Some computations regarding Foulkes' conjecture

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Abstract

We describe how certain permutation actions of large symmetric groups can be efficiently implemented on a computer. Using a specially tailored adaptation of a general technique to enumerate huge orbits, and substantial distributed computation on a cluster of workstations, we collect further evidence related to the approach to Foulkes' conjecture suggested in [Black and List, 1989].

1 Foulkes' conjecture

To state Foulkes' conjecture we first introduce some notation. Let $\mathbb{N}$ be the set of positive integers, let $\mathbb{Q}$ be the set of rational numbers, and denote by $M_n := \{1, 2, 3, \ldots, n\}$ for $n \in \mathbb{N}$ the set of natural numbers less than or equal to n. We denote the symmetric group on n points by $S_n := \{\pi : M_n \to M_n \mid \pi \text{ bijective}\}$, with concatenation of maps as product, which we denote as $\pi \circ \varphi$ meaning "apply first φ, then π".
For $m, n \in \mathbb{N}$ let $S_m \wr S_n$ be the wreath product of S_m and S_n, which is a semidirect product of the n-fold direct product $S_m^n := S_m \times \cdots \times S_m$ of copies of S_m and S_n, where the latter acts on the first by permuting the direct factors. Note that S_m^n can be identified with the set of maps $\{f : M_n \to S_m\}$. Hence, $S_m \wr S_n = S_m^n \rtimes S_n$ with product

$$(f, \pi) \cdot (f', \pi') := (f \cdot (f' \circ \pi^{-1}), \pi \circ \pi'),$$

where we multiply maps $f : M_n \to S_m$ pointwise using the product in S_m.
The wreath product $S_m \wr S_n$ has order $|S_m \wr S_n| = (m!)^n \cdot n!$, and embeds into S_{mn} by letting the i-th direct factor of S_m^n, for $i = 1, \ldots, n$, permute the points $\{(i-1)m + 1, \ldots, im\}$ and keep all other points in M_{mn} fixed, while S_n acts on M_{mn} by permuting these n blocks; for more details see [James and Kerber, 1981, Section 4.1]. We denote by $\Omega_{m,n}$ the set $\{(S_m \wr S_n) \circ \pi \mid \pi \in S_{mn}\}$ of right cosets of $S_m \wr S_n$ in S_{mn}, and by $\mathbb{Q}\Omega_{m,n}$ the associated permutation right $\mathbb{Q}S_{mn}$-module.
It is easily seen by an induction argument that for $m \geq n$ we have $|S_n \wr S_m| \leq |S_m \wr S_n|$. Thus we have $|\Omega_{m,n}| \leq |\Omega_{n,m}|$. But in fact much more is conjectured to be true:

1.1 Conjecture ([Foulkes, 1950])
Let $m, n \in \mathbb{N}$ with $m \geq n$. Then the permutation module $\mathbb{Q}\Omega_{m,n}$ is a $\mathbb{Q}S_{mn}$-submodule of the permutation module $\mathbb{Q}\Omega_{n,m}$.

An outline of this note is as follows: In Section 2 we describe how the action of S_{mn} on $\Omega_{m,n}$ can be efficiently implemented on a computer. This implementation will be used for calculations connected to the approach to Foulkes' conjecture suggested in [Black and List, 1989]. Our description uses the notion of Schur bases, which are introduced in Section 3, while in Section 4 the approach of Black and List is discussed. In Section 5 our particular computational techniques are explained, and in the final Section 6 actual computational results are presented. There we also describe, for which values of m and n the conjecture has been verified computationally so far.

2 Implementation of the action of S_{mn} on $\Omega_{m,n}$

For this section let $m, n \in \mathbb{N}$ be fixed. We consider the following set of maps:

$$V_{m,n} := \{v : M_{mn} \to M_n \mid v \text{ takes every value exactly } m \text{ times}\}.$$

One can imagine these maps as tuples of length mn with entries in M_n, each one occuring exactly m times. Hence we will denote such maps as tuples $v = (v_1, v_2, \ldots, v_{mn})$. On the computer they are stored exactly in this way. By way of concatenation of maps, we have two transitive actions on $V_{m,n}$, one on the left and one on the right: The group S_n acts regularly on the left by renaming the entries:

$$S_n \times V_{m,n} \to V_{m,n}, (\pi, v) \mapsto \pi \circ v.$$

The group S_{mn} acts on the right by permuting the entries:

