Generalisations of Small Cancellation Theory

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Status of the Project

Joint work with:

Steven Linton

Richard Parker

Colva Roney-Dougal

Richard already worked for several years on this.

The whole project is still in its early stages.

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Decide whether or not a word $w \in F(X)$ lies in N, that is, whether or not it represents the identity in G.

Of course:

$$w \in N \quad \Longleftrightarrow \quad w = \prod_{i=1}^{k} u_i r_i u_i^{-1} \quad \text{in } F(X)$$

for some $k \in \mathbb{N} \cup \{0\}$, $r_i \in R \cup R^{-1}$ and $u_i \in F(X)$.

Want: An algorithm to solve this Word Problem in G.

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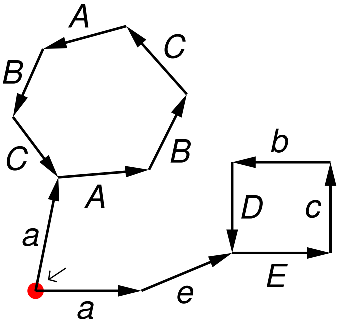

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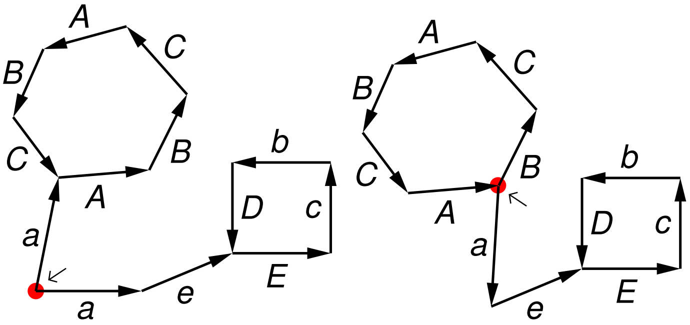

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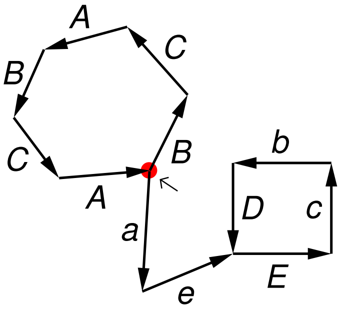

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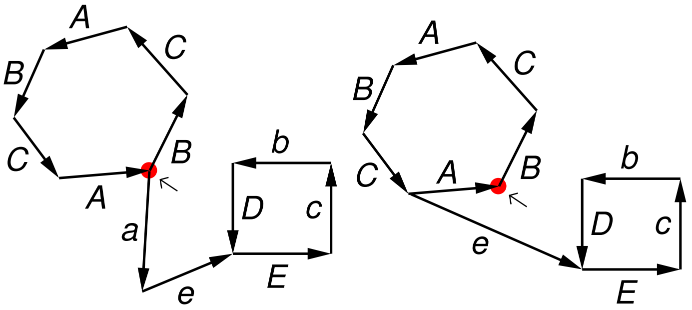

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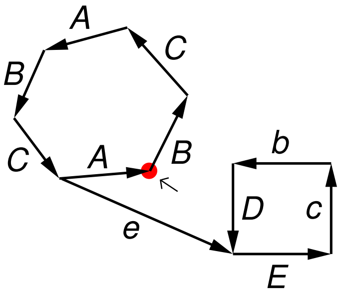

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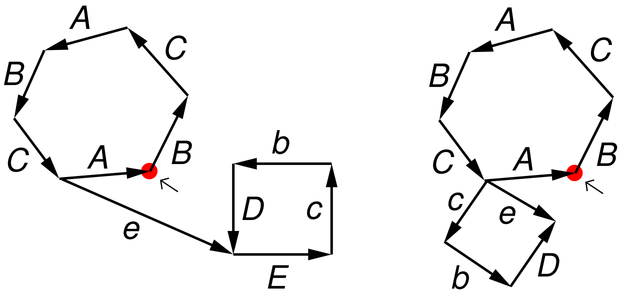

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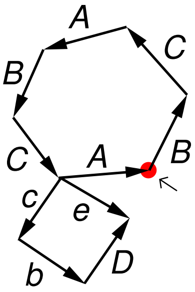

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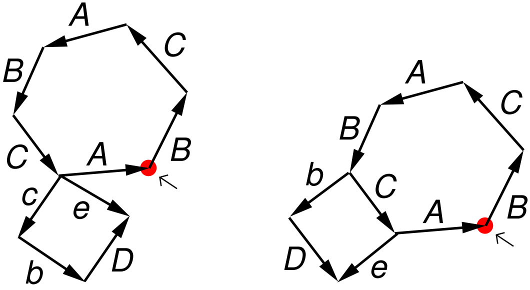

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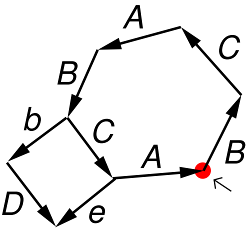

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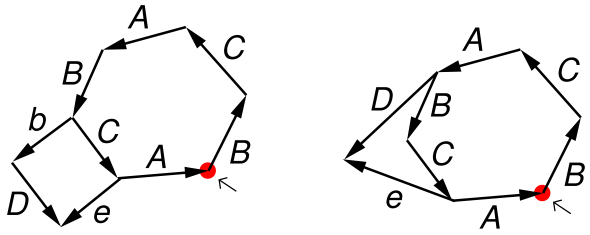

