Lie Algebras

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Contents

Chapter 1

A survival kit of linear algebra

In this chapter we recall some elementary facts of linear algebra, which are needed throughout the course, in particular to set up notation.

1 Vector spaces and subspaces

Reminder 1.1 (Vector space)
Let $\mathbb{F}$ be a field. An $\mathbb{F}$-vector space V is a set with two operations

$$\begin{array}{ccc} V \times V & \to & V \\ (v, w) & \mapsto & v + w \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathbb{F} \times V & \to & V \\ (\lambda, w) & \mapsto & \lambda \cdot w \end{array}$$

called **addition** and **multiplication by scalars** with the usual axioms (see MT3501 for details).

Example 1.2 (Complex row space)
The set of row vectors of length n containing complex numbers is denoted by

$$\mathbb{C}^{1\times n} := \{[\alpha_1, \ldots, \alpha_n] \mid \alpha_1, \ldots, \alpha_n \in \mathbb{C}\}.$$

It is a $\mathbb{C}$-vector space, we add vectors and multiply them by scalars as exhibited in the following examples:

$$[1, -2, 3] + [4, 5, 6] = [1 + 4, -2 + 5, 3 + 6] = [5, 3, 9]$$

and

$$(-3) \cdot [4, 1/2, -2] = [-3 \cdot 4, (-3) \cdot (1/2), (-2) \cdot (-3)] = [-12, -3/2, 6] = (-1)[12, 3/2, 6].$$

Remarks: Multiplication by -1 is additive inversion, we often leave out the dot $\cdot$ for multiplication.

Reminder 1.3 (Linear combinations, span, spanning set)
If V is a $\mathbb{C}$-vector space, $v_1, \ldots, v_k \in V$ and $\lambda_1, \ldots, \lambda_k \in \mathbb{C}$, then

$$w := \sum_{i=1}^{k} \lambda_i v_i = \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_k v_k$$

is called a **linear combination** of the v_i, we say that "$w \in V$ **is a linear combination** of the v_i", the **coefficients** $\lambda_1, \ldots, \lambda_k$ are **not necessarily uniquely defined**!
The **set of linear combinations**

$$\mathrm{Span}(v_1, \ldots, v_k) := \{w \in V \mid w \text{ is a linear combination of the } v_i\}$$

is called the **span** of the vectors $v_1, \dots, v_k$.
If $M \subseteq V$ is a (possibly infinite) subset, then its span is the **union**

$$\mathrm{Span}(M) := \bigcup_{n\in\mathbb{N}} \left(\bigcup_{v_1,\dots,v_n\in M} \mathrm{Span}(v_1, \dots, v_n) \right),$$

of all spans of all finite sequences of vectors in M.
Remember: Linear combinations are **always finite sums**.

Reminder 1.4 (Subspace)
Let V be a $\mathbb{C}$-vector space. A non-empty subset $W \subseteq V$ is called a **subspace**, if

$$u + v \in W \qquad \text{and} \qquad \lambda u \in W \qquad \text{for all } u, v \in W \text{ and all } \lambda \in \mathbb{C}.$$

In particular, a subspace is itself a $\mathbb{C}$-vector space. In fact, every subspace W is the span of some vectors $v_1, \dots, v_k$ for some k, and every such span is a subspace.

Example 1.5 (Sub-row space)
The following is a subspace of $\mathbb{C}^{1\times 3}$:

$$\mathrm{Span}([1, 0, -1], [0, 2, 1], [1, 2, 0]) = \left\{ [x, y, z] \in \mathbb{C}^{1\times 3} \middle| z = y/2 - x \right\}.$$

Exercise: Prove this equality (consider an arbitrary linear combination of the three row vectors)!

2 Bases, dimension and linear maps

Definition 2.1 (Linear independence)
A tuple $(v_1, \dots, v_k)$ of vectors in a $\mathbb{C}$-vector space V are called **linearly independent**, if one of the following equivalent statements is true:

(i) For arbitrary numbers $\lambda_1, \dots, \lambda_k \in \mathbb{C}$ the following implication holds:

$$\sum_{i=1}^{k} \lambda_k v_k = 0 \qquad \Longrightarrow \qquad \lambda_1 = \lambda_2 = \cdots = \lambda_k = 0.$$

(ii) Every vector in $\mathrm{Span}(v_1, \dots, v_k)$ can be expressed as a linear combination of the vectors $v_1, \dots, v_k$ **in a unique way**.

(iii) No vector v_i is contained in the span of the others:

$$v_i \notin \mathrm{Span}(v_j \mid 1 \le j \le k, j \ne i) \qquad \text{for all } i.$$

Otherwise the tuple is called **linearly dependent**. Linear dependence is a property of **the tuple** and **not** of **the individual vectors**.

Example 2.2 (Linear independent vectors)
The tuple of vectors

$$([5, 0, 2], [2, 3, 0], [-1, 0, 0])$$

is linearly independent.

Definition 2.3 (Basis of a vector space)
Let V be a $\mathbb{C}$-vector space. A tuple $(v_1, \dots, v_n)$ of vectors in V is called a **basis of** V, if

- $V = \mathrm{Span}(v_1, \ldots, v_n)$ and
- $(v_1, \ldots, v_n)$ is linearly independent.

Theorem 2.4 (Dimension)
In a $\mathbb{C}$-vector space V any two bases have the same number of elements. The number of elements in an arbitrary basis of V is called the **dimension** of V.

Note: In this course, we only deal with **finite-dimensional** vector spaces.

Example 2.5
The $\mathbb{C}$-vector space $\mathbb{C}^{1\times n}$ is n-dimensional because

$$([1, 0, \ldots, 0], [0, 1, 0, \ldots, 0], \ldots, [0, \ldots, 0, 1])$$

is a basis of length n, it is called the **standard basis**.

Reminder 2.6 (Linear maps)
Let V and W be $\mathbb{C}$-vector spaces. A map $\varphi : V \to W$ is called **$\mathbb{C}$-linear**, if

$$(u + v)\varphi = u\varphi + v\varphi \quad \text{and} \quad (\lambda v)\varphi = \lambda(v\varphi)$$

for all $u, v \in V$ and all $\lambda \in \mathbb{C}$. We write all maps on the right hand side.

Theorem 2.7 (Linear map determined by values on a basis)
Let $(v_1, \ldots, v_n)$ be a basis of a $\mathbb{C}$-vector space V and let W be a $\mathbb{C}$-vector space. Then for every tuple $(w_1, \ldots, w_n)$ of vectors in W, there is a unique linear map $\varphi : V \to W$ with $v_i\varphi = w_i$ for $1 \le i \le n$, it maps

$$\left(\sum_{i=1}^{n} \lambda_i v_i\right) \qquad \text{to} \qquad \left(\sum_{i=1}^{n} \lambda_i v_i\right)\varphi := \sum_{i=1}^{n} \lambda_i w_i \in W.$$

Example 2.8 (Example for a linear map)
The map

$$\mathbb{C}^{1\times 3} \to \mathbb{C}^{1\times 3}, \quad [x, y, z] \mapsto [2x - y + 3z, x + z, -x + 7z + 6y]$$

is $\mathbb{C}$-linear. It is uniquely defined by doing

$$[1, 0, 0] \mapsto [2, 1, -1] \quad \text{and} \quad [0, 1, 0] \mapsto [-1, 0, 6] \quad \text{and} \quad [0, 0, 1] \mapsto [3, 1, 7].$$

Theorem 2.9 (Matrix of a linear map)
Let V and W be $\mathbb{C}$-vector spaces, and $(v_1, \ldots, v_m)$ and $(w_1, \ldots, w_n)$ be bases of V and W respectively. Then there is a $\mathbb{C}$-linear bijection between the set of $\mathbb{C}$-linear maps from V to W and the set $\mathbb{C}^{m\times n}$ of $m \times n$-matrices with entries in $\mathbb{C}$, given by

$$\varphi \mapsto [a_{i,j}]_{1\le i\le m, 1\le j\le n} \qquad \text{where } v_i\varphi = \sum_{j=1}^{n} a_{i,j} w_j \quad \text{for all } i.$$

Note that this convention might be different from what you know, it comes from the fact that we write mappings on the right hand side and use row vectors.
For three spaces, the composition $\varphi \cdot \psi$ (do φ first, then ψ) is mapped to the **matrix product** of the matrices corresponding to φ and ψ respectively, if the same basis is chosen in the range of φ and the source of ψ.

Example 2.10 (Continuation of Example 2.8)
The matrix of the linear map in Example 2.8 with respect to the standard basis (in both domain and range) is

$$\begin{bmatrix} 2 & 1 & -1 \\ -1 & 0 & 6 \\ 3 & 1 & 7 \end{bmatrix}$$

since $[1,0,0]\varphi = [2,1,-1] = 2 \cdot [1,0,0] + 1 \cdot [0,1,0] + (-1) \cdot [0,0,1]$.

Definition 2.11 (Endomorphisms)
For a $\mathbb{C}$-vector space V we denote the set of $\mathbb{C}$-linear maps from V to V by $\mathrm{End}(V)$ and call them **linear endomorphisms**. The subset (in fact, subgroup) of **invertible** endomorphisms is denoted by $\mathrm{GL}(V)$. We call an endomorphism $\varphi \in \mathrm{End}(V)$ **nilpotent**, if there is an $n \in \mathbb{N}$ with $\varphi^n = 0$.

3 Direct sums

Definition 3.1 (Direct sum)
The $\mathbb{C}$-vector space V is said to be the **direct sum** $U \oplus W$ of two subspaces U and W of V, if one and thus both of the following equivalent conditions holds:

- $V = U + W := \{u + w \mid u \in U, w \in W\}$ and $U \cap W = \{0\}$,
- every vector $v \in V$ can be written as a sum $u + w$ of a vector $u \in U$ and a vector $w \in W$ **in a unique way**.

Both statements generalise more than two subspaces: The $\mathbb{C}$-vector space V is said to be the **direct sum** $U_1 \oplus \cdots \oplus U_k$ of k subspaces $U_1, \ldots, U_k$ if one and thus both of the following equivalent conditions holds:

- $V = U_1 + \cdots + U_k := \{u_1 + u_2 + \cdots + u_k \mid u_i \in U_i\}$ and
  $U_i \cap (U_1 + \cdots + U_{i-1} + U_{i+1} + \cdots + U_k) = \{0\}$ for $1 \leq i \leq k$.
- Every vector $v \in V$ can be written as a sum $u_1 + \cdots + u_k$ of vectors $u_i \in U_i$ for $1 \leq i \leq k$ **in a unique way**.

Theorem 3.2 (Basis of a direct sum)
If $V = U \oplus W$ and $(u_1, \ldots, u_m)$ is a basis of U and $(w_1, \ldots, w_n)$ is a basis of W, then

$$(u_1, \ldots, u_m, w_1, \ldots, w_n)$$

is a basis of V and we have $\dim(V) = \dim(U) + \dim(W)$.

Example 3.3 (Direct sum decomposition)
We have

$$\mathbb{C}^{1\times 3} = \mathrm{Span}([1,2,3]) \oplus \left\{[x,y,z] \in \mathbb{C}^{1\times 3} \mid z = x - y\right\}.$$

Exercise: Prove this statement.

Remark 3.4 (Complements)
Note that for every subspace U of a $\mathbb{C}$-vector space V there is a (not necessarily unique) subspace W of V such that $V = U \oplus W$.

Chapter 2

Fundamental definitions

4 Lie algebras

Definition 4.1 (Lie algebra)
A **Lie algebra** is a vector space L over a field $\mathbb{F}$ together with a multiplication

$$L \times L \to L, (x, y) \mapsto [x, y],$$

satisfying the following axioms:

(L1) $[x + y, z] = [x, z] + [y, z]$ and $[x, y + z] = [x, y] + [x, z]$,

(L2) $[\lambda x, y] = [x, \lambda y] = \lambda[x, y]$,

(L3) $[x, x] = 0$, and

(L4) $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$,

whenever $x, y, z \in L$ and $\lambda \in \mathbb{F}$. Axiom (L4) is called the **Jacobi identity**.
Note that $[[x, y], z]$ is **not necessarily** equal to $[x, [y, z]]$, we do **not** have associativity!

Remark: In this course, we will mostly study Lie algebras over the complex field $\mathbb{C}$.

Lemma 4.2 (First properties)
Let L be a Lie algebra over a field $\mathbb{F}$. Then $[x, y] = -[y, x]$ for all $x, y \in L$. The Lie multiplication is **anticommutative**.

Proof. We have $0 = [x + y, x + y] = [x, x] + [x, y] + [y, x] + [y, y] = [x, y] + [y, x]$. ■

Example 4.3 (Abelian Lie algebras)
Every $\mathbb{F}$-vector space L with $[x, y] = 0$ for all $x, y \in L$ is a Lie algebra. Such a Lie algebra is called **abelian**. Abelian Lie algebras are somewhat boring.

Example 4.4 (Lie($\mathcal{A}$)**, the Lie algebra of an associative algebra)**
Let $\mathcal{A}$ be an associative algebra over a field $\mathbb{F}$. That is, $\mathcal{A}$ is a ring with identity together with a ring homomorphism $\iota : \mathbb{F} \to Z(\mathcal{A})$ where $Z(\mathcal{A}) := \{x \in \mathcal{A} \mid xy = yx \text{ for all } y \in \mathcal{A}\}$ is the centre of $\mathcal{A}$, the set of elements of $\mathcal{A}$ that commute with every other element. Such an $\mathcal{A}$ is then automatically an $\mathbb{F}$-vector space by setting $\lambda \cdot a := \iota(\lambda) \cdot a$ for $\lambda \in \mathbb{F}$ and $a \in \mathcal{A}$. In particular, the multiplication of $\mathcal{A}$ is associative: $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.
If you do not remember this structure, simply think of $\mathcal{A} = \mathbb{C}^{n\times n}$, the set of all $n \times n$-matrices with componentwise addition and matrix multiplication. The map ι here is the embedding of $\mathbb{C}$ into the scalar multiples of the identity matrix.

