Constructions with compass and straightedge - a worksheet

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This worksheet is intended to be a guide for you to learn about constructions with compass and straightedge and how you work with them on a computer using the program "Compass and Ruler":

`http://zirkel.sourceforge.net/`

Here is a sequence of exercises building up on each other. Therefore it is sensible to work through them in their given order. Sometimes there are hints. I suggest that you first try to solve each exercise without reading the hint and once you get stuck, try the hint. If you are still stuck, then please ask for help.

Some notation: We denote points by capital letters, lines and circles by small letters. Everything is done in the plane $\mathbb{R}^2$, usually we do not need a coordinate system, though. A line can be denoted by two of its points, so AB is the line through the points A and B. Sometimes we talk of a "ray" AB, which is the straight line beginning at A, going through B and continuing to infinity. The angle $\measuredangle ABC$ is the angle between the two rays BA and BC in the counterclockwise direction. We denote the distance between two points A and B by $d(A, B)$.

1. Start the Compass and Ruler program. You can find a link to it on the following web page:

   `http://tinyurl.com/cqb5wpf`

   Hint: This uses Java Web Start, you have to allow the execution of the program manually.

2. Try to do the elementary construction steps which were shown in the demonstration: Choose points, draw lines, draw rays, draw circles, intersect lines with lines, lines with circles and circles with circles. Finally move the original points and see what happens to your construction. Name a few of your points.

   Hint: If you move your mouse pointer over one of the pictograms and wait a short time, a little help window will pop up.

3. Given two points A and B, construct the "perpendicular bisector", that is, a line that is perpendicular to the line through A and B such that A and B have the same distance to the line.

   Hint: Draw two circles of the same radius around A and B such that they intersect in two points C and D. The perpendicular bisector is the line through C and D.

4. Given two points A and B, construct their middle point M, that is, the point on the line through A and B that has the same distance to A and B.

   Hint: Use 3. and go on.

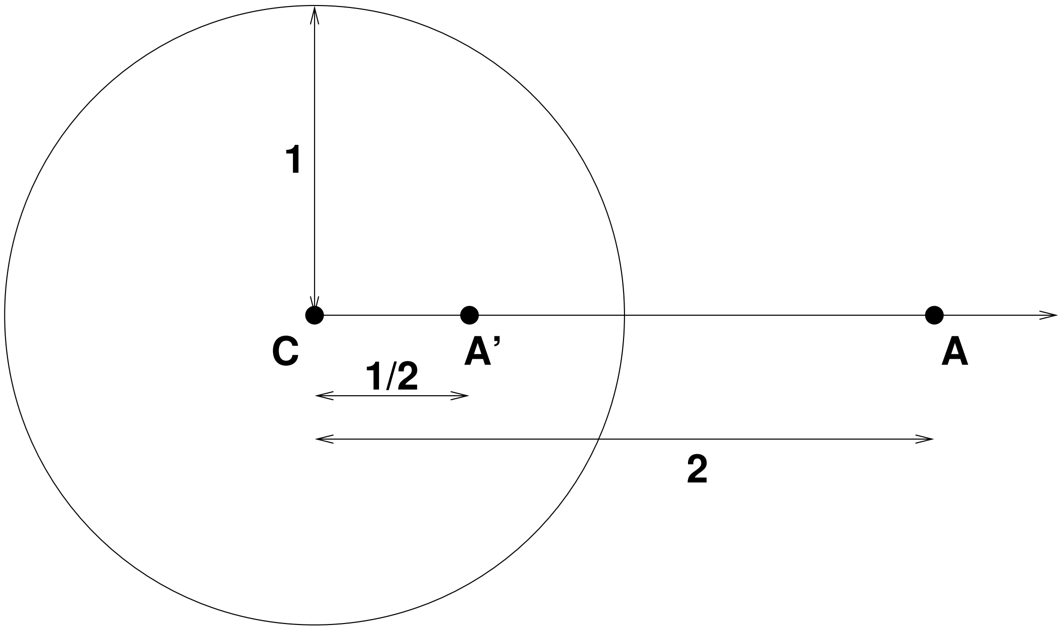
5. Practise saving and restoring a construction, and use the undo facility.

