Sudoku and Mathematics

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University of St Andrews

Edinburgh 21 January 2011

Sudoku Grids

7	9	3	6	8	4	5	1	2
4	8	6	5	1	2	9	3	7
1	2	5	9	7	3	8	4	6
9	3	2	7	5	1	6	8	4
5	7	8	2	4	6	3	9	1
6	4	1	3	9	8	7	2	5
3	1	9	4	6	5	2	7	8
8	5	7	1	2	9	4	6	3
2	6	4	8	3	7	1	5	9

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5	7	8	2	4	6	3	9	1
6	4	1	3	9	8	7	2	5
3	1	9	4	6	5	2	7	8
8	5	7	1	2	9	4	6	3
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Rule

Each row, column and 3×3-block contains the numbers 1 to 9 each exactly once.

Sudoku Puzzles

							1	
4								
	2							
				5		6		4
		8				3		
		1		9				
3			4			2		
	5		1					
			8		7			

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9	3	2	7	5	1	6	8	4
5	7	8				3	9	1
6	4	1	3	9	8	7	2	5
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5	7	8				3	9	1
6	4	1	3	9	8	7	2	5
3	1	9	4	6	5	2	7	8
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Solving Sudokus

<table>
<tr><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td>1</td><td></td></tr>
<tr><td>4</td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
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<tr><td></td><td></td><td>8</td><td></td><td></td><td></td><td>3</td><td></td><td>1</td></tr>
<tr><td></td><td></td><td>1</td><td></td><td>9</td><td></td><td></td><td></td><td></td></tr>
<tr><td>3</td><td></td><td></td><td>4</td><td>6</td><td></td><td>2</td><td></td><td></td></tr>
<tr><td></td><td>5</td><td></td><td>1</td><td>2|3</td><td></td><td></td><td></td><td></td></tr>
<tr><td></td><td></td><td></td><td>8</td><td></td><td>7</td><td></td><td></td><td></td></tr>
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Theorem

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If there are many solutions, a minor modification lets us find all of them.

$\Longrightarrow$ Solving Sudokus is mathematics!

How difficult is a Sudoku puzzle?

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There is a whole branch of CompSci/Maths called Constraint Satisfaction: deals with very hard problems similar to Solving Sudokus

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Number of 9×9 latin squares

There are altogether

$$5\,524\,751\,496\,156\,892\,842\,531\,225\,600 \approx 5.525 \cdot 10^{27}$$

latin squares of size 9×9.

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Not by brute force: If our computer enumerates one in each clock cycle, it would need 90000 years to finish!

Counting full Sudoku grids

Consider only the first block row:

1. We can renumber to get this left hand 3×3-block:

1	2	3	?	?	?	?	?	?
4	5	6	?	?	?	?	?	?
7	8	9	?	?	?	?	?	?

Count those, multiply by $9! = 362880$ in the end.

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The answer

There are altogether

$$6\,670\,903\,752\,021\,072\,936\,960 \approx 6.671 \cdot 10^{21}$$

different full Sudoku grids. (Felgenhauer/Jarvis 2006)

`http://www.afjarvis.staff.shef.ac.uk/sudoku/`

Equivalence of Sudokus

7	9	3	6	8	4	5	1	2
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1	2	5	9	7	3	8	4	6
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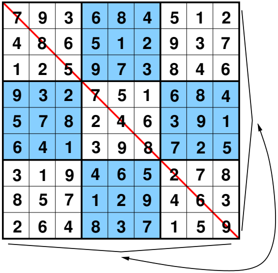
Equivalence transformations:

- Permute: rows in a block, columns in a block, block-rows, block-columns
- Renumber: entries

Equivalence of Sudokus



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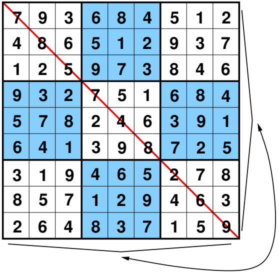

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$\longrightarrow$ All concatenations of these form a group.

Equivalence of Sudokus

Definition: Equivalent Sudokus

Two Sudoku grids/puzzles are called equivalent if one arises from the other by applying a sequence of equivalence transformations.