Every associative algebra $\mathcal{A}$ becomes a Lie algebra by defining the Lie product in this way:

$$[x, y] := x \cdot y - y \cdot x \qquad \text{for all } x, y \in \mathcal{A}.$$

We check the axioms:

(L1) $[x + y, z] = (x + y) \cdot z - z \cdot (x + y) = x \cdot z + y \cdot z = [x, z] + [y, z]$.

(L2) $[\lambda x, y] = (\lambda x) \cdot y - y \cdot (\lambda x) = x \cdot (\lambda y) - (\lambda y) \cdot x = [x, \lambda y]$ and this is equal to $\lambda(x \cdot y - y \cdot x)$.

(L3) $[x, x] = x \cdot x - x \cdot x = 0$.

(L4) $[[x, y], z] = [xy - yx, z] = xyz - yxz + zxy - zyx$, permuting cyclically and adding up everything shows the Jacobi identity.

For a $\mathbb{C}$-vector space V, the set of endomorphisms $\mathrm{End}(V)$ (linear maps of V into itself) is an associative algebra with composition as multiplication. The map ι here is the embedding of $\mathbb{C}$ into the scalar multiples of the identity map.
We set $\mathrm{gl}(V) := \mathrm{Lie}(\mathrm{End}(V))$. By choosing a basis of V this is the same as $\mathrm{Lie}(\mathbb{C}^{n\times n})$ if $\dim_{\mathbb{C}}(V) = n$ (see 4.8 below). Thus, we can compute in $\mathrm{gl}(\mathbb{C}^{1\times 2})$

$$\left[\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}\right] = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix},$$

using the standard basis.

Example 4.5 (Vector product)
Let $L := \mathbb{R}^{1\times 3}$ be the 3-dimensional real row space with the following product:

$$[[a, b, c], [x, y, z]] := [a, b, c] \times [x, y, z] := [bz - cy, cx - az, ay - bx].$$

This is a Lie algebra over the field $\mathbb{R}$ of real numbers.
Note: The vector $v := [a, b, c] \times [x, y, z]$ is zero if and only if the vectors $[a, b, c]$ and $[x, y, z]$ are parallel. Otherwise v is orthogonal to both $[a, b, c]$ and $[x, y, z]$ and its length is equal to the area of the parallelogram spanned by $[a, b, c]$ and $[x, y, z]$.
Exercise: Check the Jacobi identity for this Lie algebra.

Example 4.6 (sl_2)
Let sl_2 be the subspace of $\mathbb{C}^{2\times 2}$ containing all matrices of trace 0:

$$\mathrm{sl}_2 := \left\{M \in \mathbb{C}^{2\times 2} \mid \mathrm{Tr}(M) = 0\right\}$$

(remember, the trace $\mathrm{Tr}(M)$ of a square matrix M is the sum of the main diagonal entries).
Then sl_2 with

$$[A, B] := A \cdot B - B \cdot A \qquad \text{for all } A, B \in \mathrm{sl}_2$$

as Lie product is a Lie algebra, since $\mathrm{Tr}(A \cdot B) = \mathrm{Tr}(B \cdot A)$ for arbitrary square matrices A and B. This Lie algebra will play a **major role** in this whole theory! It is somehow the **smallest interesting building block**.

Definition 4.7 (Homomorphisms, isomorphisms)
Let L_1 and L_2 be Lie algebras over the same field $\mathbb{F}$. A **homomorphism of Lie algebras** from L_1 to L_2 is a linear map $\varphi : L_1 \to L_2$, such that

$$[x, y]\varphi = [x\varphi, y\varphi] \qquad \text{for all } x, y \in L_1.$$

If φ is bijective, then it is called an **isomorphism of Lie algebras**.

Example 4.8 ($\mathrm{gl}(\mathbb{C}^{1\times n})$ and $\mathrm{Lie}(\mathbb{C}^{n\times n})$ are isomorphic)
Choosing a basis $(v_1, \ldots, v_n)$ of the $\mathbb{C}$-vector space $\mathbb{C}^{1\times n}$ gives rise to an isomorphism of Lie algebras $\mathrm{gl}(\mathbb{C}^{1\times n}) \cong \mathrm{Lie}(\mathbb{C}^{n\times n})$ by mapping a linear map $\varphi : \mathbb{C}^{1\times n} \to \mathbb{C}^{1\times n}$ to its matrix with respect to the basis $(v_1, \ldots, v_n)$ as in Theorem 2.9.

Definition 4.9 (Subalgebras and ideals)
Let L be a Lie algebra over a field $\mathbb{F}$ and let H and K be subspaces of L. We then set

$$[H, K] := \mathrm{Span}\left(\left\{[h, k] \in L \;\middle|\; h \in H, k \in K\right\}\right).$$

Note $[H, K] = [K, H]$ and that we have to use Span here to ensure that this is a subspace of L.
A **Lie subalgebra** or short **subalgebra** of L is a subspace H with $[H, H] \leq H$.
A **Lie ideal** or short **ideal** of L is a subspace K with $[K, L] \leq K$.
Obviously, every ideal is a subalgebra.

Example 4.10 (Centre and derived subalgebra)
Let L be a Lie algebra over a field $\mathbb{F}$. Then the **centre** $Z(L) := \{x \in L \mid [x, y] = 0 \text{ for all } y \in L\}$ and the **derived algebra** $[L, L] := \mathrm{Span}(\{[x, y] \mid x, y \in L\})$ are ideals in L.

Definition 4.11 (Normaliser and centraliser)
Let L be a Lie algebra over a field $\mathbb{F}$ and let H be a subspace of L (not necessarily a subalgebra!). We then define the **normaliser** $N_L(H)$ of H in L to be the space

$$N_L(H) := \{x \in L \mid [x, H] \subseteq H\}.$$

We define the **centraliser** $C_L(H)$ of H in L to be the space

$$C_L(H) := \{x \in L \mid [x, H] = 0\}.$$

Exercise: Use the Jacobi identity to show that both the normaliser and the centraliser are Lie subalgebras of L.

Proposition 4.12 (Properties of subalgebras)
Let H and K be subspaces of a Lie algebra L over $\mathbb{F}$ and let $H + K := \{h + k \mid h \in H, k \in K\}$ be their sum as subspaces.

(i) If H and K are subalgebras, then $H \cap K$ is.

(ii) If H and K are ideals, then $H \cap K$ is.

(iii) If H is an ideal and K is a subalgebra, then $H + K$ is a subalgebra of L.

Proof. Left as an exercise for the reader. ■

Example 4.13 ($\mathrm{Lie}(\mathbb{C}^{n\times n})$ revisited)
Let $L = \mathbb{C}^{n\times n}$ with $n \geq 2$.
The subspace H of matrices with trace 0 is an ideal since $\mathrm{Tr}(A \cdot B) = \mathrm{Tr}(B \cdot A)$ for arbitrary matrices A and B and thus $\mathrm{Tr}([A, B]) = 0$ for all $A \in H$ and $B \in L$.
The subspace K of skew-symmetric matrices, i.e. $\left\{A \in \mathbb{C}^{n\times n} \mid A^t = -A\right\}$ where A^t is the transposed matrix of A, is a subalgebra but not an ideal: If $A, B \in K$, then

$$[A, B]^t = (A \cdot B - B \cdot A)^t = B^t \cdot A^t - A^t \cdot B^t = (-B) \cdot (-A) - (-A) \cdot (-B) = [B, A] = -[A, B].$$

Exercise: Show that K is not an ideal.

Lemma 4.14 (Kernels of homomorphisms are ideals and images are subalgebras)
Let L and H be Lie algebras over a field $\mathbb{F}$ and $\varphi : L \to H$. Then the image $I := \mathrm{im}\,(\varphi)$ is a Lie subalgebra of H and the kernel $K := \ker(\varphi)$ is a Lie ideal of L.

Proof. Since φ is $\mathbb{F}$-linear, both I and K are subspaces.
If $x, y \in I$, there are $\tilde{x}, \tilde{y} \in L$ with $\tilde{x}\varphi = x$ and $\tilde{y}\varphi = y$. Then $[x, y] = [\tilde{x}, \tilde{y}]\varphi \in I$ as well.
If $x \in K$, that is, $x\varphi = 0$, then for $y \in L$ we have $[x, y]\varphi = [x\varphi, y\varphi] = [0, y\varphi] = 0$ and thus $[x, y] \in K$ as well. ■

Note: In fact, Lie ideals are exactly the kernels of Lie algebra homomorphisms, as we will see next.

Definition 4.15 (Quotient Lie algebra)
Let L be a Lie algebra over a field $\mathbb{F}$ and K an ideal of L. Then the **quotient space**

$$L/K := \{x + K \mid x \in L\}$$

consisting of the **cosets** $x + K := \{x + k \mid k \in K\}$ for $x \in L$, is a Lie algebra by defining

$$(x+K)+(y+K) := (x+y)+K \text{ and } \lambda\cdot(x+K) := (\lambda\cdot x)+K \text{ and } [x+K, y+K] := [x, y]+K$$

for all $x, y \in L$ and all $\lambda \in \mathbb{F}$. This is well-defined because K is an ideal, and it inherits all the axioms directly from L. There is a surjective homomorphism $\pi : L \to L/K, x \mapsto x + K$ of Lie algebras called the **canonical map**.

Proof. Lots of little details have to be checked here. Most of it is just the standard construction of the quotient vector space, which can be found in every book on linear algebra and we do not repeat them here. The most important additional one is the well-definedness of the Lie product: Assume $x + K = \tilde{x} + K$ and $y + K = \tilde{y} + K$, that is, $\tilde{x} = x + k_1$ and $\tilde{y} = y + k_2$ for some $k_1, k_2 \in K$. Then

$$[x + k_1, y + k_2] = [x, y] + \underbrace{[x, k_2] + [k_1, y] + [k_1, k_2]}_{\in K}$$

but all three latter products lie in K because K is an ideal. All statements about π are routine verifications. ■

Theorem 4.16 (First Isomorphism Theorem)
Let $\varphi : L \to H$ a homomorphism of Lie algebras over a field $\mathbb{F}$ and $K := \ker(\varphi)$. Then

$$\begin{array}{rccc} \psi : & L/K & \to & \mathrm{im}\,(\varphi) \\ & x+K & \mapsto & x\varphi \end{array}$$

is an isomorphism of Lie algebras.

Proof. The map ψ is well-defined since $x + K = y + K$ is equivalent to $x - y \in K = \ker(\varphi)$ and thus $x\varphi = y\varphi$. This also proves that ψ is injective, and surjectivity to the image of φ is obvious. The map ψ is clearly $\mathbb{F}$-linear and a homomorphism of Lie algebras because φ is. ■

Theorem 4.17 (Second Isomorphism Theorem)
Let L be a Lie algebra, K an ideal and H a subalgebra. Then $H \cap K$ is an ideal of H and the map

$$\begin{array}{rccc} \psi : & H/(H\cap K) & \to & (H+K)/K \\ & h+(H\cap K) & \mapsto & h+K \end{array}$$

is an isomorphism of Lie algebras.

Proof. The ideal K of L is automatically an ideal of the subalgebra $H + K$ (see Proposition 4.12.(iii)). Define a map $\tilde{\psi} : H \to (H + K)/K$ by setting $h\tilde{\psi} := h + K$. This is clearly linear and a homomorphism of Lie algebras. Its image is all of $(H + K)/K$ since every coset in there has a representative in H. The kernel of $\tilde{\psi}$ is exactly $H \cap K$ and it follows that this is an ideal in H. The First Isomorphism Theorem 4.16 then does the rest.
Alternative proof (to get more familiar with quotient arguments): The subspace $H + K$ is a subalgebra by Proposition 4.12.(iii) and K is an ideal in $H + K$ because it is one even in L. The subspace $H \cap K$ is an ideal in H because $[h, l] \in H \cap K$ for all $h \in H$ and $l \in H \cap K$. Thus we can form both quotients.
The map ψ is well-defined, since if $h + (H \cap K) = \tilde{h} + (H \cap K)$, that is, $h - \tilde{h} \in H \cap K$, then in particular $h - \tilde{h} \in K$ and thus $h + K = \tilde{h} + K$. In fact, this reasoning immediately shows that ψ is injective. The map ψ is clearly linear and a homomorphism of Lie algebras by routine verification. It is surjective since every coset in $(H + K)/K$ has a representative in H. ■

5 Nilpotent and soluble Lie algebras

Definition 5.1 (Simple and trivial Lie algebras)
A Lie algebra L is called **simple**, if it is non-abelian (that is, the Lie product is not constant zero) and has no ideals other than 0 and L. A one-dimensional Lie algebra is automatically abelian and is called **the trivial Lie algebra**.

Example 5.2 (sl_2 **revisited)**
The 3-dimensional Lie algebra $L := \mathrm{sl}_2$ from Section 4.6 is simple. Let

$$x := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad y := \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \qquad \text{and} \qquad h := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

we then have $L = \mathrm{Span}(x, y, h)$ and the relations

$$[x, y] = h \qquad \text{and} \qquad [h, x] = 2x = -[x, h] \qquad \text{and} \qquad [h, y] = -2y = -[y, h].$$

Let $0 \neq K \leq L$ be an ideal of L and $0 \neq z := ax + by + ch \in K$ with $a, b, c \in \mathbb{C}$. Then we have $[[z, x], x] = -2bx$ and $[[z, y], y] = 2ay$ and thus, if either a or b is non-zero, then $K = L$. If otherwise $a = b = 0$, then $c \neq 0$ since $z \neq 0$ and thus $K = L$ as well because $[z, x] = 2cx$ and $[z, y] = -2cy$. Thus the only ideals of L are 0 and L itself.