6. Given an angle (that is, three points A, B and C and the two rays BA and BC), construct the angle bisector of $\measuredangle ABC$. You need a point D such that $\measuredangle ABD = \measuredangle DBC$.

7. Given a line AB (by two points A and B) and a point C outside of AB, construct its reflection on the line. That is, the point C' on the other side of the line with the same distance to AB as C, such that the line CC' is perpendicular to AB.

8. Given a point C (the "centre") and a point A, construct the reflection A' of A on C. That is, the point A' on the line AC that has the same distance to C as A but is not equal to A'.

We briefly stop the exercises here to introduce an important concept, the **inversion on a circle**. Let C be a point and c be a circle with radius r and centre C. Then we define a map $i : \mathbb{R}^2 \setminus \{C\} \to \mathbb{R}^2 \setminus \{C\}$ from the set of all points other than C to itself in the following way: Any point A is mapped to the point A' that lies on the ray CA such that the distance $d(A', C)$ from A' to C is $r^2/d(A, C)$.
Note that the centre C cannot be inverted on the circle c, that the points on the circle line are not moved by inversion on c and that A' is mapped back to A by the same inversion. Everything within the circle is mapped to the outside and vice versa.



9. Given a circle c with centre C and a point A, construct the image of A' under the inversion on the circle c.

   Hint: Two triangles are called **similar**, if one is a scaled and rotated version of the other. This means that the three angles of the first each have a corresponding angle in the second that is the same. In two similar triangles ABC and $A'B'C'$ the ratios between corresponding sides are the same: $d(A, B)/d(B, C) = d(A', B')/d(B', C')$. First take an A outside of c and draw a circle a around A running through C. Intersect this new circle a with c and draw two circles c_1 and c_2 around the intersection points and through C. Find some similar triangles in the figure you get and prove that you have found A'. Then invent a procedure for points A within the circle c.

10. Now that we can construct the inversion on a circle, we want to investigate its properties. Choose A on some line b and track A' as A moves along the line b. On what curve moves A'? Can you prove this?

    Hint: The Compass and Ruler program has a tracking facility (14th icon in row 2, next to the "Move point" icon). If you choose A on a given line in the beginning, you can only move it on that line.

11. Choose A on some circle d containing C and track A' as A moves along the circle d. On what curve moves A'? Can you prove this?

12. Choose A on some circle e not containing C and track A' as A moves along the circle e. On what curve moves A'? Can you prove this?

13. Assume you are given a circle c and a line $b = AB$, however, you are only given the two points A and B on b and not the line itself. Assume further that you have forgotten your straightedge at home. Drawing only circles, can you construct the intersection points of the circle with the line b without drawing a single line? (That is, you only use the compass and draw circles!)

    Hint: Invert A and B on c and construct the circle containing A', B' and C. You need its centre D', it is the inversion on c of D, which is the reflection of C on the line AB.

    Note: You can now intersect a line given by two points and a circle just by using the compass and not using a straightedge!

14. Assume you are given two lines AB and CD just by being given four points A, B, C and D. Can you construct the intersection point E of the two lines, only using the compass?

    Hint: Draw additionally a suitable circle.

    Note: You have now proved that everything that can be constructed with a compass and straightedge can also be constructed with the compass alone!

15. Given two points A and B, construct two more points C and D, such that $ABCD$ is a square. Do the same but only use a compass and no straightedge.

    Hint: You are on your own with this one... ☺

16. Construct a regular pentagon.

    Hint: This is a traditional exercise. If you get stuck, search the internet for a solution.

We are now leaving constructions only with the compass. The following two are probably rather difficult.

17. Construct a triangle ABC if you are given the three median lengths.

    Hint: The medians trisect each other.

18. Construct a triangle ABC if you are given the three altitudes h_a, h_b and h_c. **Hint:** Do you know a formula for the area of a triangle involving the altitudes? What does this tell you about the ratio of the altitudes?

    Hint: Choose an arbitrary distance between two points to be 1, your unit interval. First construct distances h'_a, h'_b and h'_c with $h'_a = 1/h_a$ and $h'_b = 1/h_b$ and $h'_c = 1/h_c$. Construct a triangle with edge lengths h'_a, h'_b and h'_c. This triangle is similar to the one you want to construct. Scale it as needed to get the result.