Algorithmic Generalisations of Small Cancellation Theory

Max Neunhöffer

University of St Andrews

9 September 2011

All this is joint work with:

- Stephen Linton,
- Richard Parker,
- Colva Roney-Dougal.

We are about to hire Jeffrey Burdges to help.

The project is already ongoing for $>$ 3 years.
However: no publications and no publishable software (yet).

This talk: Overview over the main ideas

Everybody else who wants to take part is welcome to do so!

What is Classical Small Cancellation Theory?

Assumption

Let $F = \langle A \rangle$ be a finitely-generated free group and $R \subseteq A^*$ be a finite set of relators that is inverse closed.

What is Classical Small Cancellation Theory?

Assumption

Let $F = \langle A \rangle$ be a finitely-generated free group and $R \subseteq A^*$ be a finite set of relators that is inverse closed.

Small Cancellation Theory gives

- a fixed set of conditions on R that imply

What is Classical Small Cancellation Theory?

Assumption

Let $F = \langle A \rangle$ be a finitely-generated free group and $R \subseteq A^*$ be a finite set of relators that is inverse closed.

Small Cancellation Theory gives

- a fixed set of conditions on R that imply
  - that $G := F / \langle\langle R \rangle\rangle$ is infinite,

What is Classical Small Cancellation Theory?

Assumption

Let $F = \langle A \rangle$ be a finitely-generated free group and $R \subseteq A^*$ be a finite set of relators that is inverse closed.

Small Cancellation Theory gives

- a fixed set of conditions on R that imply
  - that $G := F / \langle\langle R \rangle\rangle$ is infinite,
  - that G is word-hyperbolic

What is Classical Small Cancellation Theory?

Assumption

Let $F = \langle A \rangle$ be a finitely-generated free group and $R \subseteq A^*$ be a finite set of relators that is inverse closed.

Small Cancellation Theory gives

- a fixed set of conditions on R that imply
  - that $G := F/\langle\langle R \rangle\rangle$ is infinite,
  - that G is word-hyperbolic
    (i.e. every freely-reduced word $w \in \langle\langle R \rangle\rangle$ of length n is a product of at most $K \cdot n$ conjugates of relators for some $K > 0$), and

What is Classical Small Cancellation Theory?

Assumption

Let $F = \langle A \rangle$ be a finitely-generated free group and $R \subseteq A^*$ be a finite set of relators that is inverse closed.

Small Cancellation Theory gives

- a fixed set of conditions on R that imply
  - that $G := F/\langle\langle R \rangle\rangle$ is infinite,
  - that G is word-hyperbolic
    (i.e. every freely-reduced word $w \in \langle\langle R \rangle\rangle$ of length n is a product of at most $K \cdot n$ conjugates of relators for some $K > 0$), and
  - that an explicitly given rewrite system solves the word problem in linear time (Dehn's Algorithm).

Assumption

Let Γ be a groupoid, i.e. a small category in which every morphism is invertible. Let

$$A := \bigcup_{X,Y \in \mathrm{Ob}_\Gamma} \mathrm{Mor}_\Gamma(X, Y)$$

be our alphabet. Then A^* is a monoid.

Assumption

Let Γ be a groupoid, i.e. a small category in which every morphism is invertible. Let

$$A := \bigcup_{X,Y \in \mathrm{Ob}_\Gamma} \mathrm{Mor}_\Gamma(X, Y)$$

be our alphabet. Then A^* is a monoid.
The multiplication in Γ defines a terminating and confluent RW-system.

Assumption

Let Γ be a groupoid, i.e. a small category in which every morphism is invertible. Let

$$A := \bigcup_{X,Y \in \mathrm{Ob}_\Gamma} \mathrm{Mor}_\Gamma(X, Y)$$

be our alphabet. Then A^* is a monoid.

The multiplication in Γ defines a terminating and confluent RW-system.

Let $F := A^*/\sim$ where $\sim$ is rewrite-equivalence. Then F is a group.

Assumption

Let Γ be a groupoid, i.e. a small category in which every morphism is invertible. Let

$$A := \bigcup_{X,Y \in \mathrm{Ob}_\Gamma} \mathrm{Mor}_\Gamma(X, Y)$$

be our alphabet. Then A^* is a monoid.
The multiplication in Γ defines a terminating and confluent RW-system.
Let $F := A^*/\sim$ where $\sim$ is rewrite-equivalence. Then F is a group.

Problem

Let $R \subseteq A^*$ be a finite set of relators. Devise an algorithm $\mathcal{SC}$ that:

Assumption

Let Γ be a groupoid, i.e. a small category in which every morphism is invertible. Let

$$A := \bigcup_{X,Y \in \mathrm{Ob}_\Gamma} \mathrm{Mor}_\Gamma(X, Y)$$

be our alphabet. Then A^* is a monoid.
The multiplication in Γ defines a terminating and confluent RW-system.
Let $F := A^*/\sim$ where $\sim$ is rewrite-equivalence. Then F is a group.

Problem

Let $R \subseteq A^*$ be a finite set of relators. Devise an algorithm $\mathcal{SC}$ that:

- *delivers and proves correct an algorithm $\mathcal{WP}$ that decides whether or not a $w \in F$ is a product of conjugates of relators, and*

Assumption

Let Γ be a groupoid, i.e. a small category in which every morphism is invertible. Let

$$A := \bigcup_{X,Y \in \mathrm{Ob}_\Gamma} \mathrm{Mor}_\Gamma(X, Y)$$

be our alphabet. Then A^* is a monoid.
The multiplication in Γ defines a terminating and confluent RW-system.
Let $F := A^*/\sim$ where $\sim$ is rewrite-equivalence. Then F is a group.

