to the multiplicative semigroup of R.

The setwise stabilizer J of $\mathrm{Ims}(S)$ in $\mathcal{S}_{16}$ has 3870720 elements, and the setwise stabilizer I of $\mathrm{Kers}(S)$ in J has 4096 elements. There are 16 elements in the stabilizer $I_{\{X\}}$ of the generators X in I.

As it turns out, $\mathrm{Inn}\, S = \mathrm{Aut}\, S \cong C_2 \times D_8$; see Section 6 for more details.

The overall aim is to compute $\mathrm{Aut}\, S$ for an arbitrary S. In conjunction with Algorithm 2, the following theorem gives us a method to do this in one special case.

Theorem 3.3. *Let S be a semigroup of mappings on $\{1, 2, \ldots, n\}$ such that for all $s, t \in S$ there exists a constant mapping $k \in S$ such that $ks \neq kt$. Then Aut $S = \mathrm{Inn}\, S$.*

Proof. For a proof see [39, Theorem 1] . □

Corollary 3.4. *If S contains all the constant mappings, then Aut $S = \mathrm{Inn}\, S$.*

The converse of Theorem 3.3 is not true. For example, if S is the semigroup from Example 3.2, then the mapping with constant value 1 is the only constant in S. However, the generators do not satisfy the condition of Theorem 3.3 and $\mathrm{Aut}\, S = \mathrm{Inn}\, S$.

4. The main algorithm

In this section we give the main algorithm for computing the automorphism group $\mathrm{Aut}\, S$ of a finite transformation semigroup S. Throughout the remainder of this section we assume that S is a finite transformation semigroup. Of course, since every finite semigroup is isomorphic to a finite transformation semigroup the algorithm described in this section can be used to compute the automorphism group of an arbitrary finite semigroup.

As mentioned in Section 1 the algorithm consists of searching through a space of candidates and testing if the elements are automorphisms. Our principal focus in this section is to reduce the size of the search space by considering certain structural aspects of S that are preserved by automorphisms. The main aspect we consider is Green's $\mathcal{D}$-relation. Let S^1 denote the semigroup S with a new identity adjoined, that is, an element 1 that acts as an identity on the elements of S. Then *Green's $\mathcal{L}$-relation* is the set of pairs $(x, y) \in S \times S$ such that $S^1x = \{\, sx \,:\, s \in S^1 \,\} = S^1y$; denoted by $x\mathcal{L}y$. *Green's $\mathcal{R}$-relation* is defined dually and denoted by $x\mathcal{R}y$. Although both Green's $\mathcal{L}$- and $\mathcal{R}$-relations are preserved by automorphisms of S (see Lemma 4.1(i)), we are interested in their composition $\mathcal{D} = \mathcal{L} \circ \mathcal{R}$. Like $\mathcal{L}$ and $\mathcal{R}$, *Green's $\mathcal{D}$-relation* is an equivalence relation and as such partitions the set of elements of S into $\mathcal{D}$- classes.

Using the fact that S is finite, it can be shown (see [21, Proposition 2.1.4]) that $x\mathcal{D}y$ if and only if $S^1xS^1 = S^1yS^1$. This alternative formulation leads to a natural partial order on the $\mathcal{D}$-classes of S: $D_1 \leqslant_{\mathcal{D}} D_2$ if $S^1xS^1 \subseteq S^1yS^1$ for some $x \in D_1$ and $y \in D_2$.