Definition 5.3 (Lower central series, nilpotent Lie algebra)
Let L be a Lie algebra over a field $\mathbb{F}$. We define the **lower central series** as $L^0 := L$ and $L^i := [L^{i-1}, L]$ for $i \geq 1$. This gives a descending sequence of ideals

$$L = L^0 \supseteq L^1 = [L, L] \supseteq L^2 \supseteq \cdots$$

The Lie algebra L is called **nilpotent**, if there is an $n \in \mathbb{N}$ with $L^n = 0$.
Exercise: Convince yourself that all L^i are in fact ideals of L. Remember that $[L, L]$ is the **span** of all bracket expressions $[x, y]$ for $x, y \in L$.

Example 5.4 (Strictly lower triangular matrices)
Every abelian Lie algebra L is nilpotent, since $L^1 = [L, L] = 0$. No simple Lie algebra L is nilpotent, since $[L, L]$ is a non-zero ideal and thus equal to L. In particular, sl_2 from 4.6 is not nilpotent.
Let L be the subalgebra of $\mathrm{Lie}(\mathbb{C}^{n\times n})$ of stricly lower triangular matrices (with zeros on the diagonal). Then L has dimension $n(n-1)/2$ and L^i is strictly smaller than L^{i-1} for $i \geq 1$ and $L^n = 0$, thus L is nilpotent. This is proved by proving that

$$L^i = \left\{ \left(m_{j,k}\right) \in \mathbb{C}^{n\times n} \mid m_{j,k} = 0 \text{ if } j \leq k + i \right\}$$

using induction. (**See exercise sheet 1**.)

Definition 5.5 (Derived series, soluble Lie algebra)
Let L be a Lie algebra over a field $\mathbb{F}$. We then define the **derived series** as $L^{(0)} := L$ and $L^{(i)} := [L^{(i-1)}, L^{(i-1)}]$ for $i \geq 1$. This gives a descending sequence of ideals

$$L = L^{(0)} \supseteq L^{(1)} = [L, L] \supseteq L^{(2)} \supseteq \cdots$$

The Lie algebra L is called **soluble**, if there is an $n \in \mathbb{N}$ with $L^{(n)} = 0$.
Exercise: Convince yourself that all $L^{(i)}$ are in fact ideals of L. Remember that $[L, L]$ is the **span** of all bracket expressions $[x, y]$ for $x, y \in L$.

Example 5.6 (Lower triangular matrices)
Every abelian Lie algebra L is soluble, since $L^{(1)} = [L, L] = 0$. No simple Lie algebra L is soluble, since $[L, L]$ is a non-zero ideal and thus equal to L. In particular, sl_2 from 4.6 is not soluble.
Let L be the subalgebra of sl_2 (see 4.6) of lower triangular matrices

$$L := \left\{ \begin{bmatrix} a & 0 \\ b & -a \end{bmatrix} \,\middle|\, a, b \in \mathbb{C} \right\}.$$

Then $L^{(1)} = [L, L]$ are the subset of matrices with zeros on the diagonal and $L^{(2)} = 0$, thus L is soluble. Note that it is not nilpotent.

Theorem 5.7 (Nilpotent and soluble Lie algebras)
Let L be a Lie algebra H a subalgebra and K an ideal. Then the following hold:

(i) If L is abelian, nilpotent or soluble, then the same is true for H and L/K.

(ii) If K and L/K are soluble then L is soluble, too.

(iii) If $L/Z(L)$ is nilpotent, then so is L.

(iv) $[L^k, L^m] \leq L^{k+m}$ for all $k, m \in \mathbb{N}$.

(v) $L^{(m)} \leq L^{2^{m-1}}$ for all $m \in \mathbb{N}$.

(vi) Every nilpotent Lie algebra is soluble.

(vii) If L is nilpotent and $L \neq 0$, then $Z(L) \neq 0$.

Proof. If L is abelian, then clearly all Lie products in H and L/K are zero as well which proves (i) for "abelian". For nilpotent and soluble the inclusions

$$H^i \subseteq L^i \qquad \text{and} \qquad H^{(i)} \subseteq L^{(i)}$$

and the equations

$$(L/K)^i = \{x + K \mid x \in L^i\} \qquad \text{and} \qquad (L/K)^{(i)} = \{x + K \mid x \in L^{(i)}\}$$

immediately imply (i).
For (ii) assume that both K and L/K are soluble, that is, there are $m, k \in \mathbb{N}$ such that $(L/K)^{(m)} = \{0 + K\}$ and $K^{(k)} = 0$. But the former directly implies $L^{(m)} \subseteq K$ and thus $L^{(m+k)} = 0$ which shows that L is soluble. Note that the same proof for nilpotent does not work!
To prove (iii) assume $L^n \subseteq Z(L)$, then $L^{n+1} = [L^n, L] \subseteq [Z(L), L] = 0$.

Now we consider (iv). The statement holds for $m = 1$ by definition of $L^{k+1} = [L^k, L]$ since $[L^k, L^1] = [L^k, [L, L]] \leq [L^k, L] = L^{k+1}$. Next we use induction on m. Suppose (iv) is true for all $m \leq r$ and all k. Then

$$\begin{aligned} [L^k, L^{r+1}] &= [L^k, [L^r, L]] = [[L^r, L], L^k] \overset{(1)}{\leq} [[L^k, L^r], L] + [[L, L^k], L^r] \\ &\overset{(2)}{\leq} [L^{k+r}, L] + [L^{k+1}, L^r] \overset{(3)}{\leq} L^{k+r+1}, \end{aligned}$$

where the inequality (1) follows from the Jacobi identity $[[x, y], z] = -[[z, x], y] - [[y, z], x]$ for all $x, y, z \in L$ and (2) and (3) follow by induction.

Statement (v) now follows by induction on m. Namely, we have $L^{(1)} = [L, L] = L^1 = L^{2^0}$ for the induction start. Suppose $L^{(i)} \leq L^{2^{i-1}}$, then $L^{(i+1)} = [L^{(i)}, L^{(i)}] \leq [L^{2^{i-1}}, L^{2^{i-1}}] \leq L^{2^{i-1}+2^{i-1}} = L^{2^i}$ by using (iv).

For statement (vi) suppose that L is nilpotent, that is, there is an n such that $L^n = 0$ and thus all $L^k = 0$ for all $k \geq n$. But then $L^{(n)} \leq L^{2^{n-1}} = 0$ as well since $2^{n-1} \geq n$ for all $n \geq n$. Thus L is soluble.

Finally for statement (vii) note that the last non-zero term in the lower central series is contained in the centre $Z(L)$. ■

Theorem 5.8 (Radical)
Let L be a Lie algebra and H_1, H_2 soluble ideals of L. Then $H_1 + H_2$ is a soluble ideal of L, too. Furthermore, if L is finite-dimensional, there is a soluble ideal $\mathrm{rad}(L)$ of L that contains every soluble ideal of L. It is called the **radical** or L.

Proof. Suppose H_1 and H_2 are soluble ideals. Then $(H_1 + H_2)/H_1 \cong H_2/(H_1 \cap H_2)$ by the Second Isomorphism Theorem 4.17 and it follows from Proposition 5.7.(i) that this is soluble as quotient of the soluble Lie algebra H_2. But then H_1 is an ideal of $H_1 + H_2$ such that both the quotient $(H_1 + H_2)/H_1$ and the ideal are soluble, so by Proposition 5.7.(ii) the Lie algebra $H_1 + H_2$ is soluble as well.

If L is finite-dimensional, then there is a soluble ideal K of maximal dimension. By the above reasoning and maximality this ideal contains every other soluble ideal and is thus uniquely determined. It is called the radical and denoted by $\mathrm{rad}(L)$. ■

Definition 5.9 (Semisimple Lie algebra)
A Lie algebra L over a field $\mathbb{F}$ is called semisimple if it has no soluble ideals other than 0.

Lemma 5.10 (Radical quotient is semisimple)
For every finite-dimensional Lie algebra L, the quotient Lie algebra $L/\mathrm{rad}(L)$ is semisimple.

Proof. The preimage of any soluble ideal of $L/\mathrm{rad}(L)$ under the canonical map $L \to L/\mathrm{rad}(L)$ would be a soluble ideal of L that properly contains $\mathrm{rad}(L)$ (use Proposition 5.7.(ii) again). ■

Example 5.11 (Direct sums of simple Lie algebras are semisimple)
Every simple Lie algebra L is semisimple, since it contains no ideals other than L and 0 and L is not soluble (see 5.6). The direct sum $L_1 \oplus \cdots \oplus L_k$ of simple Lie algebras $L_1, \ldots, L_k$ is semisimple.

Proof. By the direct sum we mean the direct sum of vector spaces with component-wise Lie product. It is a routine verification that this makes the direct sum into a Lie algebra, such that every summand L_i is an ideal, since $[L_i, L_j] = 0$ for $i \neq j$ in this Lie algebra.

Assume now that K is any ideal of the sum $L_1 \oplus \cdots \oplus L_k$. We claim that for every summand L_i we either have $L_i \subseteq K$ or $L_i \cap K = \{0\}$. This is true, because $L_i \cap K$ is an ideal in L_i and L_i is simple. Thus, K is the (direct) sum of some of the L_i. However, if $K \neq \{0\}$, then K is not soluble, since $[L_i, L_i] = L_i$ for all i and at least one L_i is fully contained in K in this case. ■

In fact, we will prove the following theorem later in the course:

Theorem 5.12 (Characterisation of semisimple Lie algebras)
A Lie algebra L over $\mathbb{C}$ is semisimple if and only if it is the direct sum of minimal ideals which are simple Lie algebras.

Proof. See later. ■

We can now formulate the ultimate goal of this course:

$$\boxed{\textbf{Classify all finite-dimensional, semisimple Lie algebras over } \mathbb{C} \textbf{ up to isomorphism.}}$$

In view of the promised Theorem 5.12, this amounts to proving this theorem and classifying the simple Lie algebras over $\mathbb{C}$ up to isomorphism.

6 Lie algebra representations

Definition 6.1 (Lie algebra representation)
Let L be a Lie algebra over the field $\mathbb{F}$. A **representation** of L is a Lie algebra homomorphism

$$\rho : L \to \mathrm{Lie}(\mathrm{End}(V))$$

for some $\mathbb{F}$-vector space V of dimension $n \in \mathbb{N}$, which is called the **degree** of ρ. This means nothing but: ρ is a linear map and

$$[x, y]\rho = [x\rho, y\rho] = (x\rho) \cdot (y\rho) - (y\rho) \cdot (x\rho)$$

for all $x, y \in L$.
Two representations $\rho : L \to \mathrm{Lie}(\mathrm{End}(V))$ and $\rho' : L \to \mathrm{Lie}(\mathrm{End}(V'))$ of degree n are called **equivalent**, if there is an invertible linear map $T : V \to V'$ such that $(x\rho) \cdot T = T \cdot (x\rho')$ for all $x \in L$ (the dot $\cdot$ denotes composition of maps).

Definition 6.2 (Lie algebra module)
Let L be a Lie algebra over a field $\mathbb{F}$. An **L-module** is a finite-dimensional $\mathbb{F}$-vector space V together with an action

$$V \times L \to V, (v, l) \mapsto vl$$

such that

- $(v + w)x = vx + wx$ and $(\lambda v)x = \lambda(vx)$ (the action is linear),
- $v(x + y) = vx + vy$, and
- $v[x, y] = (vx)y - (vy)x$.

for all $v, w \in V$ and all $x, y \in L$ and all $\lambda \in \mathbb{F}$ respectively.

Lemma 6.3 (Representations and modules are the same thing)
Let L be a Lie algebra over a field $\mathbb{F}$. A representation $\rho : L \to \mathrm{Lie}(\mathrm{End}(\mathbb{F}^{1\times n}))$ makes the row space $\mathbb{F}^{1\times n}$ into an L-module by setting $vx := v(x\rho)$. Conversely, if V is an L-module then expressing the linear action as endomorphisms defines a representation of L of degree n.
Thus, the two concepts are two aspects of the same thing.

Proof. The first axiom in Definition 6.2 is needed to make the action of elements of L on V into linear maps. The other two axioms are needed to make the map $L \to \mathrm{Lie}(\mathrm{End}(V))$ a Lie algebra homomorphism. The remaining details of this proof are left as an exercise to the reader. ■

Example 6.4 (A representation)
Let L be the Lie subalgebra of $\mathrm{Lie}(\mathbb{C}^{n\times n})$ of lower triangular matrices. The map

$$\begin{array}{cccc} \pi_2 & : & L & \to & \mathrm{End}(\mathbb{C}^{1\times 2}) \\ & & [a_{i,j}]_{1\le i,j\le n} & \mapsto & \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} \end{array}$$

(and then viewing the 2×2-matrices as endomorphisms of $\mathbb{C}^{1\times 2}$) is a Lie algebra homomorphism and thus a representation. This makes $\mathbb{C}^{1\times 2}$ into an L-module.
We notice that a representation might not "see all of L". The map π_2 for example has a non-trivial kernel.

Example 6.5 (The adjoint representation)
Let L be any Lie algebra over a field $\mathbb{F}$. The **adjoint representation** of L is its **action on itself**:

$$\begin{array}{cccc} \mathrm{ad}: & L & \to & \mathrm{Lie}(\mathrm{End}(L)) \\ & x & \mapsto & x^{\mathrm{ad}} := (y \mapsto [y,x]) \end{array}.$$

Note that we denote the image of an element $x \in L$ under the map ad by x^{ad} throughout. The map ad is in fact a Lie algebra homomorphism and thus a representation. To verify this, we first check that x^{ad} is a linear map from L to L for every $x \in L$:

$$(y+\lambda z)x^{\mathrm{ad}} = [y+\lambda z, x] = [y,x] + \lambda[z,x] = yx^{\mathrm{ad}} + \lambda(zx^{\mathrm{ad}})$$

for $y, z \in L$ and $\lambda \in \mathbb{F}$. The map ad itself is linear, since

$$z(x+\lambda y)^{\mathrm{ad}} = [z, x+\lambda y] = [z,x] + \lambda[z,y] = zx^{\mathrm{ad}} + z(\lambda y^{\mathrm{ad}})$$

for all $x, y, z \in L$ and all $\lambda \in \mathbb{F}$. Finally, the Jacobi identity shows that ad is a homomorphism of Lie algebras:

$$z[x,y]^{\mathrm{ad}} = [z,[x,y]] = -[x,[y,z]] - [y,[z,x]] = [[z,x],y] - [[z,y],x] = (zx^{\mathrm{ad}})y^{\mathrm{ad}} - (zy^{\mathrm{ad}})x^{\mathrm{ad}}$$

for all $x, y, z \in L$.