Problem

Let $R \subseteq A^*$ be a finite set of relators. Devise an algorithm $\mathcal{SC}$ that:

- *delivers and proves correct an algorithm $\mathcal{WP}$ that decides whether or not a $w \in F$ is a product of conjugates of relators, and*
- *delivers a function $f : \mathbb{N} \to \mathbb{N}$ and proves for it that every reduced such $w \in F$ of length n needs at most $f(n)$ factors,*

Assumption

Let Γ be a groupoid, i.e. a small category in which every morphism is invertible. Let

$$A := \bigcup_{X,Y \in \mathrm{Ob}_\Gamma} \mathrm{Mor}_\Gamma(X, Y)$$

be our alphabet. Then A^* is a monoid.
The multiplication in Γ defines a terminating and confluent RW-system.
Let $F := A^*/\sim$ where $\sim$ is rewrite-equivalence. Then F is a group.

Problem

Let $R \subseteq A^*$ be a finite set of relators. Devise an algorithm $\mathcal{SC}$ that:

- *delivers and proves correct an algorithm $\mathcal{WP}$ that decides whether or not a $w \in F$ is a product of conjugates of relators, and*
- *delivers a function $f : \mathbb{N} \to \mathbb{N}$ and proves for it that every reduced such $w \in F$ of length n needs at most $f(n)$ factors,*
- *or fails.*

Construction

For $1 \le i \le n$ let $\mathcal{O}_i$ be pairwise disjoint finite sets and H_i groups. Set

$$\mathrm{Ob}_\Gamma := \bigcup_{i=1}^{n} \mathcal{O}_i, \quad \mathrm{Mor}_\Gamma(A, B) := \begin{cases} \{(A, h, B) \mid h \in H_i\} & \text{if } A, B \in \mathcal{O}_i \\ & \end{cases}$$

Construction

For $1 \leq i \leq n$ let $\mathcal{O}_i$ be pairwise disjoint finite sets and H_i groups. Set

$$\mathrm{Ob}_\Gamma := \bigcup_{i=1}^{n} \mathcal{O}_i, \ \mathrm{Mor}_\Gamma(A, B) := \begin{cases} \{(A, h, B) \mid h \in H_i\} & \text{if } A, B \in \mathcal{O}_i \\ \emptyset & \text{if } A \in \mathcal{O}_i \text{ and } B \in \mathcal{O}_j \text{ with } i \neq j \end{cases}$$

Construction

For $1 \leq i \leq n$ let $\mathcal{O}_i$ be pairwise disjoint finite sets and H_i groups. Set

$$\mathrm{Ob}_\Gamma := \bigcup_{i=1}^{n} \mathcal{O}_i, \quad \mathrm{Mor}_\Gamma(A, B) := \begin{cases} \{(A, h, B) \mid h \in H_i\} & \text{if } A, B \in \mathcal{O}_i \\ \emptyset & \text{if } A \in \mathcal{O}_i \text{ and } B \in \mathcal{O}_j \text{ with } i \neq j \end{cases}$$

Reduction: $(A, h, B) \cdot (C, k, D) := (A, hk, D)$ if $B = C$.

Construction

For $1 \le i \le n$ let $\mathcal{O}_i$ be pairwise disjoint finite sets and H_i groups. Set

$$\mathrm{Ob}_\Gamma := \bigcup_{i=1}^{n} \mathcal{O}_i, \;\; \mathrm{Mor}_\Gamma(A,B) := \begin{cases} \{(A,h,B) \mid h \in H_i\} & \text{if } A, B \in \mathcal{O}_i \\ \emptyset & \text{if } A \in \mathcal{O}_i \text{ and } B \in \mathcal{O}_j \text{ with } i \neq j \end{cases}$$

Reduction: $(A,h,B)\cdot(C,k,D) := (A,hk,D)$ if $B = C$.

Example (Finitely generated free groups)

$\mathcal{O}_1 := \{1,2,\ldots,r+1\}$ and $H_1 := \{1\}$ gives the free group F_r

Construction

For $1 \leq i \leq n$ let $\mathcal{O}_i$ be pairwise disjoint finite sets and H_i groups. Set

$$\mathrm{Ob}_\Gamma := \bigcup_{i=1}^{n} \mathcal{O}_i, \ \mathrm{Mor}_\Gamma(A, B) := \begin{cases} \{(A, h, B) \mid h \in H_i\} & \text{if } A, B \in \mathcal{O}_i \\ \emptyset & \text{if } A \in \mathcal{O}_i \text{ and } B \in \mathcal{O}_j \text{ with } i \neq j \end{cases}$$

Reduction: $(A, h, B) \cdot (C, k, D) := (A, hk, D)$ if $B = C$.

Example (Finitely generated free groups)

$\mathcal{O}_1 := \{1, 2, \ldots, r+1\}$ and $H_1 := \{1\}$ gives the free group F_r

Example (The modular group)

$\mathcal{O}_1 = \{1\}$, $H_1 = C_3 = \left\langle s \mid s^3 = 1 \right\rangle$ and $\mathcal{O}_2 = \{2\}$, $H_2 = C_2 = \left\langle t \mid t^2 = 1 \right\rangle$ gives the modular group $\mathrm{PSL}_2(\mathbb{Z}) \cong C_3 * C_2$.