$$V_{m,n} \times S_{mn} \to V_{m,n}, (v, \psi) \mapsto v \circ \psi.$$

These actions commute because of the associativity of concatenation: $(\pi \circ v) \circ \psi = \pi \circ (v \circ \psi)$.
Therefore we obtain an induced action of S_{mn} on the S_n-orbits in $V_{m,n}$. In the sequel we omit the "$\circ$" symbol in the notation of S_n orbits, denote the set $\{S_n v \mid v \in V_{m,n}\}$ of S_n-orbits in $V_{m,n}$ by $S_n \backslash V_{m,n}$, and the action of S_{mn} on it by $(S_n v) \circ \psi := S_n(v \circ \psi)$.
From now on let $x \in V_{m,n}$ be the tuple

$$x := (\underbrace{1, \ldots, 1}_{m \text{ times}}, \underbrace{2, \ldots, 2}_{m \text{ times}}, \ \ldots \ , \underbrace{n, \ldots, n}_{m \text{ times}}),$$

i.e. the map, which maps $k \in M_{mn}$ to $\lceil k/m \rceil$, the smallest integer greater or equal to k/m. Then the stabilizer $\mathrm{Stab}_{S_{mn}}(x)$ of x in S_{mn} is equal to S_m^n, and the stabilizer $\mathrm{Stab}_{S_{mn}}(S_n x)$ of $S_n x \in S_n \backslash V_{m,n}$ in S_{mn} is equal to $S_m \wr S_n$. Thus the action of S_{mn} on $S_n \backslash V_{m,n}$ is equivalent to the action of S_{mn} on $\Omega_{m,n}$. Hence we identify $\Omega_{m,n}$ and $S_n \backslash V_{m,n}$ in the sequel.
Passing from S_{mn}-sets to $\mathbb{Q}S_{mn}$-modules, we can consider $\mathbb{Q}V_{m,n}$ as a $\mathbb{Q}S_n$-$\mathbb{Q}S_{mn}$-bimodule, and thus the permutation $\mathbb{Q}S_{mn}$-module $\mathbb{Q}\Omega_{m,n}$ is identified with the $\mathbb{Q}S_{mn}$-submodule $(\mathbb{Q}V_{m,n})^{S_n}$ whose permutation basis consists of the sums $\overline{S_n v} := \sum_{w \in S_n v} w$ over S_n-orbits $S_n v \subseteq V_{m,n}$. Note that $(\mathbb{Q}V_{m,n})^{S_n}$ is the set of elements in $\mathbb{Q}V_{m,n}$ invariant under the left action of S_n.
We introduce the following definition to distinguish one tuple in each S_n-orbit:

2.1 Definition (S_n-minimal tuples)
*In the above situation we call the lexicographically smallest tuple in each S_n-orbit **S_n-minimal**. For each $v \in V_{m,n}$ we call the S_n-minimal tuple in the orbit $S_n v$ the **S_n-minimalization of v**. We denote by $V_{m,n}^{\min}$ the set of S_n-minimal tuples in $V_{m,n}$.*

It follows readily from the above, that the action of S_{mn} on $\Omega_{m,n}$ can be implemented on a computer by identifying $\Omega_{m,n}$ with $V_{m,n}^{\min}$, and acting with a map $\psi \in S_{mn}$ on $v \in V_{m,n}^{\min}$ by just S_n-minimalizing $v \circ \psi \in V_{m,n}$. Note the runtime needed to compute an S_n-minimalization, and hence the ψ-image of v, is proportional to the length mn of the tuples.
We note the following characterization of S_n-minimality for later reference:

2.2 Proposition (Equivalent characterization of S_n-minimality)
A tuple $v \in V_{m,n}$ is S_n-minimal, if and only if it has the following property: For all i, j with $1 \leq i < j \leq n$ the first occurence of i in v is before the first occurence of j.

Proof: Let v be S_n-minimal. If the above property would not hold, we could rename some i and j and get a lexicographically smaller tuple in the same S_n-orbit, a contradiction.
Let v have the above property, and assume v is not S_n-minimal. Then there is a tuple v' in the same S_n-orbit that is lexicographically smaller than v: Let p be the first position where both tuples differ, and let $v_p = j$ and $v'_p = i$ with $i < j$. Because v and v' are in the same S_n-orbit, p is the first position in v with value j and the first position in v' with value i. By the assumed property, the first occurence of i in v is before p. However, v and v' are equal at positions before p, therefore we have a contradiction. $\square$

3 Schur bases

To describe the approach in [Black and List, 1989], we recall a few facts about permutation modules and homomorphisms between them. For our purposes we give a slightly more general description as can be found e.g. in [Landrock, 1983, Ch.II.12].