Example 6.6 (One-dimensional representation)
A one-dimensional representation of a Lie algebra L over $\mathbb{F}$ is simply a linear map $\rho : L \to \mathbb{F}$ with

$$[x,y]\rho = (x\rho)\cdot(y\rho) - (y\rho)\cdot(x\rho) = 0$$

for all $x, y \in L$ since $\mathbb{F}$ is commutative. So the one-dimensional representations of L are precisely the $\mathbb{F}$-linear maps to $\mathbb{F}$ that vanish on the subspace $L^1 = L^{(1)} = [L, L]$. This shows for example that the simple Lie algebra sl_2 from 4.6 has only one one-dimensional representation which is the zero map:

$$\mathrm{sl}_2 \to \mathbb{C}, x \mapsto 0.$$

Anyway, the kernel of such a representation is an ideal so it can only be 0 or sl_2 because sl_2 is simple. Since $\mathbb{C}$ is one-dimensional, the kernel cannot be 0 because of the dimension formula for linear maps.

Definition 6.7 (Submodules, irreducible modules)
Let L be a Lie algebra over a field $\mathbb{F}$ and V be an L-module. A subspace W of V is called a **submodule**, if it is invariant under the action of L:

$$wx \in W \qquad \text{for all } w \in W \text{ and } x \in L.$$

A module V is called **irreducible**, if it has no submodules other than 0 and V itself. A module V is the direct sum $W_1 \oplus \cdots \oplus W_k$ of submodules $W_1, W_2, \ldots, W_k$, if it is the direct vector space direct sum of the W_i. A module V is called **indecomposable** if it is not the direct sum of two non-trivial submodules.

Remark 6.8 (Irreducible implies indecomposable)
An irreducible L-module is clearly indecomposable. However, the reverse implication does **not hold** in general. There are Lie algebras with modules V that have a proper submodule $0 < W < V$, for which there is no other submodule U with $V = W \oplus U$.

Remark 6.9 (Irreducible adjoint representation)
Let $V := L$ be the L-module given by the adjoint representation (see 6.5). A submodule of V is the same as an ideal of L. The module V is irreducible if and only if L is a simple Lie algebra.

Definition 6.10 (Homomorphisms of modules)
Let L be a Lie algebra over a field $\mathbb{F}$. A **homomorphism of L-modules** is an $\mathbb{F}$-linear map

$$T : V \to V'$$

between two L-modules V and V', such that $(vT)x = (xv)T$ for all $v \in V$ and all $x \in L$. It is called an **isomorphism** if there is a homomorphism $S : V' \to V$ of L-modules with $TS = \mathrm{id}_V$ and $ST = \mathrm{id}_{V'}$.

Definition/Proposition 6.11 (Eigenvectors and eigenvalues)
Let V be an $\mathbb{F}$-vector space and $T : V \to V$ a linear map. Then an **eigenvalue** is an element $\lambda \in \mathbb{F}$, for which a vector $v \in V \setminus \{0\}$ exists with

$$vT = \lambda \cdot v.$$

Every such v is called an **eigenvector for the eigenvalue** λ. The set of eigenvectors for the eigenvalue λ together with the zero vector is called the **eigenspace for the eigenvalue** λ. Note that an eigenvector v has to be non-zero, otherwise every $\lambda \in \mathbb{F}$ would be an eigenvalue.
For $\mathbb{F} = \mathbb{C}$, every endomorphism T has an eigenvalue, since the characteristic polynomial of T has a root ($\mathbb{C}$ is algebraically closed).

Proof. See any linear algebra book and use the fundamental theorem of algebra. ■

Lemma 6.12 (Schur I)
Let V and V' be irreducible L-modules for a Lie algebra L over $\mathbb{F}$ and let $T : V \to V'$ be an L-module homomorphism. Then either T maps every element of V to zero or it is an isomorphism.

Proof. The image $\operatorname{im} T$ and the kernel $\ker T$ of T are submodules of V' and V respectively. Since both V and V' are irreducible, either $\operatorname{im} T = 0$ and $\ker T = V$, or $\operatorname{im} T = V'$ and $\ker T = 0$. ■

Corollary 6.13 (Schur II)
Let V be an irreducible L-module for a Lie algebra L over $\mathbb{C}$ and $T : V \to V$ be an L-module homomorphism (or shorter L-endomorphism). Then T is a scalar multiple of the identity map (possibly the zero map).

Proof. Let $T : V \to V$ be any L-endomorphism. Then T is in particular a linear map from V to V so it has an eigenvalue λ with corresponding eigenvector $v \in V$ by Proposition 6.11. Thus, the linear map $T - \lambda \cdot \mathrm{id}_V$ has $v \neq 0$ in its kernel, and it is an L-endomorphism, since both T and id_V are. By Lemma 6.13, this linear map $T - \lambda \cdot \mathrm{id}_V$ must be equal to zero and thus $T = \lambda \cdot \mathrm{id}_V$. Note that λ (and thus T) can be equal to 0. ■

Theorem 6.14 (Weyl)
Let L be a semisimple Lie algebra over $\mathbb{C}$ and V a finite-dimensional L-module. Then V has irreducible submodules $W_1, W_2, \ldots, W_k$, such that $V = W_1 \oplus \cdots \oplus W_k$, for some $k \in \mathbb{N}$. That is, V is the direct sum of irreducible submodules.

Proof. Omitted. ■

Chapter 3

Representations of sl_2

For the whole chapter let sl_2 from Example 4.6, which is the $\mathbb{C}$-span of the three elements

$$e := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad f := \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad h := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

with the usual commutator $[a, b] := a \cdot b - b \cdot a$ as Lie product. We know that it is a simple Lie algebra and the following relations hold (see Example 5.2):

$$[e, f] = h \qquad \text{and} \qquad [h, e] = 2e = -[e, h] \qquad \text{and} \qquad [h, f] = -2f = -[f, h].$$

We want to classify all its finite-dimensional modules. Since sl_2 is simple, it is semisimple (see Example 5.11). Thus by Weyl's Theorem 6.14 it is enough to classify the irreducible modules, because all others are direct sums of irreducible ones.

7 The irreducible sl_2-modules introduced

Proposition 7.1 (The modules V_d)
Let $d \in \mathbb{N} \cup \{0\}$ and let $\mathbb{C}[X, Y]$ be the polynomial ring over $\mathbb{C}$ in two indeterminates X and Y. Let

$$V_d := \mathrm{Span}(X^d, X^{d-1}Y, \ldots, XY^{d-1}, Y^d),$$

this is a $\mathbb{C}$-vector space of dimension $d + 1$, actually, V_d is the set of homogeneous polynomials of total degree d. For $d = 0$, the vector space V_0 consists of the constant polynomials and $\dim(V_0) = 1$. The following equations together with linear extension make V_d into an sl_2-module:

$$\begin{aligned} (X^aY^b)e &:= Y \cdot \frac{\partial}{\partial X}(X^aY^b) = a \cdot X^{a-1}Y^{b+1}, \\ (X^aY^b)f &:= X \cdot \frac{\partial}{\partial Y}(X^aY^b) = b \cdot X^{a+1}Y^{b-1}, \\ (X^aY^b)h &:= (a-b) \cdot X^aY^b \end{aligned}$$

all for $a + b = d$ and $0 \le a, b \le d$.

Proof. Since we can prescribe a linear map from V_d into itself arbitrarily on a basis, this defines endomorphisms for e, f and h uniquely. Linear extension gives us a $\mathbb{C}$-linear map

$$\varphi : \mathrm{sl}_2 \to \mathrm{Lie}(\mathrm{End}(V_d)).$$

To check that this is a representation of Lie algebras we only have to check that it respects the Lie product, that is:

$$v([x, y]\varphi) = (v(x\varphi))(y\varphi) - (v(y\varphi))(x\varphi)$$

for all $v \in V_d$ and all $x, y \in \mathrm{sl}_2$. Since φ is $\mathbb{C}$-linear and all $(x\varphi)$ are $\mathbb{C}$-linear it is enough to check all this for basis elements, that is, we have to check

$$
\begin{aligned}
(X^aY^b)[e,f] &= ((X^aY^b)e)f - ((X^aY^b)f)e \quad \text{and} \\
(X^aY^b)[h,e] &= ((X^aY^b)h)e - ((X^aY^b)e)h \quad \text{and} \\
(X^aY^b)[h,f] &= ((X^aY^b)h)f - ((X^aY^b)f)h
\end{aligned}
$$

for all $0 \le a, b \le d$ with $a + b = d$. This is left as an exercise for the reader. ■

Illustration 7.2 (The action on V_d)
Pictorially, this means:

$$
\begin{tikzcd}
0 & X^d \arrow[l, bend left, "f", "\cdot 0"'] \arrow[r, bend left, "e"', "\cdot d"] \arrow[loop above, "h", "\cdot d"'] & X^{d-1}Y \arrow[l, bend left, "f", "\cdot 1"'] \arrow[r, bend left, "e"', "\cdot (d-1)"] \arrow[loop above, "h", "\cdot (d-2)"'] & X^{d-2}Y^2 \arrow[l, bend left, "f", "\cdot 2"'] \arrow[r, bend left, "e"', "\cdot (d-2)"] \arrow[loop above, "h", "\cdot (d-4)"'] & \cdots & X^2Y^{d-2} \arrow[l, bend left, "f", "\cdot (d-2)"'] \arrow[r, bend left, "e"', "\cdot 2"] \arrow[loop above, "h", "\cdot (4-d)"'] & XY^{d-1} \arrow[l, bend left, "f", "\cdot (d-1)"'] \arrow[r, bend left, "e"', "\cdot 1"] \arrow[loop above, "h", "\cdot (2-d)"'] & Y^d \arrow[l, bend left, "f", "\cdot d"'] \arrow[r, bend left, "e"', "\cdot 0"] \arrow[loop above, "h", "\cdot (-d)"'] & 0
\end{tikzcd}
$$

Illustration 7.3 (The action as matrices)
If we express the action of e, f and h by matrices with respect to the monomial basis

$$(X^d, X^{d-1}Y, \ldots, XY^{d-1}, Y^d)$$

in row convention, we get:

$$
e \leftrightarrow \begin{bmatrix}
0 & d & 0 & \cdots & 0 \\
0 & 0 & d-1 & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & 0 \\
\vdots & 0 & \ddots & \ddots & 1 \\
0 & \cdots & \cdots & 0 & 0
\end{bmatrix}
$$

$$
f \leftrightarrow \begin{bmatrix}
0 & 0 & \cdots & \cdots & 0 \\
1 & 0 & \ddots & 0 & \vdots \\
0 & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & d-1 & 0 & 0 \\
0 & \cdots & 0 & d & 0
\end{bmatrix}
$$

$$
h \leftrightarrow \begin{bmatrix}
d & 0 & \cdots & 0 \\
0 & d-2 & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & -d
\end{bmatrix}
$$

Proposition 7.4 (All V_d are irreducible)
For all $d \in \mathbb{N} \cup \{0\}$, the module V_d is irreducible.

Proof. Assume $0 < W \le V_d$ is a non-zero subspace that is invariant under the action of sl_2. The endomorphism of W induced by the action of h has an eigenvalue λ with a corresponding eigenvector $0 \ne w \in W$ (see Proposition 6.11). Since h has 1-dimensional eigenspaces spanned by the monomials $X^d, XY^{d-1}, \ldots, Y^d$, the vector w is a scalar multiple of one of these. But then the subspace W contains all such monomials since successive applications of e and f map one to some non-zero scalar multiple of every other one. Thus $W = V_d$ and we have proved that V_d is irreducible. ■

8 Every irreducible sl_2-module is isomorphic to one of the V_d

Lemma 8.1 (Eigenvectors to different eigenvalues are linearly independent)
Let V be an $\mathbb{F}$-vector space and $\varphi \in \mathrm{End}(V)$ an arbitrary endomorphism. Let $(v_1, v_2, \ldots, v_n)$ be a tuple of eigenvectors of φ to pairwise different eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ respectively. Then $(v_1, \ldots, v_n)$ is linearly independent.

Proof. Assume for a contradiction that $(v_1, \ldots, v_n)$ is linearly dependent. Let $k \in \mathbb{N}$ be minimal such that $(v_1, \ldots, v_k)$ is linearly independent and $v_{k+1} \in \mathrm{Span}(v_1, \ldots, v_k)$. We have $k \geq 1$ because eigenvectors are non-zero and $k < n$ because of our assumption. If $v_{k+1} = \sum_{i=1}^{k} \mu_i v_i$ for some $\mu_i \in \mathbb{F}$, then

$$\sum_{i=1}^{k} \lambda_i \cdot \mu_i v_i = \sum_{i=1}^{k} (\mu_i v_i)\varphi = v_{k+1}\varphi = \lambda_{k+1} \cdot v_{k+1} = \sum_{i=1}^{k} \lambda_{k+1} \cdot \mu_i v_i,$$

which is a contradiction since $(v_1, \ldots, v_k)$ is linearly independent and the eigenvalues are pairwise different. ■

Lemma 8.2 (Eigenvectors in sl_2-modules)
Let V be an sl_2-module over $\mathbb{C}$ and λ be an eigenvalue of h with eigenvector $v \in V$.

- Either $ve = 0$ or ve is an eigenvector of h for the eigenvalue $\lambda - 2$.
- Either $vf = 0$ or vf is an eigenvector of h for the eigenvalue $\lambda + 2$.