Construction

For $1 \le i \le n$ let $\mathcal{O}_i$ be pairwise disjoint finite sets and H_i groups. Set

$$\mathrm{Ob}_\Gamma := \bigcup_{i=1}^{n} \mathcal{O}_i, \quad \mathrm{Mor}_\Gamma(A, B) := \begin{cases} \{(A, h, B) \mid h \in H_i\} & \text{if } A, B \in \mathcal{O}_i \\ \emptyset & \text{if } A \in \mathcal{O}_i \text{ and } B \in \mathcal{O}_j \text{ with } i \neq j \end{cases}$$

Reduction: $(A, h, B) \cdot (C, k, D) := (A, hk, D)$ if $B = C$.

Example (Finitely generated free groups)

$\mathcal{O}_1 := \{1, 2, \ldots, r+1\}$ and $H_1 := \{1\}$ gives the free group F_r

Example (The modular group)

$\mathcal{O}_1 = \{1\}$, $H_1 = C_3 = \langle s \mid s^3 = 1 \rangle$ and $\mathcal{O}_2 = \{2\}$, $H_2 = C_2 = \langle t \mid t^2 = 1 \rangle$
gives the modular group $\mathrm{PSL}_2(\mathbb{Z}) \cong C_3 * C_2$.
Set $S := (1, s, 1)$ and $R := (1, s^2, 1)$ and $T := (2, t, 2)$.

Construction

For $1 \leq i \leq n$ let $\mathcal{O}_i$ be pairwise disjoint finite sets and H_i groups. Set

$$\mathrm{Ob}_\Gamma := \bigcup_{i=1}^{n} \mathcal{O}_i, \quad \mathrm{Mor}_\Gamma(A, B) := \begin{cases} \{(A, h, B) \mid h \in H_i\} & \text{if } A, B \in \mathcal{O}_i \\ \emptyset & \text{if } A \in \mathcal{O}_i \text{ and } B \in \mathcal{O}_j \text{ with } i \neq j \end{cases}$$

Reduction: $(A, h, B) \cdot (C, k, D) := (A, hk, D)$ if $B = C$.

Example (Finitely generated free groups)

$\mathcal{O}_1 := \{1, 2, \ldots, r + 1\}$ and $H_1 := \{1\}$ gives the free group F_r

Example (The modular group)

$\mathcal{O}_1 = \{1\}$, $H_1 = C_3 = \langle s \mid s^3 = 1 \rangle$ and $\mathcal{O}_2 = \{2\}$, $H_2 = C_2 = \langle t \mid t^2 = 1 \rangle$
gives the modular group $\mathrm{PSL}_2(\mathbb{Z}) \cong C_3 * C_2$.
Set $S := (1, s, 1)$ and $R := (1, s^2, 1)$ and $T := (2, t, 2)$.

$\Longrightarrow$ Actually covers all free products of finitely generated groups

Recall: $F = C_3 * C_2 = \langle S, R, T \mid SR = 1 = S^3 = T^2 \rangle$
RW-System: $SR \to \epsilon$, $RS \to \epsilon$, $TT \to \epsilon$, $SS \to R$, $RR \to S$.

What is cancellation?

Recall: $F = C_3 * C_2 = \langle S, R, T \mid SR = 1 = S^3 = T^2 \rangle$
RW-System: $SR \to \epsilon$, $RS \to \epsilon$, $TT \to \epsilon$, $SS \to R$, $RR \to S$.

What is cancellation?

$$(TSTSTSTS) \cdot (RTSTSTSTR) = TSTSTRTSTSTR$$

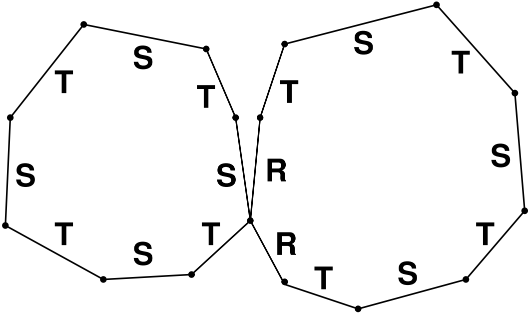
SR cancels, *TT* cancels, *SS* consolidates to *R*.

Recall: $F = C_3 * C_2 = \langle S, R, T \mid SR = 1 = S^3 = T^2 \rangle$
RW-System: $SR \to \epsilon$, $RS \to \epsilon$, $TT \to \epsilon$, $SS \to R$, $RR \to S$.

What is cancellation?

$$(TSTSTSTS) \cdot (RTSTSTSTR) = TSTSTRTSTSTR$$

SR cancels, *TT* cancels, *SS* consolidates to *R*.

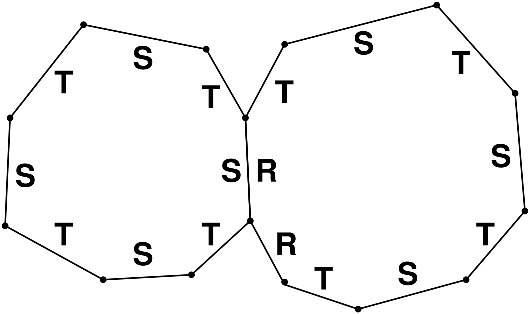


Recall: $F = C_3 * C_2 = \langle S, R, T \mid SR = 1 = S^3 = T^2 \rangle$
RW-System: $SR \to \epsilon$, $RS \to \epsilon$, $TT \to \epsilon$, $SS \to R$, $RR \to S$.

What is cancellation?

$$(TSTSTSTS) \cdot (RTSTSTSTR) = TSTSTRTSTSTR$$

SR cancels, TT cancels, SS consolidates to R.