For this section let G be a finite group, acting transitively from the right on the sets Ω and Ω'. Let $\omega_1 \in \Omega$ and $\omega'_1 \in \Omega'$ as well as $H := \mathrm{Stab}_G(\omega_1)$ and $H' := \mathrm{Stab}_G(\omega'_1)$ be the corresponding stabilizers. As above, let $\mathbb{Q}\Omega$ and $\mathbb{Q}\Omega'$ denote the associated permutation modules. The space $\mathrm{Hom}_{\mathbb{Q}G}(\mathbb{Q}\Omega, \mathbb{Q}\Omega')$ of $\mathbb{Q}G$-homomorphisms from $\mathbb{Q}\Omega$ to $\mathbb{Q}\Omega'$ has a distinguished basis, which can be described as follows:
We decompose Ω' into H-orbits, by choosing $s_1 = 1_G, s_2, \ldots, s_l \in G$ such that

$$\Omega' = \omega'_1 s_1 H \cup \omega'_1 s_2 H \cup \cdots \cup \omega'_1 s_l H$$

is a disjoint union. Note that hence $\{s_1, s_2, \ldots, s_l\}$ is a set of H'-H-double coset representatives in G.
Using the diagonal action of G on $\Omega' \times \Omega$, and considering the intersection of each G-orbit in $\Omega' \times \Omega$ with $\Omega' \times \{\omega_1\}$, we get the decomposition of $\Omega' \times \Omega$ into G-orbits by

$$\Omega' \times \Omega = (\omega'_1 s_1, \omega_1)G \cup (\omega'_1 s_2, \omega_1)G \cup \cdots \cup (\omega'_1 s_l, \omega_1)G.$$

We describe a homomorphism $\varphi \in \mathrm{Hom}_{\mathbb{Q}G}(\mathbb{Q}\Omega, \mathbb{Q}\Omega')$ by a matrix with respect to the natural bases of $\mathbb{Q}\Omega$ and $\mathbb{Q}\Omega'$, respectively, where the rows are indexed by Ω' and the columns are indexed by Ω. Denoting the (ω', ω)-entry of the matrix of φ by $\varphi_{\omega',\omega}$, we get $\varphi_{\omega',\omega g} = \varphi_{\omega' g^{-1},\omega}$, or equivalently $\varphi_{\omega' g,\omega g} = \varphi_{\omega',\omega}$, for all $\omega \in \Omega$, $\omega' \in \Omega'$ and $g \in G$, because φ is a $\mathbb{Q}G$-module homomorphism. Thus, the matrix of φ is a unique $\mathbb{Q}$-linear combination of the matrices $A^{(1)}, A^{(2)}, \ldots, A^{(l)}$ defined by

$$A^{(i)}_{\omega',\omega} = \begin{cases} 1 & \text{if } (\omega', \omega) \in (\omega'_1 s_i, \omega_1)G, \\ 0 & \text{if } (\omega', \omega) \notin (\omega'_1 s_i, \omega_1)G. \end{cases}$$

We call $\mathcal{A} := (A^{(1)}, A^{(2)}, \ldots, A^{(l)})$ and the associated $\mathbb{Q}G$-module homomorphisms $(\varphi^{(1)}, \varphi^{(2)}, \ldots, \varphi^{(l)})$ the **Schur basis** of $\mathrm{Hom}_{\mathbb{Q}G}(\mathbb{Q}\Omega, \mathbb{Q}\Omega')$, which hence is in bijection with the G-orbits in $\Omega' \times \Omega$. In particular for $\omega = \omega_1 g \in \Omega$, where $g \in G$, and thus $H^g = \mathrm{Stab}_G(\omega_1 g)$, we have:

$$\varphi^{(i)} : \omega = \omega_1 g \mapsto \sum_{\omega' \in \omega'_1 s_i g H^g} \omega'.$$

We now turn to the concatenation of homomorphisms. For a G-set Ω'' let $H'' := \mathrm{Stab}_G(\omega''_1)$ for some $\omega''_1 \in \Omega''$, and as above we choose a set $\{t_1 = 1_G, t_2, \ldots\}$ of H''-H'-double coset representatives in G, and a set $\{u_1 = 1_G, u_2, \ldots\}$ of H''-H-double coset representatives in G. Let $\mathcal{B} := (B^{(1)}, B^{(2)}, \ldots)$ and $\mathcal{C} := (C^{(1)}, C^{(2)}, \ldots)$ denote the Schur bases of $\mathrm{Hom}_{\mathbb{Q}G}(\mathbb{Q}\Omega', \mathbb{Q}\Omega'')$ and $\mathrm{Hom}_{\mathbb{Q}G}(\mathbb{Q}\Omega, \mathbb{Q}\Omega'')$, respectively. We can now write the concatenation $B^{(j)} \circ A^{(i)}$, i.e. the matrix product, in terms of the Schur basis $\mathcal{C}$ of $\mathrm{Hom}_{\mathbb{Q}G}(\mathbb{Q}\Omega, \mathbb{Q}\Omega'')$:

$$\begin{aligned}
(B^{(j)} \circ A^{(i)})_{\omega''_1 u_k, \omega_1} &= \sum_{\omega' \in \Omega'} B^{(j)}_{\omega''_1 u_k, \omega'} \cdot A^{(i)}_{\omega', \omega_1} \\
&= \left| \{ \omega' \in \Omega' \mid (\omega''_1 u_k, \omega') \in (\omega''_1 t_j, \omega'_1)G \text{ and } (\omega', \omega_1) \in (\omega'_1 s_i, \omega_1)G \} \right| \\
&= \left| \{ \omega' \in \omega'_1 s_i H \mid (\omega''_1 u_k, \omega') \in (\omega''_1 t_j, \omega'_1)G \} \right| \\
&= \left| \left\{ \omega' \in \omega'_1 s_i H \mid (\omega''_1, \omega' u_k^{-1}) \in (\omega''_1, \omega'_1 t_j^{-1})G \right\} \right| \\
&= \left| \omega'_1 s_i H u_k^{-1} \cap \omega'_1 t_j^{-1} H'' \right| \\
&= \left| \omega'_1 s_i H \cap \omega'_1 t_j^{-1} H'' u_k \right|.
\end{aligned}$$

4 The approach of Black and List

In [Black and List, 1989], an approach to prove Foulkes' conjecture is described which is based on a certain $\mathbb{Q}S_{mn}$-module homomorphism $\varphi^{(m,n)} : \mathbb{Q}\Omega_{m,n} \to \mathbb{Q}\Omega_{n,m}$. Using the language of the previous section, we first introduce a $\mathbb{Q}S_{mn}$-module homomorphism $\widetilde{\varphi}^{(m,n)} : \mathbb{Q}V_{m,n} \to \mathbb{Q}V_{n,m}$, with a view towards efficient implementation.

For a tuple $v \in V_{m,n}$ let $\tilde{v} := (\tilde{v}_1, \tilde{v}_2, \ldots, \tilde{v}_{mn})$, where $\tilde{v}_k := |\{l \in M_k \mid v_l = v_k\}|$. The tuple $\tilde{v}$ has the following property ($\star$): In those positions where v has the number i, for $i \in M_n$, all the numbers from M_m occur exactly once in $\tilde{v}$; hence we have $\tilde{v} \in V_{n,m}$. Obviously, the set of all such tuples coincides with $\tilde{v} \circ \mathrm{Stab}_{S_{mn}}(v) \subseteq V_{n,m}$, and $\tilde{v}$ is the lexicographically smallest of them. In particular we have

$$\tilde{x} = (\underbrace{\underbrace{1,2,\ldots,m}, \underbrace{1,2,\ldots,m}, \;\ldots\;, \underbrace{1,2,\ldots,m}}_{n \text{ times}}).$$

For $v \in V_{m,n}$ and $i = 1, \ldots, n$ let $1 \leq p_{i,1} < p_{i,2} < \cdots < p_{i,m} \leq mn$ be the positions such that $v_{p_{i,j}} = i$, and let $\psi_v \in S_{mn}$ defined as $\psi_v : p_{i,j} \mapsto (i-1)m + j$, for $i = 1, \ldots, n$ and $j = 1, \ldots, m$. Hence we have $x \circ \psi_v = v$ and $\tilde{x} \circ \psi_v = \tilde{v}$. Thus we conclude that all pairs $(\tilde{v}, v)$, for $v \in V_{m,n}$, belong to one and the same G-orbit in $V_{n,m} \times V_{m,n}$, and hence let $\widetilde{\varphi}^{(m,n)} \in \mathrm{Hom}_{\mathbb{Q}S_{mn}}(\mathbb{Q}V_{m,n}, \mathbb{Q}V_{n,m})$ be the corresponding Schur basis element. As $\mathrm{Stab}_{S_{mn}}(x) = S_m^n$ acts regularly on its orbit $\tilde{x} \circ S_m^n \subseteq V_{n,m}$, for $v \in V_{m,n}$ we have

$$\widetilde{\varphi}^{(m,n)} : v \mapsto \sum_{w \in \tilde{v} \circ \mathrm{Stab}_{S_{mn}}(v)} w = \sum_{\eta \in \mathrm{Stab}_{S_{mn}}(v)} \tilde{v} \circ \eta.$$

Note that, if $\sigma_{m,n} \in S_{mn}$ is defined as $\sigma_{m,n} : (i-1)m+j \mapsto (j-1)n+i$, for $i = 1, \ldots, n$ and $j = 1, \ldots, m$, then $\widetilde{\varphi}^{(m,n)}$ is the Schur basis element corresponding to the S_m^n-S_n^m-double coset $S_m^n \circ \sigma_{m,n} \circ S_n^m$ in S_{mn}.