So far, S has been partitioned into $\mathcal{D}$-classes and arranged in a partial order $\leqslant_{\mathcal{D}}$. Let us now inspect the individual $\mathcal{D}$-classes more closely. Let D be a $\mathcal{D}$-class of S. Then define D^* such that $D^* = D$ if $st \in D$ for all $s, t \in D$ and $D^* = D \cup \{0\}$

otherwise and define multiplication on D^* by

$$s * t = \begin{cases} st & \text{if } s, t, \text{and } st \in D \\ 0 & \text{if } s, t, \text{or } st \notin D. \end{cases}$$

Then D^* is a semigroup, called the *principal factor* of D. What is more, D^* is either a zero semigroup or a simple semigroup with or without a zero [21, Theorem 3.1.6]. It follows by the Rees-Suschkewitz Theorem [21, Theorem 3.2.3] that a $\mathcal{D}$-class of S can be thought of as a Rees matrix semigroup as described in Section 2. If $st \in D$ for all $s, t \in D$, then the construction in the proof of the Rees-Suschkewitz Theorem yields an isomorphism from D to $\mathcal{M}^0[G; I, J; P] \setminus \{0\}$. However, in this case $\operatorname{Aut} \mathcal{M}^0[G; I, J; P] \cong \operatorname{Aut} \mathcal{M}^0[G; I, J; P] \setminus \{0\}$ and so without loss of generality we can ignore the distinction.

Let D_1 and D_2 be $\mathcal{D}$-classes of S. Then $\phi : D_1 \to D_2$ is an *isomorphism* if it is the restriction to D_1 of an isomorphism between D_1^* and D_2^*; we will denote this by $D_1 \cong D_2$.

The following simple lemma is our main tool for reducing the size of the search space in Algorithm 3.

Lemma 4.1. *Let S be a semigroup, let D_1 and D_2 be $\mathcal{D}$-classes of S, and $\phi \in Aut\, S$. Then the following hold*

(i) *ϕ preserves Green's $\mathcal{D}$-relation ($x\phi \mathcal{D} y\phi$ if and only if $x \mathcal{D} y$);*
(ii) *ϕ preserves the partial order $\leqslant_{\mathcal{D}}$ of $\mathcal{D}$-classes ($D_1\phi \leqslant_{\mathcal{D}} D_2\phi$ if and only if $D_1 \leqslant_{\mathcal{D}} D_2$);*
(iii) *if $D_1\phi \subseteq D_2$, then D_1 and D_2 are isomorphic.*

Let S be an arbitrary finite semigroup generated by a set X with $\mathcal{D}$-classes $D_1, D_2, \ldots, D_t$. Using Lemma 4.1, we can now define the group inside which $\operatorname{Aut} S$ lives. Let $\operatorname{Aut} \mathfrak{P}$ denote the group of automorphisms of the partial order $\mathfrak{P}$ of $\mathcal{D}$- classes of S such that $D\psi \cong D$ for all $\psi \in \operatorname{Aut} \mathfrak{P}$ and all $\mathcal{D}$-classes D, and let $\phi_{i,j} : D_i \to D_j$ be a fixed isomorphism for every pair of isomorphic $\mathcal{D}$-classes D_i and D_j such that $\phi_{i,j} \circ \phi_{j,k} = \phi_{i,k}$ for all i, j, k. Let $\Psi : \operatorname{Aut} \mathfrak{P} \to \operatorname{Aut}(\operatorname{Aut} D_1 \times \cdots \times \operatorname{Aut} D_t)$ be defined by

$$(\psi)\Psi : (\delta_1, \ldots, \delta_t) \mapsto (\phi_{1,1\psi^{-1}} \delta_{1\psi^{-1}} \phi_{1,1\psi^{-1}}^{-1}, \ldots, \phi_{t,t\psi^{-1}} \delta_{t\psi^{-1}} \phi_{t,t\psi^{-1}}^{-1}),$$

where we follow the convention that $D_i\psi = D_{i\psi}$. Then form the semidirect product of $\operatorname{Aut} D_1 \times \cdots \times \operatorname{Aut} D_t$ by $\operatorname{Aut} \mathfrak{P}$ via Ψ; denoted $(\operatorname{Aut} D_1 \times \cdots \times \operatorname{Aut} D_t) \rtimes \operatorname{Aut} \mathfrak{P}$. An element $f = (\delta_1, \delta_2, \ldots, \delta_t, \psi)$ of $(\operatorname{Aut} D_1 \times \cdots \times \operatorname{Aut} D_t) \rtimes \operatorname{Aut} \mathfrak{P}$ acts on S as follows

$$sf = s\delta_i \phi_{i,i\psi} \text{ if } s \in D_i.$$

Theorem 4.2. *$Aut\, S$ is isomorphic to a subgroup of $(Aut\, D_1 \times \cdots \times Aut\, D_t) \rtimes Aut\, \mathfrak{P}$.*