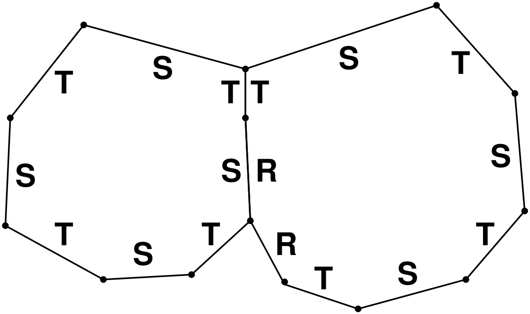


Recall: $F = C_3 * C_2 = \langle S, R, T \mid SR = 1 = S^3 = T^2 \rangle$
RW-System: $SR \to \epsilon$, $RS \to \epsilon$, $TT \to \epsilon$, $SS \to R$, $RR \to S$.

What is cancellation?

$$(TSTSTSTS) \cdot (RTSTSTSTR) = TSTSTRTSTSTR$$

SR cancels, *TT* cancels, *SS* consolidates to *R*.

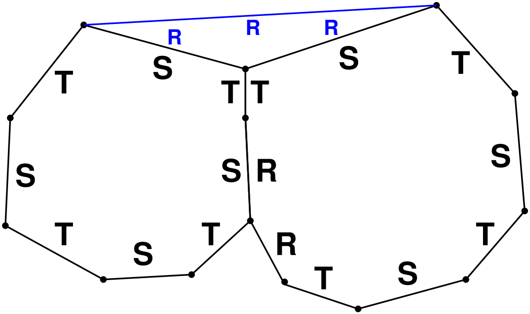


Recall: $F = C_3 * C_2 = \langle S, R, T \mid SR = 1 = S^3 = T^2 \rangle$
RW-System: $SR \to \epsilon$, $RS \to \epsilon$, $TT \to \epsilon$, $SS \to R$, $RR \to S$.

What is cancellation?

$$(TSTSTSTS) \cdot (RTSTSTSTR) = TSTSTRTSTSTR$$

SR cancels, *TT* cancels, *SS* consolidates to *R*.

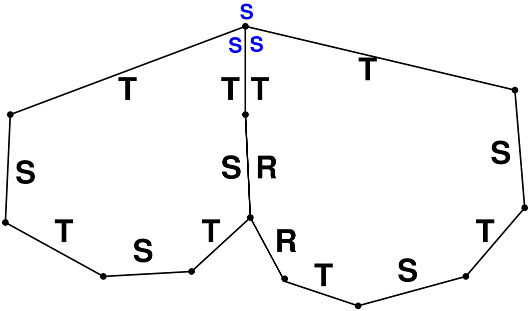


Recall: $F = C_3 * C_2 = \langle S, R, T \mid SR = 1 = S^3 = T^2 \rangle$
RW-System: $SR \to \epsilon$, $RS \to \epsilon$, $TT \to \epsilon$, $SS \to R$, $RR \to S$.

What is cancellation?

$$(TSTST\,STS) \cdot (RTS\,TSTSTR) = TSTST\,R\,TSTSTR$$

SR cancels, TT cancels, SS consolidates to R.



Principle

- Faces are labelled by relators.

Principle

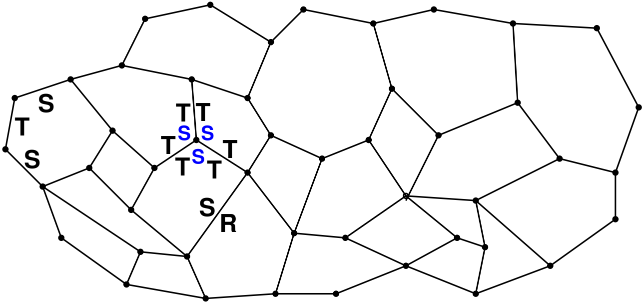
- Faces are labelled by relators.
- Diagrams are proofs that their boundary word is a product of conjugates of the relators.

Principle

- Faces are labelled by relators.
- Diagrams are proofs that their boundary word is a product of conjugates of the relators.
- Need Theorem: For every word w that is equal to a product of conjugates of the relators there is a diagram with boundary w.

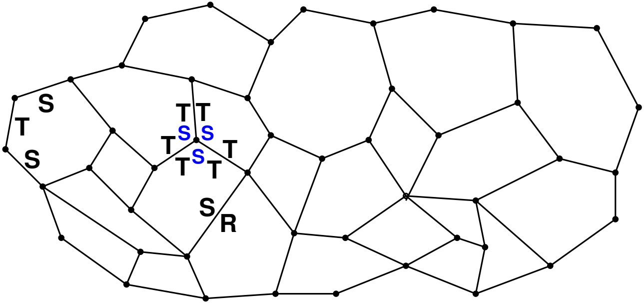
Principle

- Faces are labelled by relators.
- Diagrams are proofs that their boundary word is a product of conjugates of the relators.
- Need Theorem: For every word w that is equal to a product of conjugates of the relators there is a diagram with boundary w.



Principle

- Faces are labelled by relators.
- Diagrams are proofs that their boundary word is a product of conjugates of the relators.
- Need Theorem: For every word w that is equal to a product of conjugates of the relators there is a diagram with boundary w.



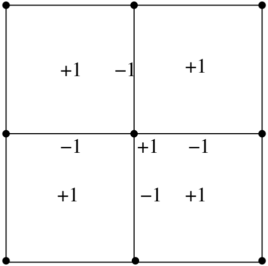
This is a generalisation of van Kampen diagrams.

Combinatorical curvature: A diagram is a planar graph. We endow

- each vertex with $+1$ unit of curvature,
- each edge with -1 unit of curvature and
- each face with $+1$ unit of curvature.

