Cryptography using Primes

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Note in particular: $a \equiv r \pmod{n}$.

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Note (Computation tricks)

If $a \equiv \hat{a} \pmod{n}$ and $b \equiv \hat{b} \pmod{n}$, then

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Fermat and Euclid

Theorem (Little Theorem of Fermat)

Let $n = p \cdot q$ be the product of two primes p and q. Then

$$a^{(p-1)(q-1)} \equiv 1 \pmod{n}$$

for all integers a that are not divisible by p or q.

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Theorem (Euclidean Algorithm)

If $d, m \in \mathbb{Z}$ do not have a common prime divisor, then it is (efficiently) possible to determine an $e \in \mathbb{Z}$, such that $de \equiv 1 \pmod{m}$.

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If Alice wants to send a secret message to Bob:

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- He then publishes n and e
- and keeps secret p, q, m and d.

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Public: n and e **Secret:** p, q, $m = (p-1)(q-1)$ and d

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- He gets back $b \equiv c^d \equiv (a^e)^d \equiv a^{de} \equiv a \pmod{n}$ since $de \equiv 1 \pmod{(p-1)(q-1)}$.

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Knowing n in principle determines p and q!

However, actually computing p and q from n is HARD.

Computing roots and discrete logarithm

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Cracking the encryption is basically solving the equation

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that is, computing e-th roots.

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- Then $a \equiv z^x \pmod{n}$ since

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However, solving discrete logarithms like $c = z^x$ is HARD.

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However: It is also not proved, that there is none!.