School of Mathematics and Statistics
MT4517 Rings & Fields
Exercises 3

Exercise 3.1. Determine which of the following are subrings of the given rings.

(i) the positive integers in $\mathbb{Z}$;

(ii) all polynomials with integer constant in $\mathbb{Q}[x]$;

(iii) all integers divisible by 3 in $\mathbb{Z}$;

(iv) all polynomials of degree at least 6 in $\mathbb{Q}[x]$;

(v) the set $\{\ 75a + 30b\ :\ a, b \in \mathbb{Z}\ \}$ in $\mathbb{Z}$;

(vi) all the zero divisors of $\mathbb{Z}/(14)$ in $\mathbb{Z}/(16)$.

Also determine which of the examples above are ideals in the respective rings.

Exercise 3.2. Let R denote the set of all subsets of a set S. Define operations $+$ and $*$ on R by

$$A + B = (A \cup B) \setminus (A \cap B) \text{ and } A * B = A \cap B,$$

where $A, B \in R$. Prove that R is a ring. [Aside: R is called a *Boolean ring*.]

Does this ring have an identity element? Which elements of the ring have multiplicative inverses? If we redefine $+$ by $A + B = A \cup B$, do we still get a ring?

Let A be a subset of S. Describe the ideal of R generated by A.

Exercise 3.3. Prove that the set of real polynomials $a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ where $a_0 = a_1 = 0$ is a subring of the polynomial ring $\mathbb{R}[x]$. Is it an ideal?

Exercise 3.4. Prove that the set of all real polynomials $a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ for which the sum $a_0 + a_1 + a_2 + \cdots + a_n = 0$ is an ideal of $\mathbb{R}[x]$.

Exercise 3.5. Prove that the set $\{\ r + s\sqrt{2}\ :\ r, s \in \mathbb{Q}\ \}$ is a field under real addition and multiplication. Prove that it is the smallest subfield of $\mathbb{R}$ which contains $\sqrt{2}$.

Exercise 3.6. What is the ideal of $\mathbb{R}$ generated by $\sqrt{2}$?

Exercise 3.7. If R is a commutative ring with identity whose only ideals are $\{0\}$ and R, prove that R is a field. If R is a commutative ring with identity, do the non-invertible elements of R form an ideal? Prove this or find a counterexample.

Exercise 3.8. Let R be the set of real matrices of the form

$$\begin{pmatrix} a & b \\ 2b & a \end{pmatrix}.$$

Prove that R is a subring of the ring of all real matrices. If we insist that the entries of R are rationals, prove that R is then a field. [Hint: a matrix with entries in a field is invertible if its determinant is non-zero.]

If the entries of R are taken from the ring $\mathbb{Z}/(3)$, prove that R is a field with 9 elements.

Exercise 3.9. Prove Lemma 5.13 from lectures.

Exercise 3.10. Prove that every field is a PID.

Exercise 3.11. Let I and J be ideals in a commutative ring R with identity. Prove that $I \cap J$, $I + J = \{ i + j \, : \, i \in I, \; j \in J \}$, and

$$IJ = \left\{ \sum_{i=1}^{n} a_i b_i \; : \; n \geq 1, \; a_i \in I, \; b_i \in J \right\}$$

are ideals in R.

Prove that $IJ \subseteq I \cap J$. Find examples of ideals I and J such that $IJ \neq I \cap J$. Is $\{ ij \, : \, i \in I, \; j \in J \}$ an ideal?

Exercise 3.12. Let

$$I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$$

be an infinite increasing sequence of ideals in a ring R. Prove that the union of the ideals is an ideal. Show that the union

$$\{ 2m \, : \, m \in \mathbb{Z} \} \cup \{ 3n \, : \, n \in \mathbb{Z} \}$$

of two ideals in $\mathbb{Z}$ is not even a subring of $\mathbb{Z}$.

Exercise 3.13. Let R be a ring with the property that every ideal $I \subseteq R$ is finitely generated, that is, there exist $r_1, \ldots, r_n \in R$ where $I = (r_1, r_2, \ldots, r_n)$. A ring with this property is called *noetherian*. Let

$$I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$$

be an infinite increasing sequence of ideals in a ring R. Prove that there exists $N \in \mathbb{N}$ such that $I_N = I_{N+1} = \cdots$.

Exercise 3.14. Let I be an ideal in a ring R. Prove that $I[x]$ is an ideal in $R[x]$.