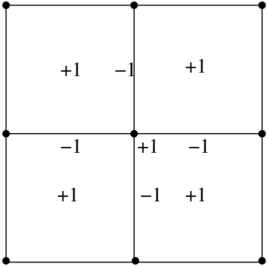
Combinatorical curvature: A diagram is a planar graph. We endow

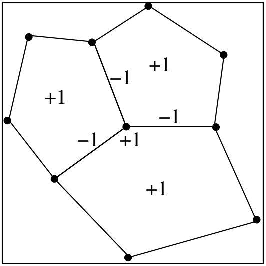
- each vertex with $+1$ unit of curvature,
- each edge with -1 unit of curvature and
- each face with $+1$ unit of curvature.



Combinatorical curvature: A diagram is a planar graph. We endow

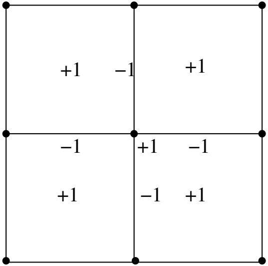
- each vertex with $+1$ unit of curvature,
- each edge with -1 unit of curvature and
- each face with $+1$ unit of curvature.



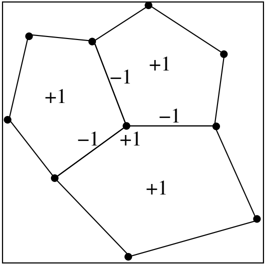


Combinatorical curvature: A diagram is a planar graph. We endow

- each vertex with $+1$ unit of curvature,
- each edge with -1 unit of curvature and
- each face with $+1$ unit of curvature.



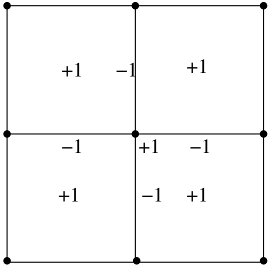
$$1 - 4 \cdot \frac{1}{2} + 4 \cdot \frac{1}{4} = 0$$



$$1 - 3 \cdot \frac{1}{2} + 3 \cdot \frac{1}{5} = \frac{1}{10}$$

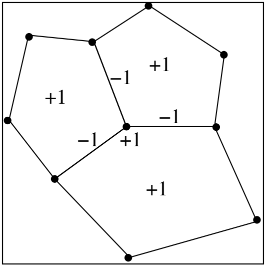
Combinatorical curvature: A diagram is a planar graph. We endow

- each vertex with $+1$ unit of curvature,
- each edge with -1 unit of curvature and
- each face with $+1$ unit of curvature.



$$1 - 4 \cdot \frac{1}{2} + 4 \cdot \frac{1}{4} = 0$$

locally flat

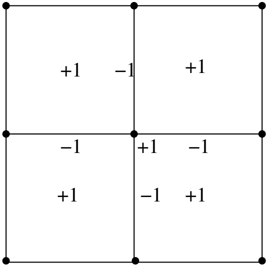


$$1 - 3 \cdot \frac{1}{2} + 3 \cdot \frac{1}{5} = \frac{1}{10}$$

locally positively curved

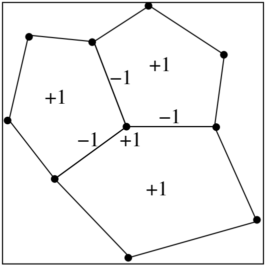
Combinatorical curvature: A diagram is a planar graph. We endow

- each vertex with $+1$ unit of curvature,
- each edge with -1 unit of curvature and
- each face with $+1$ unit of curvature.



$$1 - 4 \cdot \frac{1}{2} + 4 \cdot \frac{1}{4} = 0$$

locally flat



$$1 - 3 \cdot \frac{1}{2} + 3 \cdot \frac{1}{5} = \frac{1}{10}$$

locally positively curved

Idea

Analyse curvature locally for all possible diagrams (“instantiation”).

Idea (Curvature redistribution)

We redistribute the curvature locally in a conservative way.

Idea (Curvature redistribution)

We redistribute the curvature locally in a conservative way.
Purpose: To smear it out locally.

Idea (Curvature redistribution)

We redistribute the curvature locally in a conservative way.
Purpose: To smear it out locally.

Lemma (Euler's formula)

In a planar graph, we have: $V - E + F = +1$
(number of vertices, edges and faces, not counting the outer one).

Idea (Curvature redistribution)

We redistribute the curvature locally in a conservative way.
Purpose: To smear it out locally.

Lemma (Euler's formula)

In a planar graph, we have: $V - E + F = +1$
(number of vertices, edges and faces, not counting the outer one).
Thus: The total sum of our combinatorial curvature is always $+1$*.*

Idea (Curvature redistribution)

We redistribute the curvature locally in a conservative way.
Purpose: To smear it out locally.

Lemma (Euler's formula)

In a planar graph, we have: $V - E + F = +1$
(number of vertices, edges and faces, not counting the outer one).
Thus: The total sum of our combinatorial curvature is always $+1$.

If the local curvature (after redistribution) is negative in the interior,

- there must be some positively curved region near the boundary,

Idea (Curvature redistribution)

We redistribute the curvature locally in a conservative way.
Purpose: To smear it out locally.

Lemma (Euler's formula)

In a planar graph, we have: $V - E + F = +1$
(number of vertices, edges and faces, not counting the outer one).
Thus: The total sum of our combinatorial curvature is always $+1$.

If the local curvature (after redistribution) is negative in the interior,

- there must be some positively curved region near the boundary,
- we can disjoin positively curved cases of boundary regions,

Idea (Curvature redistribution)

We redistribute the curvature locally in a conservative way.
Purpose: To smear it out locally.

