Kazhdan-Lusztig basis and Wedderburn decomposition

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Setup

Let W, S be a finite Coxeter group, $A := \mathbb{Z}[v, v^{-1}]$
$L : W \to \mathbb{Z}$ a weight function with $L(s) > 0$ for $s \in S$
$[L(ww') = L(w) + L(w')$ if $\ell(ww') = \ell(w) + \ell(w')]$

$\mathcal{H}$ the Iwahori-Hecke algebra of W over A with parameters $v^{L(s)}$

$K \supseteq \mathbb{Q}(v)$ splitting field for the extension of scalars $\mathcal{H}_K$

Kazhdan-Lusztig:

- $\mathcal{H}$ has A-bases $(C_w)_{w \in W}$ and $(D_{z^{-1}})_{z \in W}$
- Left cells Λ_d: $\mathcal{D} \subseteq W$ and $W = \dot{\bigcup}_{d \in \mathcal{D}} \Lambda_d$ with $\Lambda_d \cap \mathcal{D} = \{d\}$
- Left cell modules: $\mathrm{LC}^{(\Lambda_d)}$ as quotient of left ideals

Lusztig:

- The asymptotic algebra $\mathcal{J}$ with A-basis $(t_z)_{z \in W}$
- A homomorphism of A-algebras $\phi : \mathcal{H} \to \mathcal{J}$
- Is isomorphism $\phi : \mathcal{H}_K \to \mathcal{J}_K$ after extension of scalars

Main results I

Proposition 1*:

$$\mathcal{B}^\vee := (C_d D_{z^{-1}})_{d\in\mathcal{D}, z\in\Lambda_d} \quad \text{is a } K\text{-basis of } \mathcal{H}_K.$$

Proposition 2*: The right regular $\mathcal{H}_K$-module decomposes as a direct sum of right ideals $\mathcal{R}_d$:

$$\mathcal{H}_K = \bigoplus_{d\in\mathcal{D}} \mathcal{R}_d \quad \text{with} \quad \mathcal{R}_d := \left\langle C_d D_{z^{-1}} \mid z \in \Lambda_d \right\rangle_K \cong \left(\mathrm{LC}_K^{(\Lambda_d)}\right)^*$$

Proposition 3*: If $\mathrm{LC}_K^{(\Lambda_d)}$ is a simple $\mathcal{H}_K$-module, then

$$\phi\left(c(d)\cdot (C_z D_{d^{-1}})^\dagger\right) = \pm t_z \quad \text{for all } z \in \Lambda_d$$

with some constant $c(d) \in K$ and an algebra involution $h \mapsto h^\dagger$.

Main results II

Proposition 4: Let Λ_d be a left cell and $\mathrm{LC}_K^{(\Lambda_d)}$ be simple. Then

$$\mathcal{B}_d := \big(c(d) C_x D_{y^{-1}}\big)_{x,y \in \Lambda_d}$$

is K-linear independent and spans the isotypic component of $\mathcal{H}_K$ corresponding to the simple module $\mathrm{LC}_K^{(\Lambda_d)}$. Further:

$$c(d) C_x D_{y^{-1}} \cdot c(d) C_u D_{w^{-1}} = \delta_{y,u} \cdot c(d) C_x D_{w^{-1}}$$

for all $x, y, u, w \in \Lambda_d$.

Proposition 5: If all $\mathrm{LC}_K^{(\Lambda_d)}$ are simple and $\mathrm{LC}_K^{(\Lambda_{d_1})}, \ldots, \mathrm{LC}_K^{(\Lambda_{d_n})}$ are representatives of the isomorphism types of simple $\mathcal{H}_K$-modules, then

$$(c(d_i) C_x D_{y^{-1}})_{1 \le i \le n; x,y \in \Lambda_i}$$

is a Wedderburn basis of $\mathcal{H}_K$.

$\mathcal{H}$ is a symmetric algebra

$\mathcal{H}$ has a symmetrizing trace map $\tau : \mathcal{H} \to A$ with:

- τ is A-linear
- $\tau(h \cdot h') = \tau(h' \cdot h)$ for all $h, h' \in \mathcal{H}$
- $(C_w)_{w \in W}$ is dual to $(D_{y^{-1}})_{y \in W}$, that is: $\tau(C_w \cdot D_{y^{-1}}) = \delta_{w,y}$
- τ is non-degenerate: every A-basis of $\mathcal{H}$ has a dual basis

If $\mathrm{Irr}(\mathcal{H}_K)$ is the set of irreducible characters of $\mathcal{H}_K$, then

$$\tau(h) = \sum_{\chi \in \mathrm{Irr}(\mathcal{H}_K)} \frac{\chi(h)}{c_\chi} \qquad \text{for } h \in \mathcal{H}_K$$

where $0 \neq c_\chi \in K$ are the Schur elements.