Next we consider $\mathbb{Q}\Omega_{m,n} = (\mathbb{Q}V_{m,n})^{S_n}$ and $\mathbb{Q}V_{n,m} = (\mathbb{Q}V_{n,m})^{S_m}$. By the description ($\star$) of the elements of $\tilde{v} \circ \mathrm{Stab}_{S_{mn}}(v) \subseteq V_{n,m}$, for $v \in V_{m,n}$, we conclude that $\tilde{v} \circ \mathrm{Stab}_{S_{mn}}(v)$ is a union of S_m-orbits. Hence by restriction we obtain a $\mathbb{Q}S_{mn}$-homomorphism

$$\varphi^{(m,n)} := \frac{1}{n!} \cdot \widetilde{\varphi}^{(m,n)}|_{\mathbb{Q}\Omega_{m,n}} : \mathbb{Q}\Omega_{m,n} \to \mathbb{Q}\Omega_{n,m}.$$

Moreover, as for $v' := \pi \circ v$, for $\pi \in S_n$, we have $\widetilde{v'} = \tilde{v}$ and $\mathrm{Stab}_{S_{mn}}(v) = \mathrm{Stab}_{S_{mn}}(v')$, we conclude that $\widetilde{\varphi}^{(m,n)}(v') = \widetilde{\varphi}^{(m,n)}(v)$. In particular we have $\varphi^{(m,n)}(\overline{S_n x}) = \sum_{\eta \in S_m^n} \tilde{x} \circ \eta$, and hence $\varphi^{(m,n)} \in \mathrm{Hom}_{\mathbb{Q}S_{mn}}(\mathbb{Q}\Omega_{m,n}, \mathbb{Q}\Omega_{n,m})$ is the Schur basis element corresponding to the $(S_m \wr S_n)$-$(S_n \wr S_m)$-double coset $(S_m \wr S_n) \circ \sigma_{m,n} \circ (S_n \wr S_m)$ in S_{mn}.

In other words, if $v \in V_{m,n}$ is an S_n-minimal tuple, then $\varphi^{(m,n)}(\overline{S_n v}) \in \mathbb{Q}\Omega_{n,m}$ is the sum of all $\overline{S_m w}$, for S_m-minimal tuples in $w \in V_{n,m}$ which have the property ($\star$). This is the original description given in [Black and List, 1989], where as the main result the following is proved:

4.1 Proposition ([Black and List, 1989])
Let $m \geq n$. If $\varphi^{(m,n)}$ is injective, then $\varphi^{(m,n-1)}$ is also injective. Thus it would be enough for proving Foulkes' conjecture to show that $\varphi^{(m,m)} \in \mathrm{End}_{\mathbb{Q}S_{m^2}}(\mathbb{Q}\Omega_{m,m})$ is injective for all $m \in \mathbb{N}$. $\square$

It has already been observed in [Black and List, 1989], that $\varphi^{(2,2)}$ and $\varphi^{(3,3)}$ indeed are injective. Moreover, it has been shown in [Jacob, 2004, 4.2] that $\varphi^{(4,4)}$ is injective. In the rest of this note we will concentrate on the question how to decide computationally whether $\varphi^{(5,5)}$ is injective or not. Due to the sheer size of this problem, it can only be tackled using particular techniques, and the answer will be given at the very end.

5 The computational approach

Since $\dim_{\mathbb{Q}}(\mathbb{Q}\Omega_{m,m}) = |\Omega_{m,m}| = \frac{(m^2)!}{(m!)^{m+1}}$, the representing matrices of the elements of $\mathrm{End}_{\mathbb{Q}S_{m^2}}(\mathbb{Q}\Omega_{m,m})$ for their natural action on $\mathbb{Q}\Omega_{m,m}$ are extremely big even for small m; e.g. for $m = 5$ we have $|\Omega_{m,m}| =$

$5\,194\,672\,859\,376 \sim 5 \cdot 10^{12}$. Hence to examine these endomorphisms, it is necessary to work in a much smaller representation of $\mathrm{End}_{\mathbb{Q}S_{m^2}}(\mathbb{Q}\Omega_{m,m})$. As long as the latter is a faithful representation, the minimum polynomials of the elements of $\mathrm{End}_{\mathbb{Q}S_{m^2}}(\mathbb{Q}\Omega_{m,m})$ are retained, and hence injectivity can be decided using the smaller representation. Motivated by the ideas in [Müller, 2003], for our computations we use the left regular representation of $\mathrm{End}_{\mathbb{Q}S_{m^2}}(\mathbb{Q}\Omega_{m,m})$, which drastically reduces the size of the representing matrices: Using the fact that $\dim_{\mathbb{Q}}(\mathrm{End}_{\mathbb{Q}S_{m^2}}(\mathbb{Q}\Omega_{m,m}))$ equals the character theoretic scalar product of the permutation character associated to $\Omega_{m,m}$ with itself, which can be evaluated with little effort using the computer algebra system GAP [GAP, 2002], we e.g. for $m = 5$ find the quite moderate size $\dim_{\mathbb{Q}}(\mathrm{End}_{\mathbb{Q}S_{m^2}}(\mathbb{Q}\Omega_{m,m})) = 1856$.