Proof. This is a straightforward corollary of Lemma 4.1. □

Of course, in order to compute $\operatorname{Aut} S$ we only have to consider the images of the generators of S under elements of $(\operatorname{Aut} D_1 \times \cdots \times \operatorname{Aut} D_t) \rtimes \operatorname{Aut} \mathfrak{P}$. Moreover, there may be elements of $\operatorname{Aut} D_i$, besides the identity, that fix the generators $X \cap D_i$ in D_i pointwise. Let $D_1, D_2, \ldots, D_r$ be the $\mathcal{D}$-classes containing generators, and let T_i be a transversal of the cosets of the pointwise stabilizer $(\operatorname{Aut} D_i)_{(X \cap D_i)}$ of $X \cap D_i$

in $\operatorname{Aut} D_i$. Then, we will search through the elements of the set

$$[T_1 \times \cdots \times T_r] \times \operatorname{Aut}\mathfrak{P}.$$

The elements in the search space will be tested to see if they induce automorphisms of S. Since every element in the search space induces a bijection from S to S it suffices to find a presentation defining S and to test if the images of the generators satisfy the relations of this presentation. The Froidure-Pin Algorithm [11] conveniently allows the $\mathcal{D}$-classes of S, the partial order of $\mathcal{D}$- classes of S, and a presentation that defines S to be calculated more or less simultaneously. Thus nothing is lost by requiring that we know a presentation for S.

The automorphism group of the partial order of $\mathcal{D}$-classes can be computed using the method given in [28] implemented in nauty [29] and available through the GAP package GRAPE [38]. Finally, since S is a transformation semigroup it is possible to verify if it is simple using [15, Proposition 2.3]. Algorithm 3 describes how to compute the automorphism group of S.

We remark that the semidirect product in Theorem 4.2 is relatively difficult to represent in the computer and so does not lend itself to backtrack search. In particular, there is no obvious way to prune the search tree in this case. Moreover, it is unlikely that we would reach the point in the algorithm in such cases that backtrack search would help, as in these cases we might be unable to compute the $\mathcal{D}$-classes or a presentation for S. We hope to address these problems in future work.

Examples 4.3, 4.4, 4.5, and 4.6 are examples of the algorithm at work; the unexplained steps can be verified using GAP.

Example 4.3. Let us return to the multiplicative semigroup S of the group ring R defined in Example 3.2. The semigroup S is not simple and S does not satisfy the hypothesis of Theorem 3.3. Using the Froidure-Pin Algorithm we compute the following presentation that defines S

$$\begin{aligned}\langle\, x, y, z \quad | \quad & yx = xy,\ zx = xz,\ zy = yz,\ z^2 = y^2,\ x^2z = x^2y,\ xyz = x,\ x^3y = x^3,\\ & x^2y^2 = x^2,\ xy^3 = xz,\ xy^2z = xy,\ x^5 = x^4,\ y^5 = y,\ y^4z = z\,\rangle.\end{aligned}$$

The $\mathcal{D}$-classes in S are: D_1 containing the generators y and z, D_2 containing the generator x, D_3 and D_4 respectively containing the mappings

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\ 1 & 7 & 5 & 13 & 1 & 7 & 1 & 13 & 5 & 13 & 5 & 7 & 1 & 7 & 5 & 13 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\ 1 & 5 & 1 & 5 & 1 & 5 & 1 & 5 & 1 & 5 & 1 & 5 & 1 & 5 & 1 & 5 \end{pmatrix}$$

and D_5 containing the constant mapping with value 1.