Proof. By the module axioms and the relations $[h, e] = 2e$ and $[h, f] = -2f$, we get:

$$\begin{aligned} (ve)h &= (vh)e - v[h, e] = \lambda \cdot (ve) - v \cdot (2e) = (\lambda - 2) \cdot (ve) \\ (vf)h &= (vh)f - v[h, f] = \lambda \cdot (vf) + v \cdot (2f) = (\lambda + 2) \cdot (vf) \end{aligned}$$

This proves the lemma, since eigenvectors have to be non-zero by definition. ■

Lemma 8.3 (Highest weights)
Let V be a finite-dimensional sl_2-module over $\mathbb{C}$. Then V contains an eigenvector w of h such that $wf = 0$.

Proof. Since we work over the complex numbers $\mathbb{C}$, the endomorphism of V induced by h has an eigenvalue λ with corresponding eigenvector v (see Proposition 6.11). We consider the sequence

$$v, vf, vf^2, \ldots, vf^k, \ldots,$$

where vf^k stands for the vector one gets by acting repeatedly with f altogether k times. By Lemma 8.2 these are all either equal to zero or are eigenvectors of h to different eigenvalues, namely $\lambda, \lambda + 2, \lambda + 4, \ldots$. If they were all non-zero, then they would all be linearly independent by Lemma 8.1, which can not be true since V is finite-dimensional. Thus there is a k with $vf^k \neq 0$ and $vf^{k+1} = 0$, the vector $w := vf^k$ is an eigenvector of h with $wf = 0$. ■

Definition 8.4 (Highest weight vector)
A vector w as in Lemma 8.3 is called a **highest weight vector** of the sl_2-module V and its corresponding eigenvalue is called a **highest weight**. We shall extend this definition later.

We are now in a position to prove the main result of this chapter:

Theorem 8.5 (Classification of finite-dimensional irreducible sl_2-modules)
Let V be an irreducible sl_2-module of dimension $d + 1$, then V is isomorphic to V_d.

Proof. Since V is finite-dimensional over $\mathbb{C}$, the endomorphism h of V has an eigenvector w with $wf = 0$ be Lemma 8.3. Let λ be the corresponding eigenvalue. We consider the sequence

$$w, we, we^2, \ldots$$

where we^k stands for the vector one gets by acting repeatedly with e altogether k times. By Lemma 8.2 these are all either equal to 0 or eigenvectors of h with eigenvalues $\lambda, \lambda-2, \lambda-4, \ldots$ respectively. As in the proof of Lemma 8.3 we conclude that there is a k with $we^{k+1} = 0$ and $we^k \neq 0$.
We claim that $W := \mathrm{Span}(w, we, we^2, \ldots, we^k)$ is an sl_2-submodule of V and that

$$\mathcal{B} := (w, we, \ldots, we^k)$$

is a basis. All these vectors are eigenvectors of h, so W is invariant under h. By construction and because of $we^{k+1} = 0$ the space W is invariant under e. Note that $\mathrm{Span}(w, we, \ldots, we^i)e = \mathrm{Span}(w, we, \ldots, we^{i+1})$.
Invariance under f comes from the fact that

$$(we^i)f = (we^{i-1})ef - (we^{i-1})fe + (we^{i-1})fe = (we^{i-1})h + ((we^{i-1})f)e \quad \text{for } 1 \le i \le k$$

and $wf = 0$ using induction by i. We have shown that W is invariant under h, e and f and thus under all elements of sl_2. Since W is non-zero and V is irreducible, we have $W = V$. Since $\mathcal{B} = (w, we, \ldots, we^k)$ is linearly independent by Lemma 8.1, it is a basis of W and thus of V and we conclude $k = d$ because $\dim(V) = d + 1$.
With respect to the basis $\mathcal{B}$ the endomorphism induced by h is a diagonal matrix with diagonal entries $\lambda, \lambda-2, \ldots, \lambda-2d$, thus its trace is equal to $\lambda\cdot(d+1) - d(d+1)$ (recall that $\sum_{i=0}^{d} = d(d+1)/2$). But since $h = [e, f]$ this trace is zero, from which follows $\lambda = d$. The eigenvalues of h in its action on V are therefore $d, d-2, d-4, \ldots, 4-d, 2-d, -d$.
Now we modify our basis $\mathcal{B}$ of V slightly to show that the action of sl_2 on V is the same as the one on V_d. Let $w_0 := w$ and $w_{i+1} := \frac{1}{d-i}\cdot w_i e$ for $0 \le i < d$, forming a new basis $\mathcal{B}' := (w_0, w_1, \ldots, w_d)$ of V.
With respect to this basis, the endomorphisms induced by the action of h and e are exactly as in Illustration 7.3, since we have

$$w_i h = (d-2i)w_i \quad \text{and} \quad w_i e = (d-i)w_{i+1}$$

for $0 \le i \le d$ where $w_{d+1} := 0$. We claim that the same holds for the endomorphism induced by the action of f. We have $w_0 f = wf = 0$ so the first row is zero. Furthermore, we claim that $w_i f = i w_{i-1}$ for $1 \le i \le d$. This follows by induction using a similar computation as above, we have

$$\begin{aligned} w_{i+1}f &= \frac{1}{d-i}w_i ef = \frac{1}{d-i}(w_i h + (w_i f)e) = \frac{1}{d-i}((d-2i)w_i + i w_{i-1}e) \\ &= \frac{d-2i+i(d+1-i)}{d-i}w_i = \frac{id-i^2+d-i}{d-i}w_i = (i+1)w_i \end{aligned}$$

for $0 \le i < d$ where $w_{-1} := 0$.
Since the action of h, e and f, and thus of all elements of sl_2, are the same with respect to the bases $(X^d, X^{d-1}Y, \ldots, XY^{d-1}, Y^d)$ of V_d and $\mathcal{B}'$ of V, the linear map $X^{d-i}Y^i \mapsto w_i$ is an isomorphism of V_d onto V, proving the theorem. ■

Because of Weyl's Theorem we have thus proved:

Theorem 8.6 (Representations of $\mathrm{sl}_2(\mathbb{C})$**)**
Let V be a finite-dimensional $\mathrm{sl}_2(\mathbb{C})$-module. Then V has irreducible submodules $W_1, W_2, \ldots, W_k$, for some $k \in \mathbb{N}$, such that $V = W_1 \oplus W_2 \oplus \cdots \oplus W_k$ and there are numbers $d_1, \ldots, d_k \in \mathbb{N} \cup \{0\}$ such that $W_i \cong V_{d_i}$. ■

Chapter 4

Engel's and Lie's Theorems

9 Engel's Theorem on nilpotent Lie algebras

Definition 9.1 (Nilpotent elements)
Let V be a vector space and $T \in \mathrm{End}(V)$ an endomorphism. Then T is called **nilpotent**, if there is a $k \in \mathbb{N}$ such that $T^k = 0$ (the zero map).
Let L be a Lie algebra and $x \in L$. Then x is called **ad-nilpotent**, if $x^{\mathrm{ad}} \in \mathrm{End}(L)$ is nilpotent.
Note that this means that $(x^{\mathrm{ad}})^k = 0$ for some $k \in \mathbb{N}$ and this uses the regular composition of maps rather than the Lie product!

Proposition 9.2 (Eigenvalues of nilpotent elements)
Let V be a finite-dimensional vector space over $\mathbb{F}$ and $T \in \mathrm{End}(V)$ be nilpotent. Then 0 is the only eigenvalue of T.

Proof. Let λ be an eigenvalue with eigenvector $0 \neq v \in V$ and let $k \in \mathbb{N}$ with $T^k = 0$. Then $0 = vT^k = \lambda^k v$ so $\lambda^k = 0$ and thus $\lambda = 0$ since $\mathbb{F}$ is a field. However, 0 is an eigenvalue since T is not invertible. ∎

In this section we want to prove the following theorem:

Theorem 9.3 (Engel)
Let L be a finite-dimensional Lie algebra over a field $\mathbb{F}$. Then L is nilpotent if and only if every element x of L is ad-nilpotent.

We only prove the "only-if"-part here, the "if"-part is proved in the rest of this section.

Proof. If L is nilpotent, then there is a k such that $L^k = 0$. This means in particular that every expression

$$[[\cdots[[x_0, x_1], x_2], \cdots], x_k] = 0$$

for arbitrary elements $x_0, x_1, \ldots, x_k \in L$. This implies immediately that

$$x_1^{\mathrm{ad}} \cdot x_2^{\mathrm{ad}} \cdot \cdots \cdot x_k^{\mathrm{ad}} = 0 \in \mathrm{End}(L)$$

and in particular that $(x^{\mathrm{ad}})^k = 0$ for all $x \in L$. So every element x of L is ad-nilpotent. ∎

We first prove some helper results:

Lemma 9.4 (Quotient modules)
Let L be a Lie algebra and V an L-module with a submodule $0 < W < V$. Then the quotient space $V/W = \{v + W \mid v \in V\}$ is an L module with the induced action

$$(v + W)x := vx + W.$$

Proof. Details omitted, but routine verification. Check well-definedness first, the module actions are directly inherited from V. ∎

Lemma 9.5 (ad-quotients)
Let L be a Lie algebra and H a subalgebra. Then we can restrict ad : $L \to \mathrm{Lie}(\mathrm{End}(L))$ to H and thus get a representation $\mathrm{ad}|_H : H \to \mathrm{Lie}(\mathrm{End}(L))$. This makes L into an H-module and H itself is an H-submodule of L. Thus the quotient space L/H is an H-module as well. If $y \in H$ is ad-nilpotent, then it acts as a nilpotent endomorphism on L/H as well.

Proof. It is clear that $\mathrm{ad}|_H$ is a Lie algebra homomorphism and thus that L is an H-module. Since H is a subalgebra (i.e. $[H, H] \leq H$), it follows that H is an H-submodule of L. By Lemma 9.4, the quotient space L/H (which is **not a Lie algebra!**) is an H-module as well with action $(x + H)h := xh^{\mathrm{ad}} + H = [x, h] + H$ for all $x \in L$ and all $h \in H$. If $(h^{\mathrm{ad}})^k = 0$ for some k, then $(x + H)(h^{\mathrm{ad}})^k = x(h^{\mathrm{ad}})^k + H = 0 + H$ for all $x \in L$. ∎

Lemma 9.6 (ad-nilpotency)
Let L be a Lie subalgebra of $\mathrm{gl}(V)$ for some finite-dimensional vector space V over $\mathbb{F}$ and suppose that L consists of nilpotent endomorphisms of V. Then for all $x \in L$ the endomorphism $x^{\mathrm{ad}} \in \mathrm{End}(L)$ is nilpotent.

Proof. If $k \in \mathbb{N}$ such that $x^k = 0$, then

$$\underbrace{[\cdots[[y, x], x], \ldots], x]}_{2k \text{ times}} = \sum_{i=0}^{2k} c_i x^i y x^{2k-i}$$

for some numbers $c_i \in \mathbb{F}$. Since for every summand in this sum there are at least k factors of x on at least one side of y, the whole sum is equal to 0. As this holds for all $y \in L$, we have proved that $(x^{\mathrm{ad}})^{2k} = 0$. ∎

Proposition 9.7 (Helper for Engel)
Let V be a finite-dimensional vector space over $\mathbb{F}$ and L a Lie subalgebra of $\mathrm{gl}(V)$ consisting of nilpotent endomorphisms. Then there is a non-zero $v \in V$ with $vx = 0$ for all $x \in L$.

Proof. We proceed by induction on $\dim(L)$. If $\dim(L) = 1$, then L consists of the scalar multiples of a single nilpotent endomorphism $x \in \mathrm{End}(V)$. By Proposition 9.2 it has 0 as eigenvalue, thus there is an eigenvector $0 \neq v \in V$ with $vx = 0$ and we are done.
Now suppose $\dim(L) > 1$ and the proposition is already proved for nilpotent Lie algebras of smaller dimension. We proceed in two steps:
Step 1: Let H be a maximal subalgebra of L (that is, H is a subalgebra such that there is no subalgebra K of L with $H < K < L$). Such an H exists and is non-zero, since every 1-dimensional subspace of L is a subalgebra and $\dim(L) < \infty$. We claim that $\dim(H) = \dim(L) - 1$ and that H is an ideal in L.
As in Lemma 9.5 we view L as H-module with submodule H and thus L/H as H-module with the action $(x + H)h := xh^{\mathrm{ad}} + H$. This gives us a representation of H on the vector space L/H and thus a homomorphism of Lie algebras $\varphi : H \to \mathrm{Lie}(\mathrm{End}(L/H))$. Since L and thus H consists of nilpotent elements we conclude that $H\varphi$ consists of nilpotent endomorphisms of L/H using Lemma 9.6. Since $\dim(H\varphi) \leq \dim(H) < \dim(L)$, we can use the induction hypothesis to conclude that there is a $y \in L \setminus H$ such that $(y + H)h = 0 + H$ for all $h \in H$, that is, $[y, H] \leq H$ but $y \notin H$. But then $H + \mathrm{Span}(y)$ is a subalgebra of L that properly contains H. By the maximality of H it follows that $H + \mathrm{Span}(y) = L$ and so $\dim(H) = \dim(L) - 1$ and H is an ideal in L.
Step 2: Now we apply the induction hypothesis to $H \leq L \leq \mathrm{gl}(V)$. We conclude that there is a $w \in V$ with $wh = 0$ for all $h \in H$. Thus $W := \{v \in V \mid vh = 0\ \forall h \in H\}$ is a non-zero

subspace of V. It is certainly invariant under H (mapped to 0 by it!) and invariant under y, since $vyh = v[y, h] + vhy = 0$ for all $v \in W$ and all $h \in H$, since $[y, h] \in H$. Since y is nilpotent on V and thus on W, it has an eigenvector $0 \neq v \in W$ with eigenvalue 0 (see Proposition 9.2), that is, $vy = 0$. However, since $vh = 0$ for all $h \in H$ and $L = H + \mathrm{Span}(y)$, it follows that $vx = 0$ for all $x \in L$. ■

Now we prove a theorem, from which Engel's Theorem 9.3 follows immediately:

Theorem 9.8 (Engel's Theorem in $\mathrm{gl}(V)$**)**
Let K be a Lie subalgebra of $\mathrm{gl}(V)$ for some finite-dimensional vector space V over a field $\mathbb{F}$, such that every element x of K is a nilpotent endomorphism. Then there is a basis $\mathcal{B}$ of V such that every element x of K corresponds to a strictly lower triangular matrix with respect to $\mathcal{B}$. It follows that K is a nilpotent Lie algebra.