Left cell modules revisited

Let Λ_d be a left cell. Then $\mathrm{LC}^{(\Lambda_d)}$ has a basis indexed by Λ_d.

The representing matrix $M(h)$ of the action of h on $\mathrm{LC}^{(\Lambda_d)}$ is:

$$M_{y,x}(h) = \tau\left(D_{y^{-1}} \cdot h \cdot C_x\right) \qquad \text{for } x, y \in \Lambda_d.$$

$(C_w)_{w \in W}$ is dual to $(D_{y^{-1}})_{y \in W}$ with respect to τ, thus we have:

$$h = \sum_{z \in W} \alpha_z C_z = \sum_{y \in W} \tau(D_{y^{-1}} \cdot h) C_y \qquad \text{for all } h \in \mathcal{H}.$$

Proof: To find α_y, multiply by $D_{y^{-1}}$ and apply τ.

Conjectures P1 to P15

Propositions 1^* to 3^* depend on Conjectures P1 to P15 in:

G. Lusztig, "*Hecke Algebras with Unequal Parameters*", Vol. 18 of CRM Monograph Series, AMS, Providence, RI, 2003.

However, these are proved at least in the "single parameter case" for finite Weyl groups, that is, W is finite and $L = \ell$.

We show Propositions 1^* to 3^* under the assumption of P1 to P15.

Proof of Proposition 4

Let Λ_d be a left cell and $\mathrm{LC}_K^{(\Lambda_d)}$ be simple with character χ.

Then a formula for Wedderburn basis elements $E_{x,y}$ yields:

$$\begin{aligned} E_{x,y} &= \frac{1}{c_\chi} \sum_{w \in W} M_{y,x}(C_w) \cdot D_{w^{-1}} \\ &= \frac{1}{c_\chi} \sum_{w \in W} \tau(D_{y^{-1}} \cdot C_w \cdot C_x) \cdot D_{w^{-1}} \\ &= \frac{1}{c_\chi} \sum_{w \in W} \tau(C_w \cdot C_x D_{y^{-1}}) \cdot D_{w^{-1}} = c_\chi^{-1} \cdot C_x D_{y^{-1}} \end{aligned}$$

Prop. 4 follows from $E_{x,y} \cdot E_{u,w} = \delta_{y,u} \cdot E_{x,w}$ for $x, y, u, w \in \Lambda_d$.

We have $c(d) = c_\chi^{-1}$.

Preimages abstractly ...

From here on, we assume P1 to P15.

Known: The formula for the Lusztig homomorphism is:

$$\phi(h^\dagger) = \sum_{\substack{d \in \mathcal{D} \\ z \in \Lambda_d}} \tau(h \cdot C_d D_{z^{-1}}) \cdot \hat{n}_z \cdot t_z \qquad \text{for all } h \in \mathcal{H},$$

where $\hat{n}_z \in \{1, -1\}$ are known signs.

Because $\phi : \mathcal{H}_K \to \mathcal{J}_K$ is an isomorphism, we have:

For every $d \in \mathcal{D}$ and $z \in \Lambda_d$ there is an $h_z \in \mathcal{H}_K$ with

$$\tau(h_z \cdot C_{d'} D_{z'^{-1}}) = \delta_{z,z'} \qquad \text{for all } d' \in \mathcal{D} \text{ and } z' \in \Lambda_d.$$

Thus, there is a basis of preimages $(h_z)_{z \in W}$ of the basis $(\hat{n}_z t_z)_{z \in W}$ and $(C_d D_{z^{-1}})_{d \in \mathcal{D}, z \in \Lambda_d}$ is a K-basis of $\mathcal{H}_K$.

... and concretely

Let Λ_d be a left cell and $\mathrm{LC}^{(\Lambda_d)}$ be simple with character χ.

For fixed $d \in \mathcal{D}$ and $z \in \Lambda_d$, we are looking for the unique element $h_z \in \mathcal{H}_K$ with

$$\tau(h_z \cdot C_{d'} D_{z'^{-1}}) = \delta_{z,z'} \qquad \text{for all } d' \in \mathcal{D} \text{ and } z' \in \Lambda_d.$$

We want to show, that $h_z = c_\chi^{-1} \cdot C_z D_{d^{-1}}$, so we want:

$$\tau(c_\chi^{-1} \cdot C_z D_{d^{-1}} \cdot C_{d'} D_{z'^{-1}}) = \delta_{z,z'}.$$

Method of proof:

- First use P1 to P15 to show that $\tau(C_z D_{d^{-1}} \cdot C_{d'} D_{z'^{-1}}) = 0$ unless $d = d'$. In that case we have $z, d, z' \in \Lambda_d$.
- Then use Proposition 4 to get

$$\tau(c_\chi^{-1} \cdot C_z D_{d^{-1}} \cdot C_d D_{z'^{-1}}) = \tau(C_z D_{z'^{-1}}) = \delta_{z,z'}.$$