The partial order P of the $\mathcal{D}$-classes is just a chain with $D_1 \geqslant_{\mathcal{D}} D_2 \geqslant_{\mathcal{D}} D_3 \geqslant_{\mathcal{D}} D_4 \geqslant_{\mathcal{D}} D_5$. Hence $\operatorname{Aut}\mathfrak{P}$ is trivial. Now, D_1^* is isomorphic to the group $C_4 \times C_2$ and D_2^* is isomorphic to a zero semigroup with 5 elements. Using Algorithm 1 it can be shown that $\operatorname{Aut} D_1$ is isomorphic to the dihedral group with 8 elements and $\operatorname{Aut} D_2$ is isomorphic to $\mathcal{S}_4$. The pointwise stabilizer of the generators in D_1 and D_2 with respect to $\operatorname{Aut} D_1$ and $\operatorname{Aut} D_2$ contain 1 and 6 elements, respectively. Thus the transversals T_1 and T_2 of cosets of these stabilizers in $\operatorname{Aut} D_1$ and $\operatorname{Aut} D_2$ are of length 8 and 4, respectively. Thus the search space contains $|T_1| \cdot |T_2| \cdot |\operatorname{Aut}\mathfrak{P}| = 8 \cdot 4 \cdot 1 = 32$ elements.

Algorithm 3 - The automorphism group of a finite transformation semigroup $S = \langle\, X \,\rangle$.

if S is simple **then**
  apply Algorithm 1 to S
else
  $A \leftarrow \operatorname{Inn} S$ from Algorithm 2 (automorphisms)
  if S satisfies Theorem 3.3 **then**
    return A
  else
    $R \leftarrow$ relations of presentation defining S
    compute $\mathcal{D}$-classes $D_1, D_2, \ldots, D_r$ containing generators and $\operatorname{Aut}\mathfrak{P}$
    find transversals $T_1, T_2, \ldots, T_r$ of pointwise stabilizers of $X \cap D_i$
    $\Omega \leftarrow [T_1 \times \cdots \times T_r] \times \operatorname{Aut}\mathfrak{P}$
    $i \leftarrow 0$ and $B \leftarrow \{\}$ (non-automorphisms)
    while $2|A| + |B| \leqslant |\Omega|$ and $i \leqslant |\Omega|$ **do**
      $i \leftarrow i + 1$ and $\Omega_i \leftarrow$ the i^{th} element of Ω
      if not Ω_i in A or B **then**
        if $X\Omega_i$ satisfies the relations R **then**
          $A \leftarrow \langle\, A, \Omega_i \,\rangle$
        else
          $B \leftarrow B \cup A\Omega_i A$
        end if
      end if
    end while
  end if
end if
return A

Recall from Example 3.2 that $\operatorname{Inn} S \cong C_2 \times D_8$. Therefore $\operatorname{Aut} S = \operatorname{Inn} S$ if and only if there is a single element in $(T_1 \times T_2 \times \operatorname{Aut}\mathfrak{P}) \setminus \operatorname{Inn} S$ that does not induce an automorphism of S.

As it turns out, such an element exists and so $\operatorname{Aut} S = \operatorname{Inn} S \cong C_2 \times D_8$.

Example 4.4. Let S be the semigroup generated by the following transformations

$$X = \left\{ x = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 4 & 8 & 8 & 8 & 8 & 4 & 8 \end{pmatrix}, y = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 2 & 8 & 2 & 5 & 5 & 8 & 8 \end{pmatrix}, \right.$$

$$\left. z = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 8 & 3 & 7 & 8 & 3 & 7 & 8 \end{pmatrix}, t = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 6 & 6 & 8 & 6 & 8 & 8 & 8 \end{pmatrix} \right\}.$$

Incidently, the semigroup S is Knast's example of a semigroup that lies in the variety **LJ** (locally $\mathcal{J}$-trivial semigroups) but not in the variety B_1 (the variety of semigroups corresponding to the dot-depth one languages) as given in the manual for [34]. The semigroup S has 30 elements.