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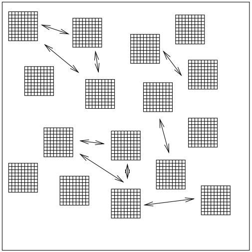

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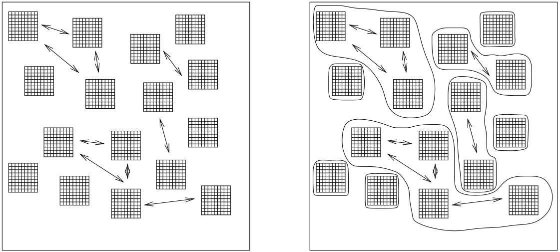

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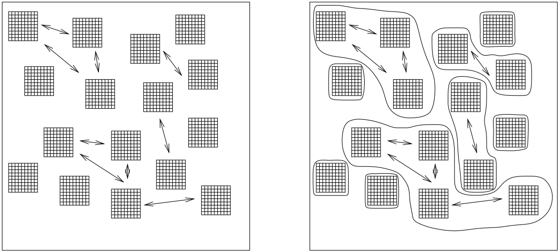

We form equivalence classes or orbits.

$\rightarrow$ There are 5 472 730 538 classes (Jarvis/Russell 2006)

http://www.afjarvis.staff.shef.ac.uk/sudoku/

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2. Distinguish cases for first row:

(Type I)

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2. Distinguish cases for first row:

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(Type II)

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4	5	6		
7	8	9		

Symmetry Breaking

We "break the symmetry" by considering exactly one from each equivalence class. Consider only the first block row:

1. We can renumber to get this left hand 3×3-block:

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3. Some more such arguments ...

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- What is the maximum number of clues in an irreducible Sudoku-puzzle?
- What is the minimum number of clues in a Sudoku-puzzle?
- We do not even have started to consider 16×16-Sudokus. . .

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Of course: such that there is a unique solution.

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- There is a set of 16 clues which allows exactly two solutions.

							1	
4								
	2							
				5		6		4
		8				3		
		1		9				
3			4			2		
	5		1					
			8		7			

This puzzle has 17 clues.

							1	
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	2							
				5		6		4
		8				3		
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This puzzle has 17 clues. None of them can be left out.

							1	
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Question:

Are there 16 clues which uniquely define a Sudoku grid?

This is an unsolved problem!

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We need constraints that the selection of 16 has to fulfill.

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- How can we find them?
- How does this help?

Unavoidable sets

7	9	3	6	8	4	5	1	2
4	8	6	5	1	2	9	3	7
1	2	5	9	7	3	8	4	6
9	3	2	7	5	1	6	8	4
5	7	8	2	4	6	3	9	1
6	4	1	3	9	8	7	2	5
3	1	9	4	6	5	2	7	8
8	5	7	1	2	9	4	6	3
2	6	4	8	3	7	1	5	9

Unavoidable sets

7	9	3	6	8	4	**5**	**1**	2
4	8	6	5	1	2	9	3	7
1	2	5	9	7	3	8	4	6
9	3	2	7	5	1	6	8	4
5	7	8	2	4	6	3	9	1
6	4	1	3	9	8	7	2	5
3	1	9	4	6	5	2	7	8
8	5	7	1	2	9	4	6	3
2	6	4	8	3	7	**1**	**5**	9

Any set of 16 clues cannot avoid the yellow positions.

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6	4	1	3	9	8	7	2	5
3				6		2	7	8
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3				6		2	7	8
8		7		2			6	
2	6		8	3	7		5	

Any set of 16 clues cannot avoid the yellow positions.

Because this puzzle has more than one solution.

The Hitting Set Problem

Problem: Hitting Set (resp. Set Covering)

Let M be a set and let $A_1, \ldots, A_k$ be subsets of M.
Find a minimal subset H of M which contains at least one element of every A_i for $1 \leq i \leq k$.

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We want to solve it to use lots of unavoidable sets to reduce the number of 16-clue sets we need to consider.

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Algorithm: Hitting Sets

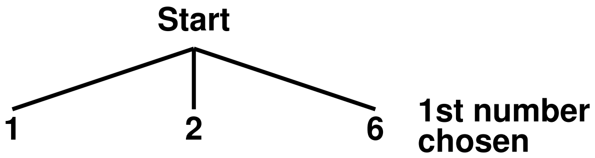
An example Hitting Set Problem

Let $M = \{1, 2, 3, \ldots, 12\}$ and

$$\mathcal{A} := \{ \quad \{1,2,6\}, \{2,3,4,8\}, \{1,7,8,9\}, \{3,4,6,9\}, \{4,6,10,12\}, \{2,10,11,12\}, \{5,7,8,9\}, \{5,7,10,12\}, \{1,3,4,5,11\} \quad \}$$

Find a 3-subset of M intersecting all members of $\mathcal{A}$ non-trivially.