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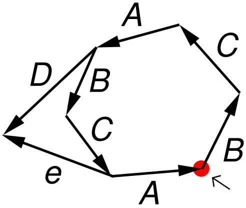

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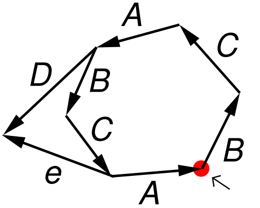

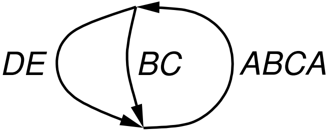

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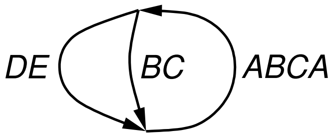

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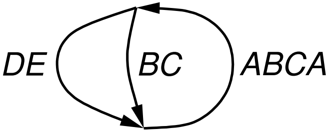

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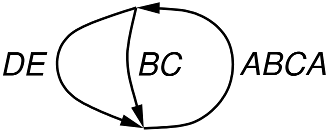

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It is a sphere if the boundary word is a relator.

Van Kampen Diagrams II

Theorem (Van Kampen, 1933)

Let $G = \langle X \mid R \rangle$ be a finitely presented group. Then a word $w \in F(X)$ is contained in the normal closure $N := \langle\langle R \rangle\rangle$ of R in $F(X)$ if and only if there is a van Kampen diagram with w as boundary word.

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Definition (Isoperimetric (or Dehn) Function)

A function $f : \mathbb{N} \to \mathbb{N}$ is called an isoperimetric function or Dehn function for $G = \langle X \mid R \rangle$ if the following holds:

for every reduced word w of length at most n representing 1 in G there is a van Kampen diagram proving this with at most $f(n)$ faces.

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Theorem (Gersten)

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Dehn's algorithm to solve the word problem

Whenever a cyclic word w contains more than half of a relator, replace this part by the inverse of the other, smaller part.

Conditions

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A word $1 \neq b \in F(X)$ is called a piece if there are relators $r_1, r_2 \in \overline{R}$ with $r_1 = bc_1$ and $r_2 = bc_2$ for $c_1, c_2 \in F(X)$ with $c_1 \neq c_2$.

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Condition $C'(\lambda)$ for $0 < \lambda < 1$

If $bc \in \overline{R}$ with a piece b, then $\ell(b) < \lambda \ell(bc)$.

Small Cancellation Theorems

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If a presentation $G = \langle X \mid R \rangle$ fulfills

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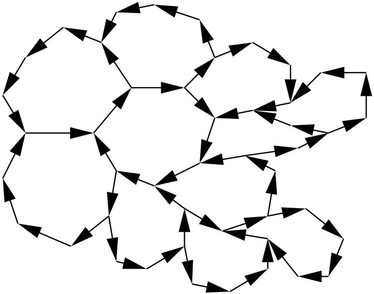

Algorithmic approach

Small Cancellation Theory is very much static.

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Idea 1: Analyse presentation dynamically

We want to write programs that analyse presentations and — if they succeed — come up with new, possibly more complicated local conditions which show that some word problem solving algorithm works.

This is a very dynamic and flexible approach.

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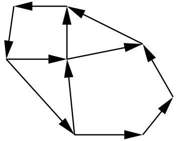

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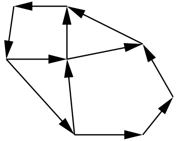

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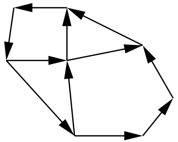

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Idea 2: Forbidden regions

We simply forbid certain subdiagrams (once we proved that they are not needed!) and check the local conditions only on the vK diagrams without forbidden regions.

Curvature redistribution

Idea 3:

To make a sphere, the surface must be curved.

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Combinatorial Curvature

If we endow every vertex with $+1$, every edge with -1 and every face with $+1$ units of "curvature", the total sum in a diagram is always equal to 1.
This way, curvature is a local phenomenon, about which we have global information.

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Idea 4: Move curvature around — Officers

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- In the end, the sum is still 1.
- If we can show that after the move, the curvature in the interior is always non-positive then
  - there are no spheres, and
  - in all vK diagrams, the curvature is at the boundary.

2-dimensional analysis by example

This analyses the interior of possible diagrams to conclude that an officer leaves no curvature in the interior.

Example: LE officer with $T(4)$ and $C'(1/4)$

Assume $T(4)$ and $C'(1/4)$ and run officer LE.

Curvature distribution (interior):

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after L	1	$< -1/2$ $(*)$	0
after E	< 0 $(**)$	0	0

$(*)$ since $-1/2 = -1 + 2 \cdot \frac{1}{4}$ using $C'(1/4)$.

$(**)$ since $0 = 1 + 4 \cdot \frac{-1/2}{2}$ using $T(4)$.

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Show that your favourite word problem solver can deal with all situations arising.

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- Leave out generators shown to be not needed to simplify relators and diagrams.
- Add generators and relations to change the geometry of diagrams (e.g. triangulation).

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We might for example:

- Add implied relators to finish spheres and thereby create new forbidden regions.
- Leave out generators shown to be not needed to simplify relators and diagrams.
- Add generators and relations to change the geometry of diagrams (e.g. triangulation).
- Do all of these to make new local small cancellation conditions applicable.

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# gens	# random rels	time [seconds]
1 000	1 140	0.03
10 000	20 010	0.77
100 000	366 500	33
500 000	2 830 000	584
1 000 000	6 850 000	2145