Proof. We proceed by induction on $\dim(V)$. If $\dim(V) = 1$ then the dimension of K is either 0 or 1 and in both cases the matrices with respect to any basis $\mathcal{B}$ are all zero because they are nilpotent 1×1-matrices.
Suppose now that $n := \dim(V) \geq 2$ and the statement is proved for all cases with smaller dimension. By Proposition 9.7 there is a vector $0 \neq v_0 \in V$ with $v_0 x = 0$ for all $x \in K$. Obviously, $W := \mathrm{Span}(v_0)$ is a K-submodule of V and thus by Proposition 9.4, the quotient space V/W is a K-module. We denote the Lie subalgebra of $\mathrm{gl}(V/W)$ induced by this action of K by $\bar{K}$. Since $\dim(V/W) = \dim(V) - 1 = n - 1$ and $\bar{K}$ consists of nilpotent endomorphisms, we can use the induction hypothesis to conclude that V/W has a basis $\bar{\mathcal{B}} = (v_1 + W, \dots, v_{n-1} + W)$ such that every element of $\bar{K}$ corresponds to a strictly lower triangular matrix with respect to $\bar{\mathcal{B}}$. But then every element of K corresponds to a strictly lower triangular matrix with respect to $\mathcal{B} := (v_0, v_1, \dots, v_{n-1})$. This implies that K is isomorphic to a subalgebra of the Lie algebra of all strictly lower triangular matrices, which was shown to be nilpotent in Example 5.4. Thus K itself is nilpotent as well. ■

We can now prove the missing implication in Engel's Theorem 9.3.

Proof. Suppose that L is a finite-dimensional Lie algebra over a field $\mathbb{F}$ such that every element of L is ad-nilpotent. Then $K := L^{\mathrm{ad}}$ is a Lie subalgebra of $\mathrm{Lie}(\mathrm{End}(L))$ fulfilling the hypotheses of Theorem 9.8 and is thus nilpotent. Since ad is a homomorphism of Lie algebras with kernel $Z(L)$ and image K, we have shown that $L/Z(L) \cong K$ is nilpotent, using the First Isomorphism Theorem 4.16. Therefore by Theorem 5.7 the Lie algebra L itself is nilpotent. ■

Remark 9.9 (A warning)
Not for every nilpotent Lie algebra contained in $\mathrm{gl}(V)$ there is a basis of V such that all elements correspond to strictly lower triangular matrices. For example $L := \mathrm{Span}(\mathrm{id}_V)$ is abelian and thus nilpotent but it contains the identity, which corresponds to the identity matrix with respect to every basis of V.

10 Lie's Theorem on soluble Lie algebras

We want to derive a similar result to Theorem 9.8 for soluble Lie algebras over $\mathbb{C}$.

Definition 10.1 (Dual space and weights)
Let L be any $\mathbb{F}$-vector space. Then we denote the set of $\mathbb{F}$-linear maps from L to $\mathbb{F}$ by L^* and call it the **dual space** of L.
Let L be a Lie algebra over $\mathbb{F}$ and V a finite-dimensional L-module. A **weight of** L **(on** V**)** is an element $\lambda \in L^*$ such that

$$V_\lambda := \{v \in V \mid vx = (x\lambda) \cdot v \text{ for all } x \in L\}$$

is not equal to $\{0\}$. The subspace V_λ for a weight λ is called a **weight space**. It consists of **simultaneous eigenvectors** of all elements of L and the zero vector.

The following lemma is crucial for what we want to do:

Lemma 10.2 (Invariance)
Let L be a Lie algebra over a field $\mathbb{F}$ of characteristic 0, V a finite-dimensional L-module and K an ideal in L. Assume that λ is a weight of K on V, that is, the weight space

$$V_\lambda := \{v \in V \mid vk = (k\lambda)v \text{ for all } k \in K\}$$

is non-zero. Then V_λ is invariant under the action of L.

Proof. Let $0 \neq v \in V_\lambda$ and $x \in L$. Then

$$vxk = v[x,k] + vkx = ([x,k]\lambda) \cdot v + (k\lambda) \cdot vx.$$

Note, that $[x,k] \in K$ since K is an ideal of L. That is, if we could show that $[x,k]\lambda = 0$ for all $k \in K$ and all $x \in L$, we would be done.
To this end, we consider the sequence of vectors

$$v, vx, vx^2, \ldots$$

and let m be the least integer, such that $(v, vx, \ldots, vx^m)$ is linearly dependent. We claim that $U := \mathrm{Span}(v, vx, \ldots, vx^{m-1})$ is invariant under K and that the matrix M_k of the action of any $k \in K$ with respect to the basis $\mathcal{B} := (v, vx, \ldots, vx^{m-1})$ is a lower triangular matrix with all diagonal entries being $k\lambda$:

$$M_k = \begin{bmatrix} k\lambda & 0 & \cdots & 0 \\ * & k\lambda & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ * & \cdots & * & k\lambda \end{bmatrix}.$$

Indeed, $vk = (k\lambda)v$ showing that the first row of M_k is $(k\lambda, 0, \ldots, 0)$. We then proceed by induction on the rows showing that $vx^ik = (k\lambda)vx^i + w$ for some $w \in \mathrm{Span}(v, vx, \ldots, vx^{i-1})$ for $1 \leq i < m$ using

$$vx^{i+1}k = vx^i[x,k] + vx^ikx = (k\lambda)vx^{i+1} + u$$

for some $u \in \mathrm{Span}(v, vx, \ldots, vx^i)$ because $[x,k] \in K$ and the induction hypothesis.
We have showed that U is invariant under K and under x, so it is invariant under the whole Lie subalgebra $K + \mathrm{Span}(x)$ of L. For every element $k \in K$, the commutator $[x,k]$ is contained in K, so the matrix $M_{[x,k]}$ of its action on U with respect to the basis $\mathcal{B}$ is lower triangular with $[x,k]\lambda$ on the diagonal. On the other hand, this matrix is the commutator of the matrices M_x and M_k, so in particular its trace is zero. Thus $[x,k]\lambda = 0$ and we have proved the Invariance Lemma.
Note that we have proved at the same time that $U \leq V_\lambda$. ■

We prove a Proposition analogous to Proposition 9.7:

Proposition 10.3 (Helper for Lie)
Let L be a soluble Lie subalgebra of $\mathrm{gl}(V)$ for some finite-dimensional $\mathbb{C}$-vector space V. Then L has a weight λ on V and thus a non-zero weight space.

Proof. We need to find a simultaneous eigenvector for all elements of L. We proceed by induction on $\dim(L)$ very similarly to the proof of Proposition 9.7. If $\dim(L) = 1$, then L consists of the scalar multiples of a single non-zero element x. This element has an eigenvalue μ with corresponding eigenvector v by Proposition 6.11 because V is over $\mathbb{C}$. Thus $\lambda : L \to \mathbb{C}, c \cdot x \mapsto c \cdot \mu$ for $c \in \mathbb{C}$ is a weight with weight space V_λ containing at least $\mathrm{Span}(v)$.
Now suppose $n := \dim(L) \geq 2$ and the statement is already proved for all Lie algebras of dimension less than n. Since L is soluble, the space $L^{(1)} = [L, L]$ is a proper ideal of L. Let K be an $(n-1)$-dimensional subspace of L containing $[L, L]$ and $x \in L \setminus K$, such that we have $L = K + \mathrm{Span}(x)$. The subspace K is an ideal of L since every commutator in L is contained in $[L, L]$ and thus in K. Therefore K is in particular a subalgebra of smaller dimension than L and thus by Theorem 5.7 itself soluble. Using the induction hypothesis we conclude that K has a weight $\tilde{\lambda} \in K^*$. Let $W := V_{\tilde{\lambda}}$ be the corresponding weight space.
Using the Invariance Lemma 10.2 we conclude that W is invariant under x and thus under all of L. Since we are working over the complex numbers $\mathbb{C}$, the endomorphism induced by x on W has an eigenvector w with eigenvalue μ, that is, $wx = \mu \cdot w$. But if we now define $\lambda : L \to \mathbb{C}$ setting

$$(k + \nu \cdot x)\lambda := k\tilde{\lambda} + \nu\mu$$

this defines a $\mathbb{C}$-linear map and thus an element $\lambda \in L^*$, such that

$$w(k + \nu \cdot x) = wk + \nu \cdot wx = (k\tilde{\lambda})w + \nu\mu w = (k + \nu \cdot x)\lambda \cdot w$$

for all $k \in K$ and all $\nu \in \mathbb{C}$ showing that λ is a weight of L such that the weight space V_λ contains w. ■

Theorem 10.4 (Lie)
Let L be a soluble Lie algebra over $\mathbb{C}$ and V is a finite-dimensional L-module. Then there is a basis $\mathcal{B}$ of V, such that the matrix the action of every element of L with respect to $\mathcal{B}$ is a lower triangular matrix.

Proof. We proceed by induction on $\dim(V)$. If $\dim(V) = 1$ then the dimension the matrices with respect to any basis $\mathcal{B}$ are lower triangular because they are 1×1-matrices.
Suppose now that $n := \dim(V) \geq 2$ and the statement is proved for all cases with smaller dimension. Being a module, V gives rise to a Lie algebra homomorphism $\varphi : L \to \mathrm{gl}(V)$ and the image $L\varphi$ is soluble using Theorem 5.7 and the First Isomorphism Theorem 4.16. By Proposition 10.3 applied to $L\varphi$ there is weight λ' of $L\varphi$ on V. However, this immediately gives rise to a weight $\lambda := \varphi\lambda'$ of L. In particular, we have a non-zero vector v_0 in the weight space V_λ. That is, $v_0 x = (x\lambda)v_0$ for all $x \in L$. Obviously, $W := \mathrm{Span}(v_0)$ is an L-submodule of V and thus by Proposition 9.4, the quotient space V/W is an L-module of smaller dimension. Since $\dim(V/W) = \dim(V) - 1 = n - 1$ we can use the induction hypothesis to conclude that V/W has a basis $\bar{\mathcal{B}} = (v_1 + W, \ldots, v_{n-1} + W)$ such that every element of L corresponds to a lower triangular matrix with respect to $\bar{\mathcal{B}}$. But then every element of L corresponds to a lower triangular matrix with respect to the basis $\mathcal{B} := (v_0, v_1, \ldots, v_{n-1})$ of V. ■

Chapter 5

Jordan decomposition and Killing form

11 Jordan decomposition

We recall some definitions and results from linear algebra:

Definition/Proposition 11.1 (Jordan normal form)
Let V be an n-dimensional vector space over $\mathbb{C}$ and $T \in \mathrm{End}(V)$. Then V has a basis $\mathcal{B}$ such that the matrix corresponding to T with respect to $\mathcal{B}$ is of the block matrix form

$$\begin{bmatrix} J_1 & 0 & \cdots & 0 \\ 0 & J_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & J_k \end{bmatrix}$$

and each J_i is of the form

$$\begin{bmatrix} \lambda_i & 0 & \cdots & \cdots & 0 \\ 1 & \lambda_i & \ddots & 0 & \vdots \\ 0 & 1 & \lambda_i & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & \lambda_i \end{bmatrix},$$

for some $\lambda_i \in \mathbb{C}$. The J_i are called **Jordan blocks**, we say that such a matrix is **in Jordan normal form**. The number of Jordan blocks with a given diagonal entry λ and a given size is equal for all choices of such a basis $\mathcal{B}$. An endomorphism T is called **diagonalisable**, if all Jordan blocks in its Jordan normal form have size (1×1), that is, the Jordan normal form is a diagonal matrix. Obviously, T is nilpotent if and only if all diagonal entries in all Jordan blocks are equal to 0.

From this result we immediately get:

Definition/Proposition 11.2 (Jordan decomposition)
Let $T \in \mathrm{End}(V)$ for a finite-dimensional $\mathbb{C}$-vector space V. The Jordan decomposition of T is an expression of T as $T = D + N$ with $D, N \in \mathrm{End}(V)$, such that D is diagonalisable, N is nilpotent and $DN = ND$. Both endomorphisms D and N are uniquely defined by these conditions. There is a polynomial $p \in \mathbb{C}[X]$ with $D = p(T)$.

Proof. We only give a rough idea here:
Choose a basis $\mathcal{B}$ of V such that the matrix of T with respect to $\mathcal{B}$ is in Jordan normal form. The matrix of D with respect to $\mathcal{B}$ is the diagonal matrix containing only the diagonal entries of the

Jordan blocks, such that $N := T - D$ is nilpotent. The endomorphisms D and N commute since for every Jordan block the two matrices

$$\begin{bmatrix} \lambda_i & 0 & \cdots & \cdots & 0 \\ 0 & \lambda_i & \ddots & 0 & \vdots \\ 0 & 0 & \lambda_i & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 & \lambda_i \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 0 & \cdots & \cdots & 0 \\ 1 & 0 & \ddots & 0 & \vdots \\ 0 & 1 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}$$

commute. One proves next the existence of the polynomial p, which we skip here.
We need to prove the uniqueness. Let $T = D + N = \tilde{D} + \tilde{N}$ be two Jordan decompositions of T. Since D and $\tilde{D}$ are polynomials in T, they commute with each other and thus can be diagonalised simultaneously. But then since $D + N = \tilde{D} + \tilde{N}$ we get $D - \tilde{D} = \tilde{N} - N$ is nilpotent which can only be if $D = \tilde{D}$. ■

Proposition 11.3 (Solubility implies zero traces)
Let L be a soluble subalgebra of $\mathrm{gl}(V)$ where V is a finite-dimensional $\mathbb{C}$-vector space. Then for all $x \in L$ and all $y \in [L, L]$ we have $\mathrm{Tr}(xy) = 0$.