Backtrack Search in Action



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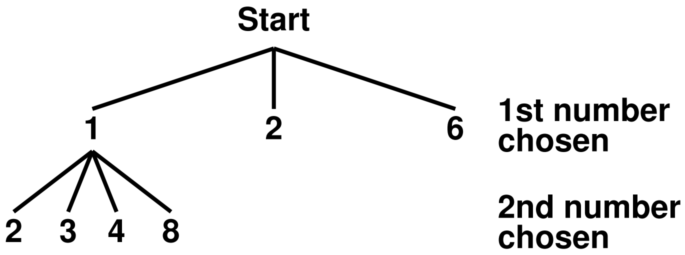

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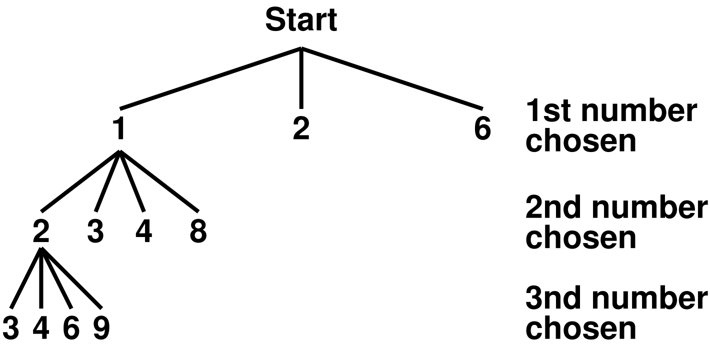

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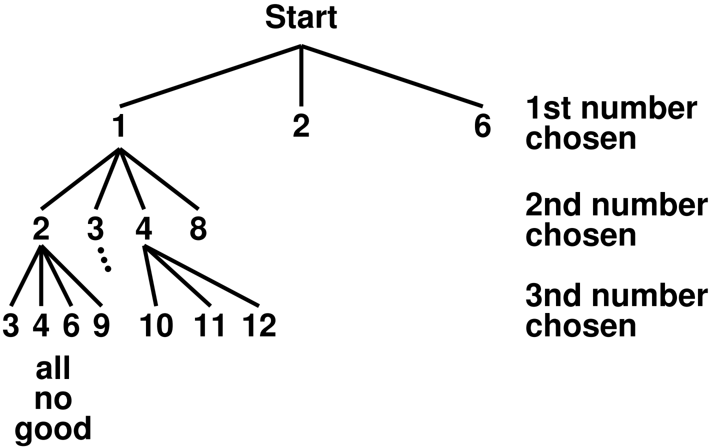

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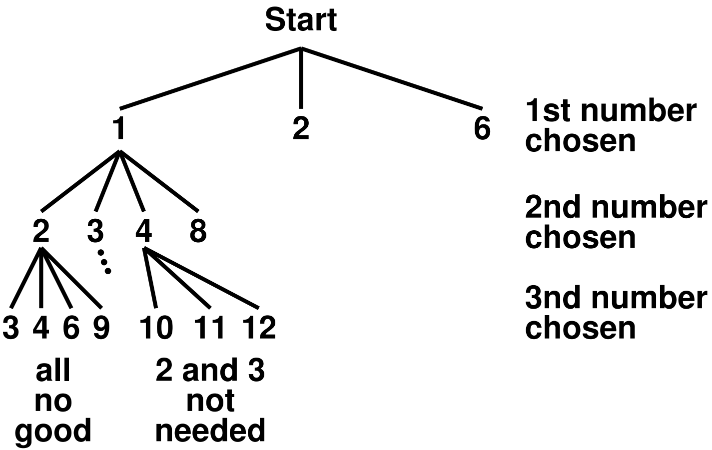

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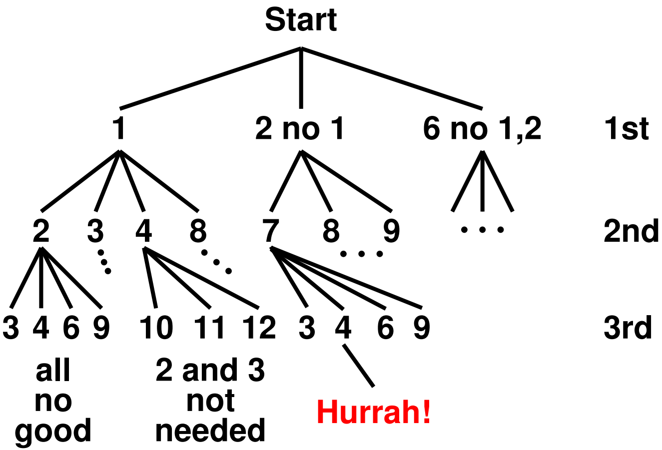

Unique solution: {2,4,7}

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I need another 300 000 CPU years.