What can we do with matrices (over finite fields)?

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Definition

Let $\mathbb{F}$ be a field, $A = (A_{i,j}) \in \mathbb{F}^{k \times m}$ and $B = (B_{j,\ell}) \in \mathbb{F}^{m \times n}$. Then the matrix product $C = (C_{i,\ell}) \in \mathbb{F}^{k \times n}$ is the matrix defined by

$$C_{i,\ell} = \sum_{j=1}^{m} A_{i,k} \cdot B_{k,\ell} \qquad \text{for } 1 \leq i \leq k \text{ and } 1 \leq \ell \leq n.$$

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Proposition

Evaluating $C = A \cdot B$ needs $k \cdot m \cdot n$ multiplications and $k \cdot (m-1) \cdot n$ additions in $\mathbb{F}$. For $d = k = m = n$ this is about $2d^3$ elementary field operations.

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Similarly: Inversion, Gaussian elimination.

For $\mathbb{F}_p$, we need $\lceil \log_2(p) \rceil$ bits of memory to store one field element.

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Pack for small p some field elements in the same machine word.

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Arithmetic can be done table driven or machine word-wise.

Operation	Time		Memory	
	C	U	C	U
Mult. in $\mathbb{F}_2^{4370\times 4370}$	320 ms	1335 s	2.3 MB	152 MB
Add. in $\mathbb{F}_2^{1\times 4370}$	240 ns	209 μs	550 B	35 kB
Mult. in $\mathbb{F}_3^{500\times 500}$	50 ms	2140 ms	78 kB	2 MB

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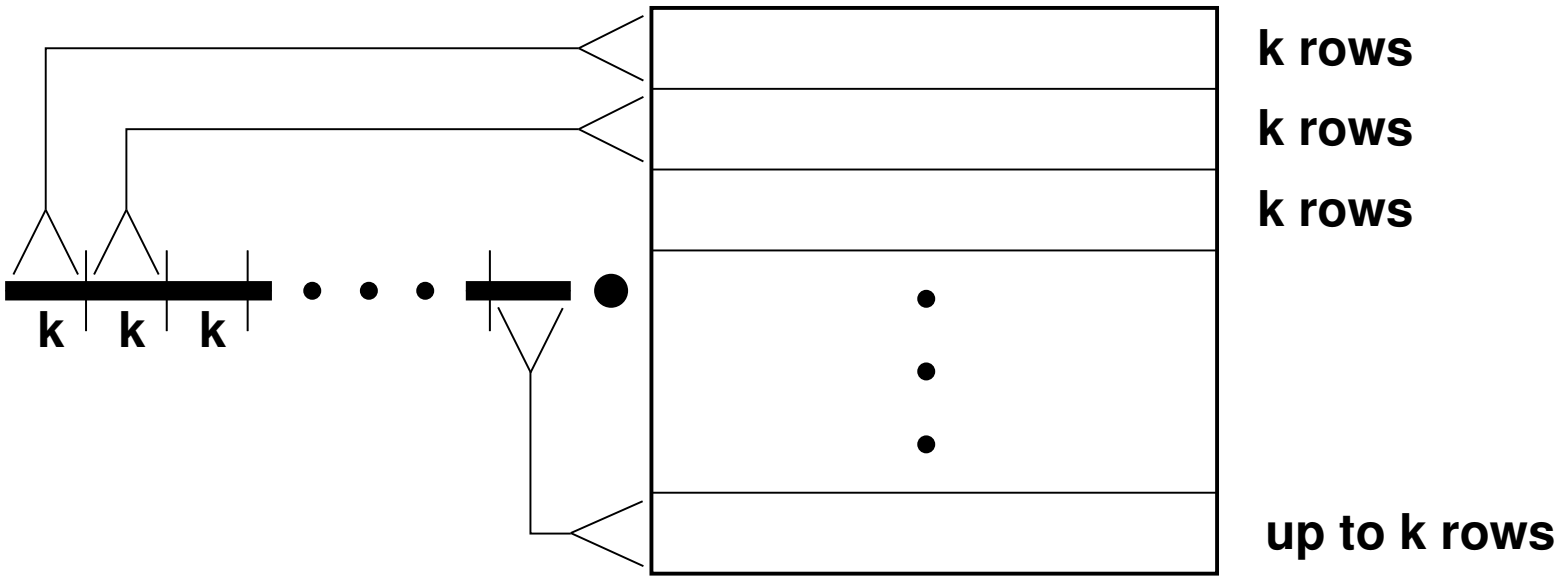

