1. The polynomial x^2+1 is irreducible over $\mathbb{F}_3$. Thus $\mathbb{F}_9 \cong \mathbb{F}_3[x]/(x^2+1)$,
let Θ be a root of x^2+1, for example $\Theta := x+(x^2+1)$.
$\mathbb{F}_9^*$ is cyclic of order 8, $\phi(8)=4$ elements are generators, i.e. primitive roots.
We have $\Theta^2=-1$, $\Theta^3=-\Theta$ and $\Theta^4=1$ thus Θ has order 4.
That is, all other elements of $\mathbb{F}_9^*$ are primitive roots: $\{\Theta+1, \Theta+2, 2\Theta+1, 2\Theta+2\}$
Take $\Theta+1$, we get its conjugates by repeatedly powering up to exponent 3:
$$(\Theta+1)^3 = \Theta^3+1 = \Theta(\Theta^2)+1 = \Theta(-1)+1 = 2\Theta+1$$
Thus, the conjugates of $\Theta+1$ are $\{\Theta+1, 2\Theta+1\}$, all are primitive roots.

2. Let $\varphi: \mathbb{C}\to\mathbb{C}$ be an automorphism of $\mathbb{C}$ which fixes $\mathbb{R}$ elementwise.
Then since $x^2+1 = (x+i)(x-i) = x^2+(i+(-i))x+(i)\cdot(-i)$ we see that φ permutes the set $\{i,-i\}$, here we have extended φ to a ring automorphism of $\mathbb{C}[x]$ by applying it to all coefficients of a polynomial.
Another proof: Let $\alpha\in\mathbb{C}$ be $\varphi(i)$. Since $-i=(-1)\cdot i$ we have $\varphi(-i)=\varphi(-1)\cdot\varphi(i) = -1\cdot\alpha = -\alpha$. Since $\underset{1=\varphi(1)}{=}\varphi(i\cdot(-i)) = \alpha\cdot(-\alpha) = -\alpha^2$
we have $\alpha\in\{i,-i\}$. Because $\{1,i\}$ is a basis of $\mathbb{C}$ as $\mathbb{R}$-vectorspace, both proofs show that there is exactly two such automorphisms:
① the identity
② complex conjugation $i\mapsto -i$

3. Assume $\alpha = \gamma^q-\gamma = \beta^q-\beta$ with $\beta,\gamma\in F \Rightarrow \beta^q-\gamma^q = (\beta-\gamma)^q = \beta-\gamma$
and thus $\beta-\gamma\in K=\mathbb{F}_q$.
Assume $\beta-\gamma\in\mathbb{F}_q \Rightarrow (\beta-\gamma)^q = \beta^q-\gamma^q = \beta-\gamma \Rightarrow \alpha = \beta^q-\beta = \gamma^q-\gamma$.

4. $N_{F/K}(\alpha) = \alpha^{\left(\frac{q^m-1}{q-1}\right)}$.
If $\alpha=\beta^{q-1}$ for some $\beta\in F$ then $N_{F/K}(\alpha) = \alpha^{\left(\frac{q^m-1}{q-1}\right)} = \beta^{q^m-1} = 1$.
~~Let $N_{F/K}(\alpha)=1$, that is: $\alpha\cdot\alpha^q\cdot\alpha^{q^2}\cdots\alpha^{q^{m-1}}=1 \Rightarrow \alpha=$~~
$N_{F/K}$ is a surjective group hom. from F^* to K^*, both are cyclic groups of orders q^m-1 and $q-1$ respectively. Let γ be a primitive root of F^* then $N_{F/K}(\gamma)$ is a primitive root of K^* since $N_{F/K}$ is surjective. Thus γ^{q-1} generates the kernel, $\langle\gamma^{q-1}\rangle$ is the subgroup

of elements of F^* which are $(q-1)$-st powers.

5. There are $q^m - 1$ possible values for the first vector b_1 in a basis (has to be $\neq 0$). The second must not lie in the $\mathbb{F}_q$-span of b_1, thus for every b_1 there are $q^m - q$ choices for the second basis vector b_2. For the i-th basis vector to be linearly independent from the first $i-1$ vectors, it must not lie in their $\mathbb{F}_q$-span. That leaves $q^m - q^{i-1}$ choices for b_i for every choice of $(b_1, \dots, b_{i-1})$. The result follows.

<u>NOTE</u>: A vector space of dimension k over $\mathbb{F}_q$ has q^k elements.