According to the description of the concatenation of homomorphisms given in Section 3, we can determine the representing matrix of $\varphi^{(m,m)}$ for its left regular action, with respect to the Schur basis of $\mathrm{End}_{\mathbb{Q}S_{m^2}}(\mathbb{Q}\Omega_{m,m})$, by counting. More precisely, we let $\Omega = \Omega' = \Omega'' = \Omega_{m,m}$ and $\omega_1 = \omega_1' = \omega_1'' = \overline{S_m x}$, as well as $G = S_{m^2}$ and $H = H' = H'' = S_m \wr S_m$, and $s_i = t_i = u_i$ and thus $A^{(i)} = B^{(i)} = C^{(i)}$, for $1 \leq i \leq l = \dim_{\mathbb{Q}}(\mathrm{End}_{\mathbb{Q}G}(\mathbb{Q}\Omega))$. Letting $s_2 := \sigma_{m,m} \in S_{m^2}$, we have $\varphi^{(m,m)} = \varphi^{(2)}$ and thus

$$A^{(2)} \circ A^{(i)} = \sum_{k=1}^{l} \left|\omega_1 \circ s_i \circ H \cap \omega_1 \circ s_2^{-1} \circ H \circ s_k\right| \cdot A^{(k)}.$$

Hence we have reduced the problem to study $\varphi^{(m,m)}$ to the following tasks:
- Classify the H-orbits of the G-orbit $\Omega_{m,m}$, and thereby find corresponding representatives $\{s_1, s_2, \ldots, s_l\}$ of the H-H-double cosets in G, where $s_1 = 1_G$ and $s_2 = \sigma_{m,m}$; note that $\sigma_{m,m}$ is an involution.
- Determine $p_{2,i,k} := \left|\overline{S_m x} \circ s_i \circ H \cap \overline{S_m x} \circ s_2^{-1} \circ H \circ s_k\right|$, by running through the H-orbit $\overline{S_m x} \circ s_2^{-1} \circ H = \overline{S_m x} \circ \sigma_{m,m} \circ H = \bigcup_{\eta \in S_m^m} \overline{S_m \tilde{x}} \circ \eta$, applying all representatives s_k respectively, and classifying the resulting elements into the H-orbits. Note that in the computer implementation, this is done with S_m-minimal tuples representing S_m-orbits.
- Decide whether the resulting matrix $M := [p_{2,i,j}]_{i,j=1,2,\ldots,l} \in \mathbb{Z}^{l \times l}$ has full $\mathbb{Q}$-rank.

As the numerical data for the case $m = 5$ given below indicate, the subtask of classifying points into H-orbits is still considerable. Its solution deserves a particular technique, which is a specially tailored adaptation of ideas in [Lübeck and Neunhöffer, 2001] and [Müller, 2003].
Let $U = S_m^m < S_m \wr S_m = H$ be as in Section 1. Thus, every H-orbit of $\Omega_{m,m}$ or $V_{m,m}$ is comprised of U-orbits. The basic idea now is to define U-minimal points in each U-orbit and store only those. To recognize the H-orbit of a point, we first find its U-minimalization and look that one up. To define the concept of U-minimality we first go back to tuples in $V_{m,m}$ again:

5.1 Definition (**U-minimal tuple**)
*In $V_{m,m}$ we call the lexicographically smallest tuple in each U-orbit **U-minimal**. For any $v \in V_{m,m}$ we call the U-minimal tuple in $v \circ U$ the **U-minimalization of v**.*

The following Lemma links the concepts of S_m-minimality and U-minimality in $V_{m,m}$:

5.2 Lemma
If $v \in V_{m,m}$ is an S_m-minimal tuple, then its U-minimalization is again S_m-minimal.

Proof: By Proposition 2.2 the tuple v is S_m-minimal, if and only if for all i, j with $1 \leq i < j \leq m$ the first occurence of i in v is before the first occurence of j in v. As the subgroup U just permutes the entries within the m-blocks, the process of U-minimalization just sorts the entries in each m-block into ascending order.
Let v' be the U-minimalization of v and $1 \leq i < j \leq m$. If the first occurence of i and that of j in v are in the same m-block, then the same will be true after the sorting within the m-blocks and S_m-minimality is not violated. If they are in different m-blocks, the same holds, because their relative order is not changed at all. □

5.3 Definition (**U-minimal S_m-orbits**)
*An S_m-orbit $S_m v \subseteq V_{m,m}$ is called **U-minimal**, if its representing S_m-minimal tuple is a U-minimal tuple.*