Proof. We use Lie's Theorem 10.4: There is a basis $\mathcal{B}$ of V such that the every element $x \in L$ corresponds to a lower triangular matrix with respect to $\mathcal{B}$. Since $y \in [L, L]$ is a sum of commutators, the diagonal entries of its matrix with respect to $\mathcal{B}$ are all zero. But then all diagonal entries of the matrix of xy are zero and thus the trace of xy is zero. ■

For the other direction, we need a slightly stronger hypothesis:

Proposition 11.4 (Zero traces imply solubility)
Let V be a finite-dimensional $\mathbb{C}$-vector space and L a Lie subalgebra of $\mathrm{gl}(V)$. Suppose that $\mathrm{Tr}(xy) = 0$ for all $x, y \in L$. Then L is soluble.

Proof. Not extremely difficult, but left out of these notes for the sake of brevity. ■

Surprisingly, these two can be put together for this result:

Theorem 11.5 (Criterion for solubility)
Let L be a finite-dimensional Lie algebra over $\mathbb{C}$. Then L is soluble if and only if $\mathrm{Tr}(x^{\mathrm{ad}} y^{\mathrm{ad}}) = 0$ for all $x \in L$ and $y \in [L, L]$.

Proof. Assume that L is soluble. Then L^{ad} is a soluble subalgebra of $\mathrm{gl}(L)$ by Theorem 5.7 and because ad is a homomorphism of Lie algebras. The statement of the theorem now follows immediately from Proposition 11.3 since $[u, v]^{\mathrm{ad}} = [u^{\mathrm{ad}}, v^{\mathrm{ad}}]$ by the Jacobi identity.
Assume conversely that $\mathrm{Tr}(x^{\mathrm{ad}} y^{\mathrm{ad}}) = 0$ for all $x \in L$ and all $y \in [L, L]$. Then Proposition 11.4 implies that $[L, L]^{\mathrm{ad}} = [L^{\mathrm{ad}}, L^{\mathrm{ad}}]$ is soluble (using our hypothesis only for $x, y \in [L, L]$. Thus L^{ad} itself is soluble since $[L^{\mathrm{ad}}, L^{\mathrm{ad}}] = (L^{\mathrm{ad}})^{(1)}$. But since $L^{\mathrm{ad}} \cong L/Z(L)$ it follows using Theorem 5.7.(ii) that L itself is soluble as $Z(L)$ is abelian. ■

12 The Killing form

Definition/Proposition 12.1 (The Killing form)
Let L be a Lie algebra over a field $\mathbb{F}$. Then the mapping

$$\begin{array}{rccc} \kappa : & L \times L & \to & \mathbb{F} \\ & (x, y) & \mapsto & \mathrm{Tr}(x^{\mathrm{ad}} y^{\mathrm{ad}}) \end{array}$$

is **bilinear**, that is, $\kappa(x + \lambda\tilde{x}, y) = \kappa(x, y) + \lambda\kappa(\tilde{x}, y)$ and $\kappa(x, y + \lambda\tilde{y}) = \kappa(x, y) + \lambda\kappa(x, \tilde{y})$ for all $x, \tilde{x}, y, \tilde{y} \in L$ and all $\lambda \in \mathbb{F}$. The map κ is called the **Killing form**. It is **symmetric**, that is,

$$\kappa(x, y) = \kappa(y, x) \qquad \text{for all } x, y \in L.$$

The Killing form is **associative**, that is,

$$\kappa([x, y], z) = \kappa(x, [y, z]) \qquad \text{for all } x, y, z \in L.$$

The latter property comes from the fact that $\mathrm{Tr}((uv - vu)w) = \mathrm{Tr}(u(vw - wv))$ for all endomorphisms $u, v, w \in \mathrm{End}(V)$ for any vector space V.

We can now restate Theorem 11.5 using this language:

Theorem 12.2 (Cartan's First Criterion)
Let L be a finite-dimensional Lie algebra over $\mathbb{C}$. Then L is soluble if and only if $\kappa(x, y) = 0$ for all $x \in L$ and $y \in [L, L]$.

The Killing form can not only "'detect solubility"', but also semisimplicity. We need a few more definitions.

Definition 12.3 (Perpendicular space, non-degeneracy)
Let V be a vector space over a field $\mathbb{F}$ and $\tau : V \times V \to \mathbb{F}$ a symmetric bilinear form. For any subspace $W \leq V$ we define

$$W^{\perp} := \{v \in V \mid \tau(v, w) = 0 \text{ for all } w \in W\}$$

and call it the **perpendicular space** of W. It is a subspace of V. We call τ **non-degenerate**, if $V^{\perp} = \{0\}$, that is, there is no $0 \neq u \in V$ with $\tau(u, v) = 0$ for all $v \in V$. Otherwise, we call τ **degenerate**. If τ is non-degenerate, then

$$\dim_{\mathbb{F}}(V) = \dim_{\mathbb{F}}(W) + \dim_{\mathbb{F}}(W^{\perp})$$

for all subspaces $W \leq V$.

Lemma 12.4 (Perpendicular space of ideals with respect to the Killing form)
Let L be a Lie algebra, K be an ideal of L and κ the Killing form of L. Then $K^{\perp}$ (with respect to κ) is an ideal of L as well.

Proof. This uses the associativity of the Killing form: Let $x \in K^{\perp}$, that is, $\kappa(x, z) = 0$ for all $z \in K$. We have $\kappa([x, y], z) = \kappa(x, [y, z]) = 0$ for all $y \in L$ and all $z \in K$ because $[y, z] \in K$. ■

Theorem 12.5 (Cartan's Second Criterion)
Let L be a finite-dimensional Lie algebra over $\mathbb{C}$. Then L is semisimple if and only if κ is non-degenerate.

Proof. Suppose that L is semisimple. By Lemma 12.4, the space $L^{\perp}$ (with respect to κ) is an ideal of L, such that $\kappa(x, y) = 0$ for all $x \in L^{\perp}$ and all $y \in [L^{\perp}, L^{\perp}]$ (indeed, even for all $y \in L$). Thus, by Theorem 12.2, the ideal $L^{\perp}$ is soluble. However, because we assumed that L is semisimple, it has no soluble ideals except $\{0\}$ and thus $L^{\perp} = 0$ and thus κ is non-degenerate.
Suppose that L is not semisimple. By Exercise 6 on Tutorial Sheet 2 it then has a non-zero abelian ideal A. Let $a \in A$ be a non-zero element. For every $x \in L$, the map $a^{\mathrm{ad}}x^{\mathrm{ad}}a^{\mathrm{ad}}$ sends all of L to 0, since $[[z, a], x] \in A$ and thus $[[[z, a], x], a] = 0$ for every $z \in L$. Thus $(a^{\mathrm{ad}}x^{\mathrm{ad}})^2 = 0$ and therefore $a^{\mathrm{ad}}x^{\mathrm{ad}}$ is a nilpotent endomorphism. However, nilpotent endomorphisms have trace 0, so a is a non-zero element of $L^{\perp}$ and κ is shown to be degenerate. ■

Lemma 12.6 (Killing form on ideal)
Let I be an ideal in a finite-dimensional Lie algebra over $\mathbb{C}$. Then I is in particular a subalgebra and thus a Lie algebra on its own. The Killing form of I is then the restriction of the Killing from of L to I:

$$\kappa_I(x, y) = \kappa(x, y) \qquad \text{for all } x, y \in I.$$

Proof. Choose a basis of I and extend it to a basis of L. Then write matrices of x^{ad} for elements $x \in I$ with respect to this basis. The result follows. ■

Lemma 12.7 (Ideals in semisimple Lie algebras)
Let I be a non-trivial proper ideal in a complex semisimple Lie algebra L, then $L = I \oplus I^{\perp}$. The ideal I is a semisimple Lie algebra in its own right.

Proof. Let κ denote the Killing form on L, it is non-degenerate by Cartan's Second Criterion 12.5 since L is semisimple. The restriction of κ to $I \cap I^{\perp}$ is identically 0, so by Cartan's First Criterion 12.2 we get $I \cap I^{\perp} = 0$ because L does not have a non-zero soluble ideal. Counting dimensions now gives $L = I \oplus I^{\perp}$.
We need to show that I is a semisimple Lie algebra. Suppose not, then its Killing form is degenerate (using Cartan's Second Criterion 12.5). Thus, there is an $0 \neq a \in I$ such that $\kappa_I(a, x) = 0$ for all $x \in I$, where κ_I is the Killing form of I. By Lemma 12.6 this means that $\kappa(a, x) = 0$ for all $x \in I$. But then $a \in L^{\perp}$ since $L = I \oplus I^{\perp}$ contradicting that L is semisimple. ■

Using Lemma 12.7 it is now relatively easy to prove Theorem 5.12:

Theorem 12.8 (Characterisation of semisimple Lie algebras)
A finite-dimensional Lie algebra L over $\mathbb{C}$ is semisimple if and only if it is the finite direct sum of minimal ideals which are simple Lie algebras.

Proof. We only give the idea for the "only if" part: Use induction by the dimension, for the induction step choose a minimal non-zero ideal I and use Lemma 12.7 to write $L = I \oplus I^{\perp}$ and to show that $I^{\perp}$ is again semisimple of lower dimension. The ideal I is a simple Lie algebra because it was chosen minimal. ■

13 Abstract Jordan decomposition

Can we have a Jordan decomposition in an abstract Lie algebra?
If L is a one-dimensional Lie algebra, then every linear map $\varphi : L \to \text{gl}(V)$ is a representation. So in general, an element $x \in L$ can be mapped to an arbitrary endomorphism of V. However, for complex semisimple Lie algebras, we can do better:

Theorem 13.1 (Abstract Jordan decomposition)
Let L be a finite-dimensional semisimple Lie algebra. Each $x \in L$ can be written uniquely as $x = d + n$, where $d, n \in L$ are such that d^{ad} is diagonalisable, n^{ad} is nilpotent, and $[d, n] = 0$. Furthermore, if $[x, y] = 0$ for some $y \in L$, then $[d, y] = 0 = [n, y]$.
The decomposition $x = d + n$ as above is called **abstract Jordan decomposition** of x.

Proof. Omitted. ■

This in fact covers all representations of L:

Theorem 13.2 (Jordan decompositions)
Let L be a finite-dimensional semisimple Lie algebra over $\mathbb{C}$ and let $\varphi : L \to \text{gl}(V)$ by any representation. Let $x = d + n$ be the abstract Jordan decomposition of x. Then the Jordan decomposition of $x\varphi \in \text{gl}(V)$ is $x\varphi = d\varphi + n\varphi$.

Proof. Omitted. ■

Chapter 6

Classification of semisimple Lie algebras

When we studied $\mathrm{sl}_2(\mathbb{C})$, we discovered that it is spanned by elements e, f and h fulfilling the relations:

$$[e,h] = -2e, \qquad [f,h] = 2f \qquad \text{and} \qquad [e,f] = h.$$

Furthermore h was diagonalisable in every irreducible representation and $H := \mathrm{Span}(h)$ is obviously an abelian subalgebra. Note that $h = h + 0$ is the abstract Jordan decomposition of h, that $H = C_L(H)$ is the weight space of H, acting on L with the adjoint action, corresponding to the weight $0 \in H^*$. Likewise, $\mathrm{Span}(e)$ is the weight space for the weight $c \cdot h \mapsto -2c$ for $c \in \mathbb{C}$, and $\mathrm{Span}(f)$ is the weight space for the weight $c \cdot h \mapsto 2c$ for $c \in \mathbb{C}$.
This approach can be generalised. Our big plan will be:

1. Find a maximal abelian subalgebra H consisting of elements that are diagonalisable in every representation.
2. Restrict the adjoint representation of L to H and show that L is the direct sum of weight spaces with respect to H ("root space decomposition").
3. Prove general results about the set of weights ("root systems").
4. Show that the isomorphism type of L is completely determined by its root system.
5. Classify such root systems.

The rest of the course will be more expository than before.

14 Maximal toral subalgebras

Definition 14.1 (Semisimple elements)
Let L be a finite-dimensional semisimple Lie algebra over $\mathbb{C}$. An element $x \in L$ is called **semisimple**, if its abstract Jordan decomposition is $x = x + 0$, that is, the nilpotent part is equal to zero (see Theorem 13.1). This means, that x acts diagonalisably on every L-module (see Theorem 13.2).

Definition 14.2 (Maximal toral subalgebras)
Let L be a finite-dimensional semisimple Lie algebra over $\mathbb{C}$. A **toral subalgebra** T is a subalgebra consisting of semisimple elements. A toral subalgebra $T \leq L$ is called a **maximal toral subalgebra** if L has no toral subalgebra properly containing T. It is clear that every finite-dimensional semisimple Lie algebra over $\mathbb{C}$ has a maximal toral subalgebra. All these are non-zero since L contains semisimple elements (because of Theorem 13.1, note that if all elements of L were equal to their nilpotent part in the abstract Jordan decomposition, then they would in particular be ad-nilpotent and thus L would be nilpotent, a contradiction to being semisimple).

Lemma 14.3 (Maximal toral subalgebras are abelian)
Let L be a finite-dimensional semisimple Lie algebra over $\mathbb{C}$. Every maximal toral subalgebra T of L is abelian.

Proof. Omitted. ■

Definition 14.4 (Cartan subalgebra)
Let L be a finite-dimensional semisimple Lie algebra over $\mathbb{C}$. A maximal abelian toral subalgebra is called **Cartan subalgebra**. By Lemma 14.3 every such L has a Cartan subalgebra since every maximal toral subalgebra is abelian.

Theorem 14.5 (Cartan subalgebras are self-centralising)
Let H be a Cartan subalgebra of a finite-dimensional semisimple Lie algebra L over $\mathbb{C}$. Then $C_L(H) = H$.

Proof. Omitted. ■

Theorem 14.6 (Simultaneous diagonalisation)
Let $T_1, T_2, \ldots, T_k \in \mathrm{End}(V)$ be endomorphisms of a finite-dimensional $\mathbb{F}$-vector space V. Suppose that all T_i are diagonalisable and that $T_iT_j = T_jT_i$ for all $1 \le i < j \le k$. Then there is a basis $\mathcal{B}$ of V such that the matrices of all T_i with respect to $\mathcal{B}$ are diagonal.