Lemma (Euler's formula)

In a planar graph, we have: $V - E + F = +1$
(number of vertices, edges and faces, not counting the outer one).
Thus: The total sum of our combinatorial curvature is always $+1$*.*

If the local curvature (after redistribution) is negative in the interior,

- there must be some positively curved region near the boundary,
- we can disjoin positively curved cases of boundary regions,
- there are no spheres, and

Idea (Curvature redistribution)

We redistribute the curvature locally in a conservative way.
Purpose: To smear it out locally.

Lemma (Euler's formula)

In a planar graph, we have: $V - E + F = +1$
(number of vertices, edges and faces, not counting the outer one).
Thus: The total sum of our combinatorial curvature is always $+1$*.*

If the local curvature (after redistribution) is negative in the interior,

- there must be some positively curved region near the boundary,
- we can disjoin positively curved cases of boundary regions,
- there are no spheres, and
- we can derive an upper bound for the number of faces in terms of the boundary length.

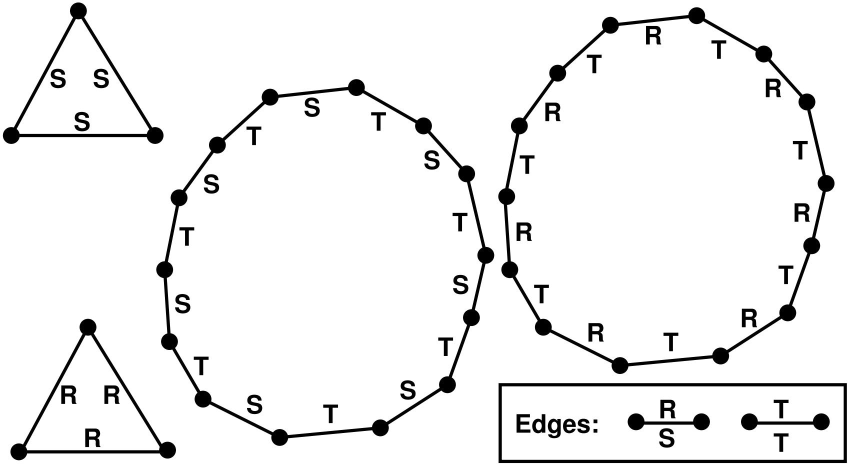
We want to study this presentation:

$$G := \left\langle S, R, T \mid S^3 = 1 = RS = T^2 = (ST)^7 \right\rangle$$

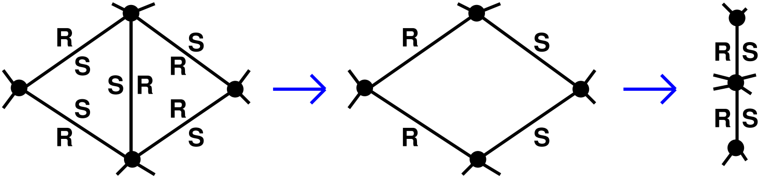
We want to study this presentation:

$$G := \langle S, R, T \mid S^3 = 1 = RS = T^2 = (ST)^7 \rangle$$

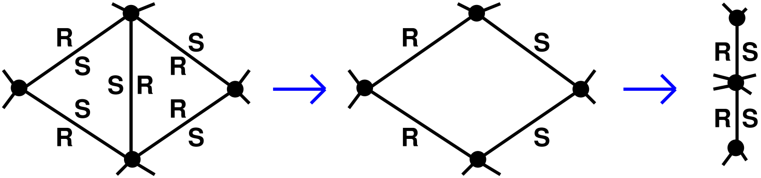
Classical van Kampen diagrams consist of:



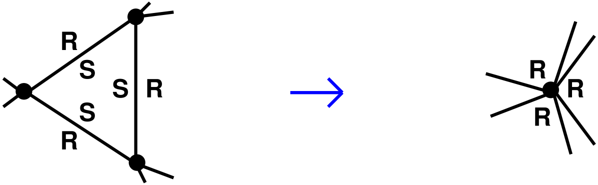
Inverse relators need never touch each other:



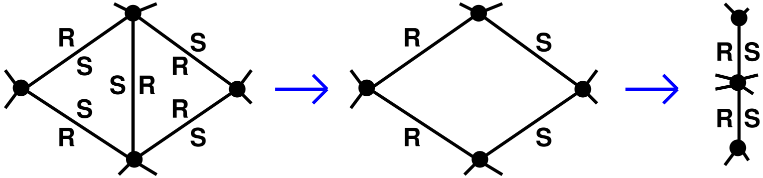
Inverse relators need never touch each other:



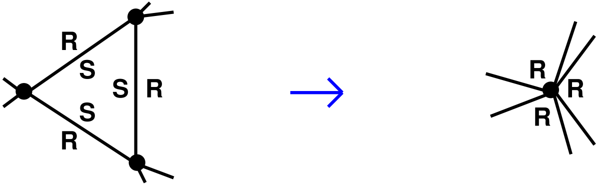
Our generalisations allow:



Inverse relators need never touch each other:



Our generalisations allow:

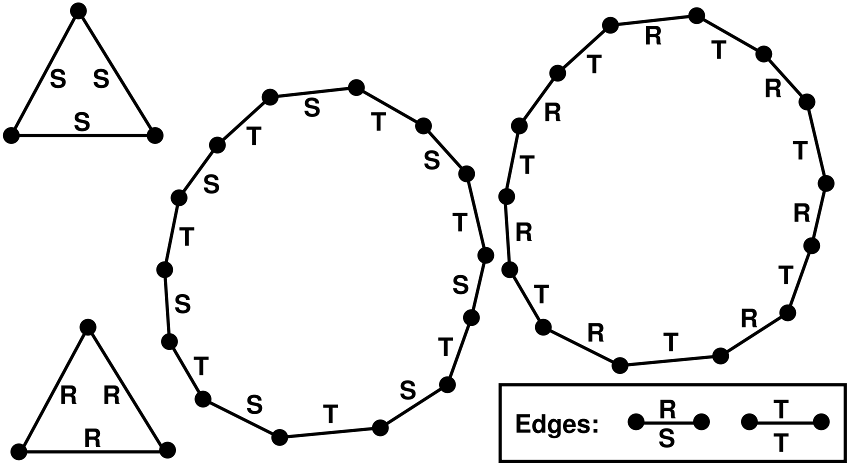


$\Longrightarrow$ Triangles not needed, the big guys do not touch.

We want to study this presentation:

$$G := \langle S, R, T \mid S^3 = 1 = RS = T^2 = (ST)^7 \rangle$$

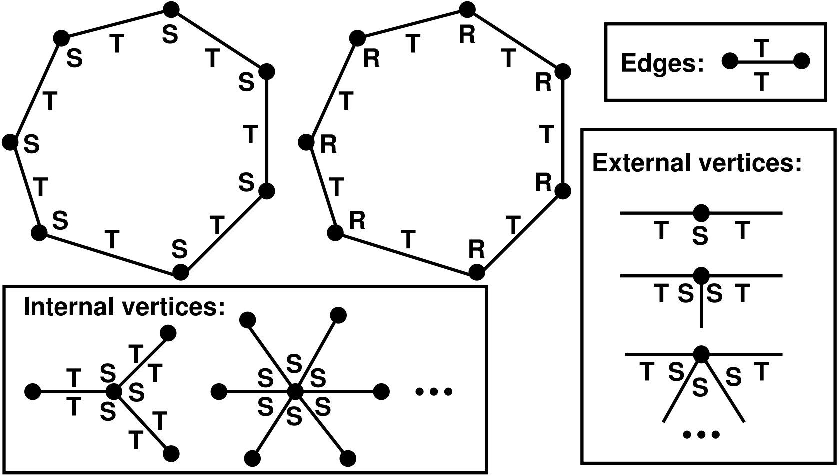
Classical van Kampen diagrams for this problem consist of:



We want to study this presentation:

$$G := \langle S, R, T \mid S^3 = 1 = RS = T^2 = (ST)^7 \rangle$$

Our van Kampen diagrams for this problem consist of:



1. Start by giving each vertex $+1$, each edge -1 and each face $+1$ unit of curvature.

1. Start by giving each vertex $+1$, each edge -1 and each face $+1$ unit of curvature.
2. Distribute the curv. at each vertex equally to all adjacent edges.

1. Start by giving each vertex $+1$, each edge -1 and each face $+1$ unit of curvature.
2. Distribute the curv. at each vertex equally to all adjacent edges.
3. Distribute the curv. at each edge equally to all adjacent faces. (External edges give all to one, internal edges distribute $\frac{1}{2} : \frac{1}{2}$.)

1. Start by giving each vertex $+1$, each edge -1 and each face $+1$ unit of curvature.
2. Distribute the curv. at each vertex equally to all adjacent edges.
3. Distribute the curv. at each edge equally to all adjacent faces. (External edges give all to one, internal edges distribute $\frac{1}{2} : \frac{1}{2}$.)

Conclusions

- After step 2 all vertices have 0 curvature.

1. Start by giving each vertex $+1$, each edge -1 and each face $+1$ unit of curvature.
2. Distribute the curv. at each vertex equally to all adjacent edges.
3. Distribute the curv. at each edge equally to all adjacent faces. (External edges give all to one, internal edges distribute $\frac{1}{2} : \frac{1}{2}$.)

Conclusions

- After step 2 all vertices have 0 curvature.
- After step 2 all internal edges have $\leq -1/3$ curvature.

1. Start by giving each vertex $+1$, each edge -1 and each face $+1$ unit of curvature.
2. Distribute the curv. at each vertex equally to all adjacent edges.
3. Distribute the curv. at each edge equally to all adjacent faces. (External edges give all to one, internal edges distribute $\frac{1}{2} : \frac{1}{2}$.)

Conclusions

- After step 2 all vertices have 0 curvature.
- After step 2 all internal edges have $\leq -1/3$ curvature.
- After step 3 all edges have 0 curvature.

1. Start by giving each vertex $+1$, each edge -1 and each face $+1$ unit of curvature.
2. Distribute the curv. at each vertex equally to all adjacent edges.
3. Distribute the curv. at each edge equally to all adjacent faces. (External edges give all to one, internal edges distribute $\frac{1}{2} : \frac{1}{2}$.)

Conclusions

- After step 2 all vertices have 0 curvature.
- After step 2 all internal edges have $\leq -1/3$ curvature.
- After step 3 all edges have 0 curvature.
- After step 3 all internal faces have $\leq -1/6$ curvature.

1. Start by giving each vertex $+1$, each edge -1 and each face $+1$ unit of curvature.
2. Distribute the curv. at each vertex equally to all adjacent edges.
3. Distribute the curv. at each edge equally to all adjacent faces. (External edges give all to one, internal edges distribute $\frac{1}{2} : \frac{1}{2}$.)