The set $\operatorname{Ims}(S)$ of images that elements of S admit is

$$\{\{2,5,8\}, \{2,8\}, \{3,7,8\}, \{3,8\}, \{4,8\}, \{5,8\}, \{6,8\}, \{7,8\}, \{8\}\}$$

and the setwise stabilizer $I_{\{\operatorname{Ims}(S)\}}$ of $\operatorname{Ims}(S)$ in $I = \mathcal{S}_8$ is the permutation group generated by $\{(4\,6), (2\,3)(5\,7), (2\,3\,5\,7)\}$ ($I_{\{\operatorname{Ims}(S)\}} \cong C_2 \times D_8$). The set $\operatorname{Kers}(S)$ of kernels that elements of S admit is

$$\{\{\{1,2,3,4,5,6,7,8\}\},\{\{1,2,3,5,6,8\},\{4,7\}\},\{\{1,2,4,5,7,8\},\{3,6\}\},$$
$$\{\{1,2,5,8\},\{3,6\},\{4,7\}\},\{\{1,2,7\},\{3,4,5,6,8\}\},\{\{1,3,5,6,7,8\},\{2,4\}\},$$
$$\{\{1,3,7,8\},\{2,4\},\{5,6\}\},\{\{1,3,7,8\},\{2,4,5,6\}\},\{\{1,4,6,7,8\},\{2,3,5\}\}\}$$

and the setwise stabilizer of $\mathrm{Kers}(S)$ in $I_{\{\mathrm{Ims}(S)\}}$ is trivial. Thus $\mathrm{Inn}\, S$ is trivial.

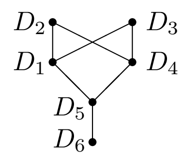
The presentation

$$\begin{aligned}\langle\, x,y,z,t \quad | \quad & xt=x^2, y^2=y, yz=x^2, tx=x^2, t^2=x^2, zy=x^2, z^2=z, x^3=x^2,\\ & x^2y=x^2, x^2z=x^2, xyx=x, xzx=x, xzt=x^2, yx^2=x^2, tyx=x^2,\\ & tyt=t, tzx=x^2, tzt=t, zx^2=x^2 \,\rangle\end{aligned}$$

defines S. The generators x,y,z,t of S lie in the distinct $\mathcal{D}$-classes $D_1,\dots D_4$, respectively, and there are two further $\mathcal{D}$-classes D_5 and D_6 with representatives

$$\begin{pmatrix}1&2&3&4&5&6&7&8\\6&6&8&8&8&8&6&8\end{pmatrix}$$

and the constant function with value 8, respectively. The Hasse diagram of the partial order of $\mathcal{D}$-classes is



and $\mathrm{Aut}\,\mathfrak{P}=\langle(2\,3),(1\,4)\rangle(\cong C_2\times C_2)$. It can be shown that $D_1\cong D_4\cong\mathcal{M}^0[G;3,3;P]$ where $G=\{1\}$ is the trivial group and P is the matrix

$$\begin{pmatrix}1&1&0\\1&0&1\\0&1&1\end{pmatrix}.$$

Furthermore, D_2 and D_3 are both isomorphic to the trivial group. Using Algorithm 1 it can be shown that $\mathrm{Aut}\,D_1\cong\mathrm{Aut}\,D_4\cong\mathcal{S}_3$ and clearly $\mathrm{Aut}\,D_2$ and $\mathrm{Aut}\,D_3$ are trivial. It turns out that the stabilizers of $X\cap D_1$ and $X\cap D_4$ under the action of the respective automorphism groups have size 2. Thus the search space has size

$$(3!/2)^2\cdot|\mathrm{Aut}\,\mathfrak{P}|=3^2\cdot 4=36.$$

As it turns out none of the non-identity elements in the search space are automorphisms and so $\mathrm{Aut}\,S$ is trivial.

Example 4.5. Let S be the semigroup generated by the following set of transformations

$$X=\left\{x=\begin{pmatrix}1&2&3&4&5&6&7&8&9\\2&3&4&5&6&7&8&9&1\end{pmatrix}, y=\begin{pmatrix}1&2&3&4&5&6&7&8&9\\4&2&3&4&5&6&7&8&9\end{pmatrix}\right\}$$

Then S has 40266 elements.