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Strassen/Winograd multiplication

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$\Longrightarrow$ needs 7 multiplications and 15 additions of $d/2 \times d/2$-matrices.

Analysis of complexity:

Let $f(d)$ be the number of elementary field operations needed to multiply two $(d \times d)$-matrices.

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The best known exponent is 2.3, that is totally useless in practice.

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Proposition (Invariant factors)

Let $A := x \cdot \mathbf{1} - M \in \mathbb{F}[x]^{d\times d}$. Then there are matrices $S, T \in \mathbb{F}[x]^{d\times d}$ with determinant 1 and

$$S \cdot A \cdot T = \operatorname{diag}(1, \ldots, 1, p_1, p_2, \ldots, p_k)$$

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with $p_1 \mid p_2 \mid \cdots \mid p_k$. The p_i are uniquely determined (up to scalars) and the matrices S and T can be computed explicitly.
We have $\mu_M = p_r$ and $\chi_M = p_1 \cdot \cdots \cdot p_k$.

Definition (Order polynomial)

$\mathbb{F}$ field, $\mathcal{A}$ f.d. $\mathbb{F}$-algebra, $V \in \mathrm{mod}\text{-}\mathcal{A}$, $v \in V$, $M \in \mathcal{A}$.
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Definition (Relative order polynomial)

If additionally $W < V$ is M-invariant, then we call $\mathrm{ord}_M(v + W)$ the relative order polynomial of $v + W \in V/W$.

Definition (Order polynomial)

$\mathbb{F}$ field, $\mathcal{A}$ f.d. $\mathbb{F}$-algebra, $V \in \mathrm{mod}\text{-}\mathcal{A}$, $v \in V$, $M \in \mathcal{A}$.
Then the order polynomial $q := \mathrm{ord}_M(v) \in \mathbb{F}[x]$ is the monic polynomial of least degree such that $v \cdot q(M) = 0$.

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Lemma (Generator of annihilator)

The order polynomial $\mathrm{ord}_M(v)$ *divides every polynomial* $q \in \mathbb{F}[x]$ *with* $v \cdot q(M) = 0$.

Let $M \in \mathbb{F}^{d\times d}$ and $v_1, \ldots, v_i \in V$ and $V_i := \langle v_1, \ldots, v_i \rangle_M$ the $\mathbb{F}[M]$-span.

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$$v_1M^{d_1} = \sum_{i=0}^{d_1-1} a_i v_1 M^i \quad \text{then} \quad \operatorname{ord}_M(v_1) = x^{d_1} - \sum_{i=0}^{d_1-1} a_i x^i.$$

