Recognising Matrix Groups

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Tucson 2006

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Lots of others contributed ideas, results, and code.

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Variant: $\{\bar{M}_1, \ldots, \bar{M}_k\} \subseteq \mathrm{PGL}_d(\mathbb{F}_q)$, $G := \langle \bar{M}_1, \ldots, \bar{M}_k \rangle$

Straight line programs

Example:

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# input:
r:= [ a, b, c ];
# program:
r[4]:= r[1]^2*r[2]*r[1]^-2;
r[5]:= r[4]*r[3]^7;
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[ r[4], r[5]^5 ]
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- can express long products memory efficiently.

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$\Longrightarrow$ this decision shortened SLPs from 500 000 steps down to 500 in examples

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$\Longrightarrow$ We work “modulo” this problem.

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- Improve algorithms

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- act with matrices on vectors or on subspaces $\longrightarrow$ gives homomorphisms to permutation groups

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- execute S on $M_1, \ldots, M_k$, get $g' \in G$ s.t. $\varphi(g) = \varphi(g')$
- $\Longrightarrow g^{-1} \cdot g' \in N$
  $\longrightarrow$ this creates a (pseudo-) random element in N

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- We have recognised G constructively!

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Choose as "nice generators" $M'_1, \ldots, M'_{k'}$ for G:

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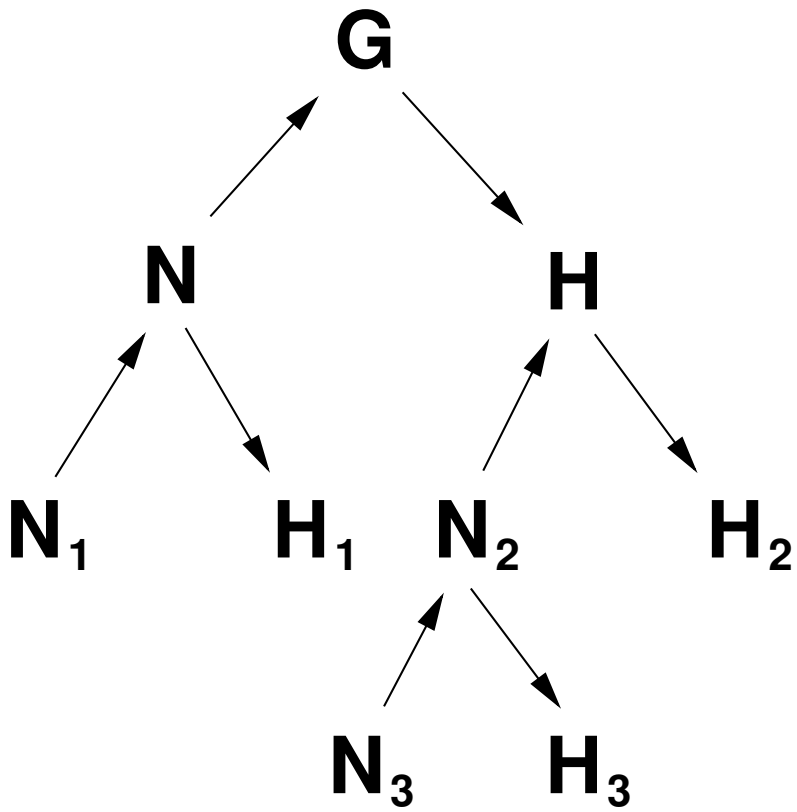
- put together S from S' and S'' plus one multiplication

A Composition Tree



Upward arrows: monomorphisms
Downward arrows: epimorphisms

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Unfortunately, it is not yet analysed to be polynomial-time!

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Theorem (Aschbacher, 1984)

If $G < \mathrm{GL}_d(q)$ then it falls under at least one of:

C1 *G leaves invariant a subspace $0 < W < \mathbb{F}_q^{1\times d}$*

C2 *G preserves a decomposition $\mathbb{F}_q^{1\times d} \cong V_1 \oplus \cdots \oplus V_j$*

C3 *G comes from a bigger field (semilinear)*

C4 *G preserves a decomposition $\mathbb{F}_q^{1\times d} \cong V_1 \otimes V_2$*

C5 *G is realizable over a subfield*

C6 *$G \leq N_{\mathrm{GL}}(r^{1+2k})$ where r^{1+2k} is an extraspecial group*

C7 *G is tensor-induced*

C8 *G contains a "classical group" like $\mathrm{SL}_d(q)$ or $\mathrm{Sp}_d(q)$*

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All classes C1 to C7 are defined "geometrically" and promise some kind of homomorphism or "simplification".

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Classify: Irred. modular representations of finite groups.

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- use collected data about subgroups
- directly recognise the group constructively:
  - use base and strong generating sets (matrix Schreier-Sims)
  - use tricks involving involution centralisers

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Authors: MN and Ákos Seress

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Authors: (currently)
Peter Brooksbank, Maska Law, Steve Linton, MN, Alice Niemeyer, Eamonn O'Brien, Ákos Seress.

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- higher level algorithms after recognition (Sylow subgroups, maximal subgroups, centralisers, normalisers, etc.)

Help is appreciated

Everybody is welcome to contribute.

We need ideas, code, and analysis.