If $I=\mathcal{S}_9$, then $I_{\{\mathrm{Ims}(S)\}}$ has 1296 elements and so does the stabilizer of $\mathrm{Kers}(S)$ in $I_{\{\mathrm{Ims}(S)\}}$. It can be shown using Algorithm 2 that $\mathrm{Inn}\, S\cong(C_9\rtimes C_3)\rtimes C_2$ (the group with identification number $[54,6]$ used in the Small Group library [4] available in GAP and MAGMA [26]).

The number of $\mathcal{D}$-classes in S is 11 with the generators x and y is different $\mathcal{D}$-classes D_x and D_y. Now, $\mathrm{Aut}\,D_x\cong C_6$ and $\mathrm{Aut}\,D_y$ is a group with 93312 elements.

The stabilizer of x in Aut D_x is trivial but the stabilizer of the generator y in Aut D_y has 5184 elements. Thus

$$6 \cdot (93312/5184) \cdot |\mathrm{Aut}\,\mathfrak{P}| = 6 \cdot 18 \cdot 1 = 108.$$

As it turns out, exactly half of the elements in this space are automorphisms and $\mathrm{Aut}\,S = \mathrm{Inn}\,S \cong (C_9 \rtimes C_3) \rtimes C_2$.

Example 4.6. Let S be the semigroup generated by the following set of transformations

$$X = \left\{ x = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 9 & 10 & 11 & 12 & 8 \end{pmatrix}, \right.$$

$$y = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 1 & 1 & 4 & 5 & 6 & 7 & 3 & 8 & 9 & 10 & 11 & 12 \end{pmatrix},$$

$$\left. z = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 1 & 1 & 4 & 3 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \end{pmatrix} \right\}.$$

Then S is a Clifford semigroup, that is, a strong semilattice Y of groups G_y, $y \in Y$, with respect to the homomorphisms $\phi_{x,y} : G_x \to G_y$ (multiplication is defined by $st = (s)\phi_{s,st}(t)\phi_{t,st}$). In particular, the semilattice in this case has 2 elements $a > b$, the groups $G_a \cong C_5$ and $G_b \cong C_5 \times \mathcal{S}_5$ correspond to the $\mathcal{D}$-classes D_x of x and $D_{y,z}$ of y and z, respectively, and the homomorphism $\phi_{a,b} : G_a \to G_b$ is defined by $x \mapsto xz^2$.

If $I = \mathcal{S}_{12}$, then $I_{\{\mathrm{Ims}(S)\}}$ has 39916800 elements and the stabilizer of $\mathrm{Kers}(S)$ in $I_{\{\mathrm{Ims}(S)\}}$ has 3628800 elements. It can be shown using Algorithm 2 that Inn S has 480 elements. Note that without the use of backtrack search in Algorithm 2 computing the inner automorphisms of S was very time consuming.

The automorphism group of the partial order $\mathfrak{P}$ of $\mathcal{D}$-classes of S is trivial. Now, $\mathrm{Aut}\,C_5 \cong C_4$ and $\mathrm{Aut}\,C_5 \times \mathcal{S}_5 \cong C_4 \times \mathcal{S}_5$, the stabilizer of x in Aut D_x is trivial and the stabilizer of y and z in Aut $D_{y,z}$ contains 4 elements. Thus the size of the search space is

$$4 \cdot (480/4) \cdot |\mathrm{Aut}\,\mathfrak{P}| = 4 \cdot 120 \cdot 1 = 480.$$

Hence $\mathrm{Aut}\,S = \mathrm{Inn}\,S$ is a group with 480 elements.

5. Small semigroups

In this section we list the isomorphism types of the groups that occur as automorphism groups of a semigroup, up to isomorphism and anti-isomorphism, with 1 to 7 elements. We also provide the number of semigroups with a given automorphism group.