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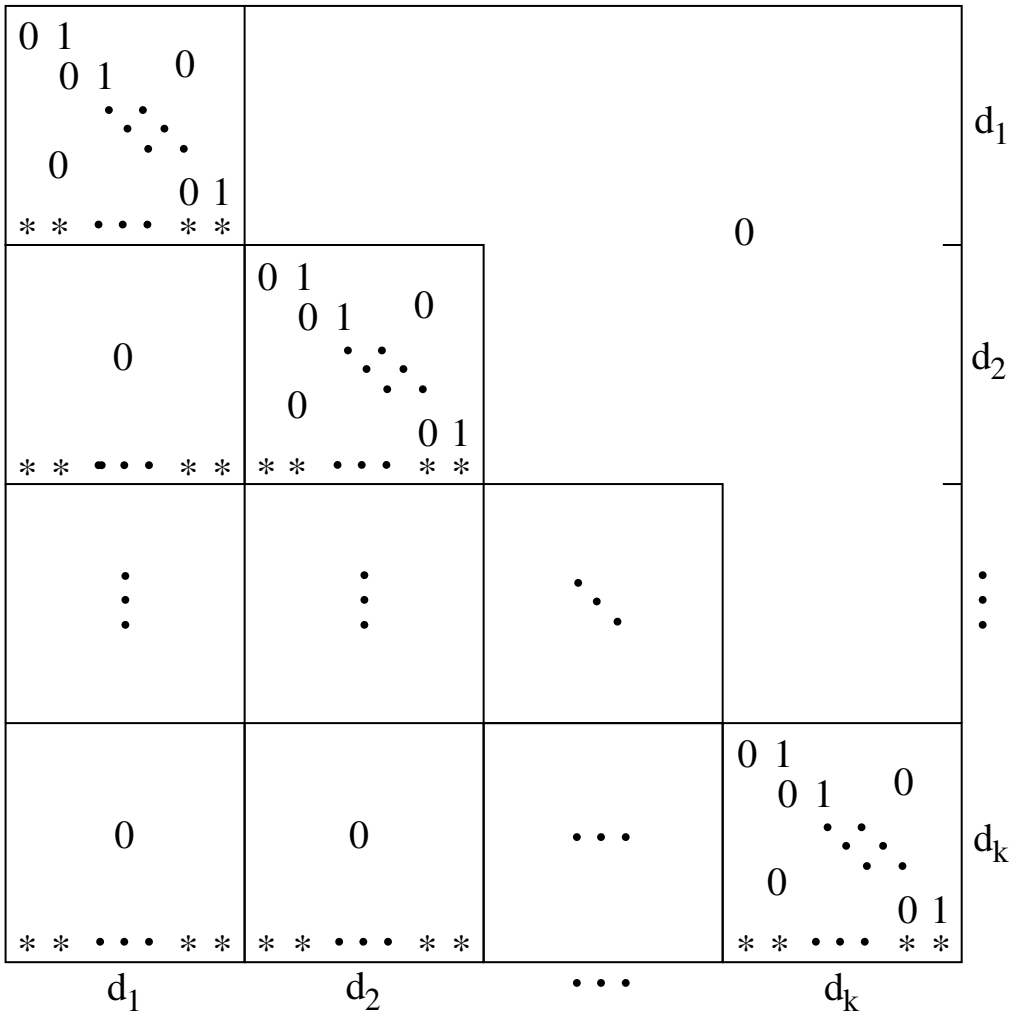
Going on like this we find an $\mathbb{F}$-basis Y of V:

$$Y := (v_1, v_1M, \ldots, v_1^{d_1-1}, \ldots, v_k, v_kM, \ldots, v_kM_k^{d_k-1}).$$

$$Y \cdot M \cdot Y^{-1} =$$



- Block lower-triangular
- with companion matrices along diagonal
- some sparse garbage below the diagonal

The minimal polynomial

→ compute the absolute order polynomials $\mathrm{ord}_M(v_i)$
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If $V = \langle v_1, \ldots, v_k\rangle_M$ *then*

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Minimal polynomial: can be done in asymptotically $\sim n^3$.

Proposition (N., Praeger (2008))

Let $\mathbb{F} = \mathbb{F}_q$, randomise $v_1, \ldots, v_u \in V$ independently and uniformly distributed, $\chi_M = \prod_{i=1}^{t} q_i^{e_i}$. Then:

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Needs asymptotically $\leq 5n^3 + \mathit{Factorisation}(n)$ field ops.

How can we compute the centraliser of an involution?

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The following method by John Bray does the job:

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And: If o is odd, then z is uniformly distributed in $C_G(x)$.