Conclusions

- After step 2 all vertices have 0 curvature.
- After step 2 all internal edges have $\leq -1/3$ curvature.
- After step 3 all edges have 0 curvature.
- After step 3 all internal faces have $\leq -1/6$ curvature.
- After step 3 all external faces have $\leq +1$ curvature.

1. Start by giving each vertex $+1$, each edge -1 and each face $+1$ unit of curvature.
2. Distribute the curv. at each vertex equally to all adjacent edges.
3. Distribute the curv. at each edge equally to all adjacent faces. (External edges give all to one, internal edges distribute $\frac{1}{2} : \frac{1}{2}$.)

Conclusions

- After step 2 all vertices have 0 curvature.
- After step 2 all internal edges have $\leq -1/3$ curvature.
- After step 3 all edges have 0 curvature.
- After step 3 all internal faces have $\leq -1/6$ curvature.
- After step 3 all external faces have $\leq +1$ curvature.
- The total sum of the curvature stays $+1$.

1. Start by giving each vertex $+1$, each edge -1 and each face $+1$ unit of curvature.
2. Distribute the curv. at each vertex equally to all adjacent edges.
3. Distribute the curv. at each edge equally to all adjacent faces. (External edges give all to one, internal edges distribute $\frac{1}{2} : \frac{1}{2}$.)

Conclusions

- After step 2 all vertices have 0 curvature.
- After step 2 all internal edges have $\leq -1/3$ curvature.
- After step 3 all edges have 0 curvature.
- After step 3 all internal faces have $\leq -1/6$ curvature.
- After step 3 all external faces have $\leq +1$ curvature.
- The total sum of the curvature stays $+1$.
- $\#$ external faces $\leq \ell(w)$ (length of boundary word)

1. Start by giving each vertex $+1$, each edge -1 and each face $+1$ unit of curvature.
2. Distribute the curv. at each vertex equally to all adjacent edges.
3. Distribute the curv. at each edge equally to all adjacent faces. (External edges give all to one, internal edges distribute $\frac{1}{2} : \frac{1}{2}$.)

Conclusions

- After step 2 all vertices have 0 curvature.
- After step 2 all internal edges have $\leq -1/3$ curvature.
- After step 3 all edges have 0 curvature.
- After step 3 all internal faces have $\leq -1/6$ curvature.
- After step 3 all external faces have $\leq +1$ curvature.
- The total sum of the curvature stays $+1$.
- $\#$ external faces $\leq \ell(w)$ (length of boundary word)
- Thus: $\#$ internal faces $< 6 \cdot \ell(w)$

1. Start by giving each vertex $+1$, each edge -1 and each face $+1$ unit of curvature.
2. Distribute the curv. at each vertex equally to all adjacent edges.
3. Distribute the curv. at each edge equally to all adjacent faces. (External edges give all to one, internal edges distribute $\frac{1}{2} : \frac{1}{2}$.)

Conclusions

- After step 2 all vertices have 0 curvature.
- After step 2 all internal edges have $\leq -1/3$ curvature.
- After step 3 all edges have 0 curvature.
- After step 3 all internal faces have $\leq -1/6$ curvature.
- After step 3 all external faces have $\leq +1$ curvature.
- The total sum of the curvature stays $+1$.
- $\#$ external faces $\leq \ell(w)$ (length of boundary word)
- Thus: $\#$ internal faces $< 6 \cdot \ell(w)$
- Thus: $\#$ faces $< 7 \cdot \ell(w)$

More ideas for generalisations:

- Forbidden subdiagrams:
  - Provided we find a sphere,
  - then we can rewrite the bigger half to the smaller one.

More ideas for generalisations:

- Forbidden subdiagrams:
  - Provided we find a sphere,
  - then we can rewrite the bigger half to the smaller one.
  - So we never need a proof containing the bigger half and
  - consider only minimal proofs.

More ideas for generalisations:

- Forbidden subdiagrams:
  - Provided we find a sphere,
  - then we can rewrite the bigger half to the smaller one.
  - So we never need a proof containing the bigger half and
  - consider only minimal proofs.
- Change generating set to improve “small cancellation”.

More ideas for generalisations:

- Forbidden subdiagrams:
  - Provided we find a sphere,
  - then we can rewrite the bigger half to the smaller one.
  - So we never need a proof containing the bigger half and
  - consider only minimal proofs.
- Change generating set to improve “small cancellation”.
- Change presentation to improve “small cancellation”.

More ideas for generalisations:

- Forbidden subdiagrams:
  - Provided we find a sphere,
  - then we can rewrite the bigger half to the smaller one.
  - So we never need a proof containing the bigger half and
  - consider only minimal proofs.
- Change generating set to improve “small cancellation”.
- Change presentation to improve “small cancellation”.
- Other algebraic structures than groups (monoids, semigroups)

More ideas for generalisations:

- Forbidden subdiagrams:
  - Provided we find a sphere,
  - then we can rewrite the bigger half to the smaller one.
  - So we never need a proof containing the bigger half and
  - consider only minimal proofs.
- Change generating set to improve “small cancellation”.
- Change presentation to improve “small cancellation”.
- Other algebraic structures than groups (monoids, semigroups)

Good news

We already have prototypes, algorithms and data structures.

More ideas for generalisations:

- Forbidden subdiagrams:
  - Provided we find a sphere,
  - then we can rewrite the bigger half to the smaller one.
  - So we never need a proof containing the bigger half and
  - consider only minimal proofs.
- Change generating set to improve “small cancellation”.
- Change presentation to improve “small cancellation”.
- Other algebraic structures than groups (monoids, semigroups)

Good news

We already have prototypes, algorithms and data structures.

Stay tuned . . .