Note that the numbers of semigroups with 8 and 9 elements are 1843120128 and 52989400714478, respectively, and the number of semigroups with 10 elements is unknown. Consequently it was not possible to compute the automorphism groups of all of the semigroups of any fixed order greater than 7. The semigroups with 1 to 8 elements are available in the smallsemi package [7] for GAP.

The fourth column in the table contains the group identification number used in the Small Group library [4] available in GAP and MAGMA [26].

n	Automorphism groups	Number of semigroups	Group Id.
2	trivial	3	(1,1)
	C_2	1	(2,1)
3	trivial	12	(1,1)
	C_2	5	(2,1)
	$\mathcal{S}_3$	1	(6,1)
4	trivial	78	(1,1)
	C_2	39	(2,1)
	$C_2 \times C_2$	3	(4,2)
	$\mathcal{S}_3$	5	(6,1)
	$\mathcal{S}_4$	1	(24,12)
5	trivial	746	(1,1)
	C_2	342	(2,1)
	C_3	2	(3,1)
	C_4	1	(4,1)
	$\mathcal{S}_3$	33	(6,1)
	D_8	1	(8,3)
	D_{12}	4	(12,4)
	$\mathcal{S}_4$	4	(24,12)
	$\mathcal{S}_5$	1	(120, 34)
6	trivial	10965	(1,1)
	C_2	4121	(2,1)
	$C_2 \times C_2$	441	(4,2)
	$C_2 \times C_2 \times C_2$	6	(8,5)
	$C_2 \times \mathcal{S}_4$	4	(48, 48)
	C_3	26	(3,1)
	C_4	7	(4,1)
	D_{12}	49	(12,4)
	D_8	17	(8,3)
	$\mathcal{S}_3$	300	(6,1)
	$\mathcal{S}_3 \times \mathcal{S}_3$	2	(36, 10)
	$\mathcal{S}_4$	30	(24,12)
	$\mathcal{S}_5$	4	(120, 34)
	$\mathcal{S}_6$	1	(720, 763)
7	trivial	746277	(1,1)
	$(\mathcal{S}_3 \times \mathcal{S}_3) \wr C_2$	1	(72,40)
	C_2	76704	(2,1)
	$C_2 \times C_2$	7314	(4,2)

n	Automorphism groups	Number of semigroups	Group Id.
	$C_2 \times C_2 \times C_2$	172	(8,5)
	$C_2 \times C_2 \times \mathcal{S}_3$	14	(24,14)
	$C_2 \times D_8$	10	(16, 11)
	$C_2 \times \mathcal{S}_4$	45	(48, 48)
	$C_2 \times \mathcal{S}_5$	4	(240, 189)
	C_3	412	(3,1)
	C_4	82	(4,1)
	$C_4 \times C_2$	4	(8,2)
	C_5	6	(5,1)
	C_6	37	(6,2)
	D_{10}	2	(10,1)
	D_{12}	790	(12,4)
	D_8	169	(8,3)
	$\mathcal{S}_3$	3638	(6,1)
	$\mathcal{S}_3 \times \mathcal{S}_3$	24	(36, 10)
	$\mathcal{S}_3 \times \mathcal{S}_4$	4	(144, 183)
	$\mathcal{S}_4$	277	(24,12)
	$\mathcal{S}_5$	30	(120, 34)
	$\mathcal{S}_6$	4	(720, 763)
	$\mathcal{S}_7$	1	-

From the values in the above table, it seems reasonable to conjecture that asymptotically almost all semigroups (up to isomorphism and anti-isomorphism) have trivial automorphism group. However, we do not know a proof of this statement.

6. Group rings

Note that Algorithm 3 can be easily modified to compute the automorphism group of a near-ring, or indeed any algebra with associative binary operations. To illustrate we compute the automorphism groups of the multiplicative semigroup of some group rings.

In the following table, G denotes the group, R the ring and S the multiplicative semigroup of the group ring over G and R. The fourth column in the table contains the group identification number used in the Small Group library [4] available in GAP and MAGMA [26].