As the S_m-orbits in $V_{m,m}$ are identified with $\Omega_{m,m}$, this also defines U-minimal elements of $\Omega_{m,m}$. But note that this does not mean that every U-orbit $S_m v \circ U$ in $V_{m,m}$ contains exactly one U-minimal S_m-orbit; e.g. for $m = 5$, there are 2 298 891 tuples in $V_{5,5}$ which are S_5-minimal and U-minimal at the same time, and therefore represent U-minimal S_5-orbits in $\Omega_{5,5}$, while there are only 190 131 U-orbits in $\Omega_{5,5}$ altogether. But still, the strategy sketched above works:

6 Actual computations

From here on, we concentrate on the case $m = 5$, and let $G = S_{25}$ and $U = S_5^5 < S_5 \wr S_5 = H$. It turns out that there are 623 360 743 125 120 $\sim 6 \cdot 10^{14}$ tuples in $V_{5,5}$ and 5 194 672 859 376 $\sim 5 \cdot 10^{12}$ points in $\Omega_{5,5}$. The H-orbit $\overline{S_5 x} \circ s_2^{-1} \circ H$ has $(5!)^4 = 207\,360\,000 \sim 2 \cdot 10^8$ points. The number of H-orbits in $\Omega_{5,5}$ is equal to $\dim_{\mathbb{Q}}(\mathrm{End}_{\mathbb{Q}G}(\mathbb{Q}\Omega_{5,5})) = 1856$. Thus it is feasible, at least by distributed computing, to run through the H-orbit $\overline{S_5 x} \circ s_2^{-1} \circ H$, and to apply the H-H-double coset representatives $s_1, s_2, \ldots, s_{1856}$, once we have found them. However, as already mentioned above, we have to recognize in which H-orbit a point $\overline{S_5 x} \circ s_2^{-1} \circ h \circ s_k$ lies. Apart from the fact that we can not enumerate $\Omega_{5,5}$ completely, we could not even store an H-orbit number for each such point, as this would need at least $2 \cdot 5\,194\,672\,859\,376 \sim 10^{13}$ Bytes. If we had to store every single tuple of $\Omega_{5,5}$, the situation would be even worse. To circumvent this the notion of U-minimality comes into play:

In a precomputation, we classify all 2 298 891 tuples in $V_{5,5}$ which are S_5-minimal and U-minimal at the same time, into the 1856 H-orbits in $\Omega_{5,5}$, build up a database containing these tuples and the associated H-orbit number, and determine suitable group elements $s_1, s_2, \ldots, s_{1856} \in G$.
A note on the classification of the S_5-minimal and U-minimal tuples into the H-orbits in $\Omega_{5,5}$ might be of interest: We first enumerate all these tuples by a standard backtrack method. Then we start with putting each of these into a class of its own and begin applying generators of H to tuples, followed by S_5-minimalization and U-minimalization. Whenever we observe that two tuples represent S_5-orbits in the same H-orbit in $\Omega_{5,5}$, we merge their classes. We repeat this, until there are only 1856 classes left. This hence is the distribution of S_5-minimal and U-minimal tuples into the H-orbits in $\Omega_{5,5}$, This approach turns out to work quite efficiently, and from this classification we can read off suitable elements $s_1, \ldots, s_{1856} \in S_{25}$.
The precomputation is implemented in the computer algebra system GAP, and takes a few minutes on a modern PC. The resulting database, and the elements $s_1, \ldots, s_{1856}$ are written out.

In the main computation, every time an S_5-orbit $S_5 v$, represented by an S_5-minimal tuple v, occurs we compute the S_5-minimal tuple $v' \in S_5 v$ by S_5-minimalization, then we determine the U-minimalization v'' of v', which also is a S_5-minimal tuple by Lemma 5.2. The tuple v'' is in our database, so we can look up the H-orbit number of $S_5 v''$, and because $S_5 v''$ is in the same U-orbit as $S_5 v$, we have determined the H-orbit number of $S_5 v$ by this method.
The main computation is done in a specially tailored C program. In this part we use distributed computing, because different instances of the program on different machines can apply different elements s_k, each having the precomputed database available. After some 14 hours of computation on about 11 modern PCs, i.e. about 150 hours of CPU time, we get the resulting matrix $M \in \mathbb{Z}^{1856 \times 1856}$, representing $\varphi^{(5,5)}$ in the left regular represention of $\mathrm{End}_{\mathbb{Q}G}(\mathbb{Q}\Omega_{5,5})$.