G	R	$\operatorname{Aut} S$	Group Id.
C_2	$GF(2)$	trivial	(1,1)
C_3	$GF(2)$	C_2	(2,1)
C_4	$GF(2)$	$C_2 \times D_8$	(16,11)
$C_2 \times C_2$	$GF(2)$	$C_2 \times (((C_2 \times C_2 \times C_2 \times C_2) \rtimes C_3) \rtimes C_2)$	(192,1538)
C_5	$GF(2)$	$C_4 \times C_2$	(8,2)
C_6	$GF(2)$	$\mathcal{S}_3 \times \mathcal{S}_3$	(36, 10)
$\mathcal{S}_3$	$GF(2)$	$\mathcal{S}_3$	(6,1)
C_7	$GF(2)$	$C_3 \times ((C_6 \times C_2) \rtimes C_2)$	(72,30)

7. What groups?

In this section we consider the class of groups that occur as automorphism groups of semigroups. It might be imagined that if this class is restricted, then we could use this fact to our advantage in the procedures described above. Such speculation is irrelevant as the following well-known theorem shows that the class of automorphism groups of semigroups is not in general restricted. Furthermore, our conjecture remains irrelevant even if we restrict our attention to some of the most important special classes of semigroup. It is worth noting that, in contrast to Theorem 7.1, it is known that certain groups do not occur as the automorphism groups of any groups; for example see [16].

Theorem 7.1. *Every finite group is isomorphic to the automorphism group of a finite semigroup of any of the following types: nilpotent, commutative, Clifford, and Rees matrix semigroups.*

Proof. We begin by proving that every finite group is isomorphic to the automorphism group of a finite semigroup with no further conditions. Frucht's Theorem [5, Section 14.7] states that every group G is the automorphism group of some simple graph Γ with vertices V. Let b and r be elements that are not in V. Form a semigroup from the set $S = V \cup \{b, r\}$ by defining the product of adjacent elements of V to equal b and all other products to equal r. The mapping $\phi : S \to S$ is an automorphism of S if and only if $\phi|_V$ is an automorphism of Γ, $b\phi = b$, and $r\phi = r$. Thus $\operatorname{Aut} S \cong \operatorname{Aut} \Gamma$. Note that the semigroup S constructed above is nilpotent, and commutative.

In [14] it was shown that every finite group is isomorphic to the automorphism group of a finite bounded lattice. (Here automorphism means order automorphism.) A lattice can be thought of as a Clifford semigroup over trivial groups. The automorphisms of this semigroup are precisely the order automorphisms of the lattice. It follows that every finite group occurs as the automorphism group of a Clifford semigroup.

To conclude the proof we consider the case of Rees matrix semigroups. We will use the same notation used in Section 2. Let $\lambda \in \operatorname{Aut} \Gamma(M)$ and let $1_{\operatorname{Aut} G}$ be the unique automorphism of the trivial group $G = \{1\}$. Then there is only one possible function $c : I \cup J \to G$, the constant mapping with value 1. The equality $p_{j,i} = (jc)(p_{j\lambda^{-1}, i\lambda^{-1}})1_{\operatorname{Aut} G}(ic)^{-1}$ holds for all $p_{j,i} \neq 0$, since both sides equal 1. Thus if $M = \mathcal{M}^0[G; I, J; P]$ is a Rees matrix semigroup over the trivial group, then, by Theorem 2.2, $\operatorname{Aut} M \cong \operatorname{Aut} \Gamma(M)$.

Now, any bipartite graph can occur as the graph $\Gamma(M)$ of some Rees matrix semigroup M. Thus the class of automorphism groups of Rees matrix semigroups contains the class of automorphism groups of bipartite graphs. In [17] it is shown that every group is the automorphism group of a bipartite graph; also see [18, Section 4.8]. □

Corollary 7.2. *Every finite group is isomorphic to the automorphism group of a finite semigroup of any of the following types: orthodox, regular, completely regular, and inverse.*

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