The source code of the GAP and C programs used can be downloaded from the following web page:

`http://www.math.rwth-aachen.de/~Max.Neunhoeffer/Mathematics/foulkes.html`

Finally, it remains to decide whether M has full $\mathbb{Q}$-rank. Actually, determining the $\mathbb{Q}$-rank or even the kernel of an integer matrix of size 1856×1856 is not a completely trivial task. An approach to find a vector $0 \neq v \in \mathbb{Q}^{1 \times 1856}$ with $v \cdot M = 0$ is by reducing M modulo p, where p is a suitable prime, and finding p-adic approximations of v inductively, until a rational lift is equal to v. This has been described in [Dixon, 1982]; an implementation e.g. is available through the function `RationalSolutionIntMat` in the GAP package EDIM [Lübeck, 2004].

It turns out that the matrix M does not have full $\mathbb{Q}$-rank. Actually, using the GAP package IntegralMeatAxe [Müller, 2004], which also employs p-adic techniques, it is possible to compute the kernel of M, which turns out to have $\mathbb{Q}$-dimension 15.

Therefore $\varphi^{(5,5)}$ is not invertible, and hence the approach in [Black and List, 1989] in general does not work. Note that this does not imply a counterexample to Foulkes' conjecture. Actually Foulkes' conjecture has already been verified in [Foulkes, 1950] for all cases $n < m = 5$. In addition, we have used the SYMMETRICA program (see [Kerber and Kohnert, 1992]) to verify the conjecture for all cases with $m \leq 14$ and $n \leq 4$ and for all cases with $m \leq 12$ and $n + m \leq 17$ as well. For bigger cases, some multiplicities of simple modules in the permutation modules are greater than 2^{31}, such that integer overflows occur on our 32 bit machines.

Addendum: In the meantime we have learned that this by [Briand, 2004, Prop.3.9] is a counterexample to Howe's conjecture [Howe, 1987], which is a strengthening of Foulkes' conjecture, and moreover also is a counterexample to Stanley's conjecture [Stanley, 2000, p.304], which is a generalisation of Foulkes' conjecture. We would like to thank Malek Abdesselam for pointing us in that direction.

References

[Black and List, 1989] Black, S. C. and List, R. J. (1989). A note on plethysm. *European J. Combin.*, 10(1):111–112.

[Briand, 2004] Briand, E. (2004). *Polynômes multisymétriques*. PhD thesis, Université de Rennes 1 and Universidad de Cantabria. `http://emmanuel.jean.briand.free.fr/publications/`.

[Dixon, 1982] Dixon, J. D. (1982). Exact solution of linear equations using p-adic expansions. *Numer. Math.*, 40(1):137–141.

[Foulkes, 1950] Foulkes, H. O. (1950). Concomitants of the quintic and sextic up to degree four in the coefficients of the ground form. *J. London Math. Soc.*, 25:205–209.

[GAP, 2002] GAP (2002). *GAP – Groups, Algorithms, and Programming, Version 4.3*. The GAP Group. `http://www.gap-system.org`.

[Howe, 1987] Howe, R. (1987). $(\mathrm{GL}_n, \mathrm{GL}_m)$-duality and symmetric plethysm. *Proc. Indian Acad. Sci. Math. Sci.*, 97(1-3):85–109 (1988).

[Jacob, 2004] Jacob, J. (2004). *Representation theory of association schemes*. PhD thesis, RWTH Aachen.

[James and Kerber, 1981] James, G. and Kerber, A. (1981). *The representation theory of the symmetric group*, volume 16 of *Encyclopedia of Mathematics and its Applications*. Addison-Wesley Publishing Co., Reading, Mass. With a foreword by P. M. Cohn, With an introduction by Gilbert de B. Robinson.

[Kerber and Kohnert, 1992] Kerber, A. and Kohnert, A. (1992). *Symmetrica 1.0*. `http://www.mathe2.uni-bayreuth.de/axel/symneu_engl.html`

[Landrock, 1983] Landrock, P. (1983). *Finite group algebras and their modules*, volume 84 of *London Mathematical Society Lecture Note Series*. Cambridge Univ. Press.

[Lübeck, 2004] Lübeck, F. (2004). *GAP package EDIM*. `http://www.math.rwth-aachen.de:8001/~Frank.Luebeck/EDIM`.

[Lübeck and Neunhöffer, 2001] Lübeck, F. and Neunhöffer, M. (2001). Enumerating large orbits and direct condensation. *Experiment. Math.*, 10(2):197–205.

[Müller, 2003] Müller, J. (2003). *On endomorphism rings and character tables*. Habilitationsschrift. RWTH Aachen.

[Müller, 2004] Müller, J. (2004). *GAP package IntegralMeatAxe*. Unpublished.

[Stanley, 2000] Stanley, R. P. (2000). Positivity problems and conjectures in algebraic combinatorics. In *Mathematics: frontiers and perspectives*, pages 295–319. Amer. Math. Soc., Providence, RI.

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