Proof. Omitted here, see Exercise 3 of tutorial sheet 3 or a text on Linear Algebra. ■

For the rest of the chapter L will always be a finite-dimensional semisimple Lie algebra over $\mathbb{C}$ and H a Cartan subalgebra. We denote the Killing form by κ.

Definition/Proposition 14.7 (Root space decomposition)
In this situation, L is an H-module by the adjoint action of H on L: The map

$$\begin{array}{rcccc} \mathrm{ad}|_H & : & H & \to & \mathrm{Lie}(\mathrm{End}(L)) \\ & & h & \mapsto & h^{\mathrm{ad}} \end{array}$$

is a representation of H. We consider all its weight spaces (see Definition 10.1). Let $\Phi \subseteq H^*$ be the set of non-zero weights, note that the zero map $(h \mapsto 0)$ is a weight and that $L_0 = H$ by Theorem 14.5.
The space L is the direct sum of the weight spaces for H:

$$L = H \oplus \bigoplus_{\alpha \in \Phi} L_\alpha.$$

This decomposition is called the **root space decomposition of L with respect to H**. As defined in Definition 10.1, we have

$$L_\alpha = \{x \in L \mid [x, h] = (h\alpha) \cdot x \text{ for all } h \in H\}.$$

The set Φ is called the **set of roots** of L with respect to H and the L_α for $\alpha \in \Phi \cup \{0\}$ are called the **root spaces**. Note that we immediately conclude from the finite dimension of L that Φ is finite!

Proof. Let $h_1, \ldots, h_k$ be a basis of H. Since H is abelian, the endomorphisms $h_1^{\mathrm{ad}}, \ldots, h_k^{\mathrm{ad}} \in \mathrm{End}(L)$ fulfill the hypothesis of Theorem 14.6. Thus L has a basis $\mathcal{B}$ of simultaneous eigenvectors of the h_i^{ad}. Since every element of $\mathcal{B}$ is contained in a root space, L is the sum of the weight spaces. The intersection of two root spaces L_α and L_β for $\alpha \neq \beta$ is equal to the zero space, since if $h\alpha \neq h\beta$, then $x \in L_\alpha \cap L_\beta$ implies $(h\alpha)x = xh = (h\beta)x$ and thus $x = 0$. A short inductive argument shows that the sum of all root spaces in the root space decomposition is in fact direct (just add in one root space at a time). ■

In the sequel we will study the set Φ of roots.

Lemma 14.8 (Properties of Φ)
Suppose that $\alpha, \beta \in \Phi \cap \{0\}$. Then:

(i) $[L_\alpha, L_\beta] \leq L_{\alpha+\beta}$.

(ii) If $\alpha + \beta \neq 0$, then $\kappa(L_\alpha, L_\beta) = \{0\}$.

(iii) The restriction of κ to L_0 is non-degenerate.

Proof. Let $x \in L_\alpha$ and $y \in L_\beta$. Then

$$[[x, y], h] = [[x, h], y] + [x, [y, h]] = (h\alpha)[x, y] + (h\beta)[x, y] = (h(\alpha + \beta))[x, y],$$

thus $[x, y] \in L_{\alpha+\beta}$ which proves (i).
For (ii), we conclude from $\alpha + \beta \neq 0$ that there is some $h \in H$ with $h(\alpha + \beta) \neq 0$. Then

$$(h\alpha)\kappa(x, y) = \kappa([x, h], y) = \kappa(x, [h, y]) = -(h\beta)\kappa(x, y),$$

and thus $(h(\alpha + \beta) \cdot \kappa(x, y) = 0$. Therefore, $\kappa(x, y) = 0$.
For (iii), suppose that $z \in L_0$ and $\kappa(z, x_0) = 0$ for all $x_0 \in L_0$. Since every $x \in L$ can be written as

$$x = x_0 + \sum_{\alpha \in \Phi} x_\alpha$$

with $x_\alpha \in L_\alpha$, we immediately get $\kappa(z, x) = 0$ for all $x \in L$ from (ii) contradicting the non-degeneracy of κ on L. ∎

Quite surprisingly, every semisimple Lie algebra over $\mathbb{C}$ contains lots of copies of $\mathrm{sl}_2(\mathbb{C})$:

Theorem 14.9 (Copies of $\mathrm{sl}_2(\mathbb{C})$ in L)
Let $\alpha \in \Phi$ and $0 \neq e \in L_\alpha$. Then $-\alpha$ is a root and there exists $f \in L_{-\alpha}$ such that $\mathrm{Span}(e, f, h)$ with $h := [e, f]$ is a Lie subalgebra of L with

$$[e, h] = -2e \qquad \text{and} \qquad [f, h] = 2f.$$

Thus, it is isomorphic to sl_2.

Note that we can replace (e, f, h) by $(\lambda e, f/\lambda, h)$ for some $0 \neq \lambda \in \mathbb{C}$ without changing the relations. However, h and $\mathrm{Span}(e, f, h)$ remains always the same!

Proof. This proof was not be presented in the class but is contained in the notes for the sake of completeness.
Since κ is non-degenerate, there is an $x \in L$ with $\kappa(e, x) \neq 0$. Write $x = \sum_{\alpha \in \Phi \cup \{0\}} x_\alpha$ with $x_\alpha \in L_\alpha$. By Lemma 14.8.(ii) we conclude that $x_{-\alpha} \neq 0$ and $\kappa(e, x_{-\alpha}) \neq 0$. Therefore, $-\alpha$ is a root. Set $\tilde{f} := x_{-\alpha}$. Since $\alpha \neq 0$ there is a $t \in h$ with $t\alpha \neq 0$. Thus

$$\kappa([e, \tilde{f}], t) = \kappa(e, [\tilde{f}, t]) = -(t\alpha) \cdot \kappa(e, \tilde{f}) \neq 0$$

showing that $\tilde{h} := [e, \tilde{f}] \neq 0$. Note that $\tilde{f} \in H = L_0$ by Lemma 14.8.(i).
We claim that $\tilde{h}\alpha \neq 0$. Namely, if $\tilde{h}\alpha$ were equal to 0, then $[e, \tilde{h}] = (\tilde{h}\alpha)e = 0$ and $[\tilde{f}, \tilde{h}] = -(\tilde{h}\alpha)\tilde{f} = 0$. Therefore by Proposition 14.10 $\tilde{h}^{\mathrm{ad}}$ would be nilpotent. However, $\tilde{h}$ is semisimple, and the only semisimple and nilpotent element is 0. We can now set $f := -2\tilde{f}/(\tilde{h}\alpha)$ and $h := [e, f] = -2\tilde{h}/(\tilde{h}\alpha)$ to get the relations in the theorem. ∎

Note that by this L is an $\mathrm{sl}_2(\mathbb{C})$-module in many ways! This allows us to use our results about the representations of $\mathrm{sl}_2(\mathbb{C})$ for every $\alpha \in \Phi$ separately!
We have used:

Proposition 14.10
Let $x, y \in \mathrm{End}(V)$ be endomorphism of the finite-dimensional complex vector space V. Suppose that both x and y commute with $[x, y] = xy - yx$. Then $[x, y]$ is a nilpotent map.

15 Root systems

We keep our general hypothesis that L is a finite-dimensional semisimple Lie algebra over $\mathbb{C}$ with Cartan subalgebra H and corresponding set of roots Φ.
For this section, let E be a finite-dimensional vector space over $\mathbb{R}$ with a positive definite symmetric bilinear form $(-|-) : E \times E \to \mathbb{R}$ ("positive definite" means $(x|x) > 0$ if and only if $x \neq 0$).

Definition 15.1 (Reflections)
For $v \in E$, the map

$$\begin{array}{cccc} s_v & : & E & \to & E \\ & & x & \mapsto & x - \frac{2(x|v)}{(v|v)} v \end{array}$$

is called the **reflection along** v. It is linear, interchanges v and $-v$ and fixes the hyperplane orthogonal to v. As an abbreviation, we use $\langle x|v\rangle := \frac{2(x|v)}{(v|v)}$ for $x, v \in E$, note that $\langle -|-\rangle$ is only linear in the first component. We have $xs_v = x - \langle x|v\rangle\, v$.

Definition 15.2 (Root system)
A subset $R \subseteq E$ is called a **root system**, if

(R1) R is finite, $\mathrm{Span}(R) = E$ and $0 \notin R$.

(R2) If $\alpha \in R$, then the only scalar multiples of α in R are α and $-\alpha$.

(R3) If $\alpha \in R$, then s_α permutes the elements of R.

(R4) If $\alpha, \beta \in R$, then $\langle \alpha|\beta\rangle \in \mathbb{Z}$.

Theorem 15.3 (Φ is a root system)
Then Φ is a root system if we take E to be the $\mathbb{R}$-span of Φ with the bilinear form induced by the Killing form κ.

Proposition 15.4 (Moving forms)
The Killing form κ restricted to H is non-degenerate by Lemma 14.8.(iii). Therefore, the linear map

$$\begin{array}{ccc} H & \to & H^* \\ h & \mapsto & (x \mapsto \kappa(h, x)) \end{array}$$

is injective and thus bijective since H and H^* have the same finite dimension. Therefore, for every $\alpha \in H^*$, there is a unique $t_\alpha \in H$ with $x\alpha = \kappa(t_\alpha, x)$ for all $x \in H$. We set $(\alpha|\beta) := \kappa(t_\alpha, t_\beta)$ for all $\alpha, \beta \in H^*$, this defines a non-degenerate bilinear form on H^*, which we call **the bilinear form on H^* induced by κ**.

The proof of Theorem 15.3 works through a series of little results always using all those $\mathrm{sl}_2(\mathbb{C})$-subalgebras and the fact that L is an $\mathrm{sl}_2(\mathbb{C})$-module in different ways. Here we just look at a few of them without proofs:

Lemma 15.5
Let $\alpha \in \Phi$. If $x \in L_{-\alpha}$ and $y \in L_\alpha$, then $[x, y] = \kappa(x, y)t_\alpha$.

Proof. For all $h \in H$, we have

$$\kappa([x, y], h) = \kappa(x, [y, h]) = (h\alpha)\kappa(x, y) = \kappa(t_\alpha, h)\kappa(x, y) = \kappa(\kappa(x, y)t_\alpha, h).$$

Thus $[x, y] - \kappa(x, y)t_\alpha \in H^\perp$ and is therefore equal to 0, since κ is non-degenerate on H. ∎

Lemma 15.6
Let $\alpha \in \Phi$ and $0 \neq e \in L_\alpha$ and $\mathrm{sl}_\alpha := \mathrm{Span}(e, f, h)$ as in Theorem 14.9. If M is an sl_α-submodule of L, then the eigenvalues of h on M are integers.

Proof. Follows immediately from Weyl's Theorem and our classification of sl_2-modules. ∎

Lemma 15.7
Let $\alpha \in \Phi$. The root spaces L_α and $L_{-\alpha}$ are 1-dimensional. Moreover, the only scalar multiples of α that are in Φ are α itself and $-\alpha$.

Note that it follows that this now identifies the copy of $\mathrm{sl}_2(\mathbb{C})$ sitting in $L_\alpha \oplus H \oplus L_{-\alpha}$ uniquely since we only have a choice for $e \in L_\alpha$ up to a scalar. All these choices give us the same sl_α. It even identifies a unique $h_\alpha \in H$!

Lemma 15.8
Suppose that $\alpha, \beta \in \Phi$ and $\beta \notin \{\alpha, -\alpha\}$. Then:

(i) $h_\alpha \beta = \frac{2(\beta|\alpha)}{(\alpha|\alpha)} = \langle \beta|\alpha \rangle \in \mathbb{Z}$.

(ii) There are integers $r, q \geq 0$ such that for all $k \in \mathbb{Z}$, we have $\beta + k\alpha \in \Phi$ if and only if $-r \leq k \leq q$. Moreover, $r - q = h_\alpha \beta$.

(iii) $\beta - (h_\alpha \beta) \cdot \alpha = \beta - \langle \beta|\alpha \rangle \alpha = \beta s_\alpha \in \Phi$.

(iv) $\mathrm{Span}(\Phi) = H^*$.

Lemma 15.9
If α and β are roots, then $\kappa(h_\alpha, h_\beta) \in \mathbb{Z}$ and $(\alpha|\beta) = \kappa(t_\alpha, t_\beta) \in \mathbb{Q}$.
It follows, that if $\{\alpha_1, \dots, \alpha_n\} \subseteq \Phi$ is a basis of H^* and $\beta \in \Phi$, then β is a linear combination of the α_i with coefficients in $\mathbb{Q}$.

Proposition 15.10
The bilinear form defined by $(\alpha|\beta) := \kappa(t_\alpha, t_\beta)$ is a positive definite symmetric bilinear form on the real span E of Φ.

16 Dynkin diagrams

In this section we will classify all possible root systems, we will only use the axioms in Definition 15.2.

Lemma 16.1 (Finiteness Lemma)
Let R be a root system in a finite-dimensional real vector space E equipped with a positive-definite symmetric bilinear form $(-|-) : E \times E \to \mathbb{R}$. Let $\alpha, \beta \in R$ with $\beta \notin \{\alpha, -\alpha\}$. Then

$$\langle \alpha|\beta \rangle \cdot \langle \beta|\alpha \rangle \in \{0, 1, 2, 3\}.$$

Proof. By (R4), the product is an integer. We have

$$(x|y)^2 = (x|x) \cdot (y|y) \cdot \cos^2(\theta)$$

if θ is the angle between two non-zero vectors $x, y \in E$. Thus $\langle x|y \rangle \cdot \langle y|x \rangle = 4\cos^2\theta$ and this must be an integer. If $\cos^2\theta = 1$, then θ is an integer multiple of π and so α and β are linearly dependent which is impossible because of our assumptions and (R2). ∎

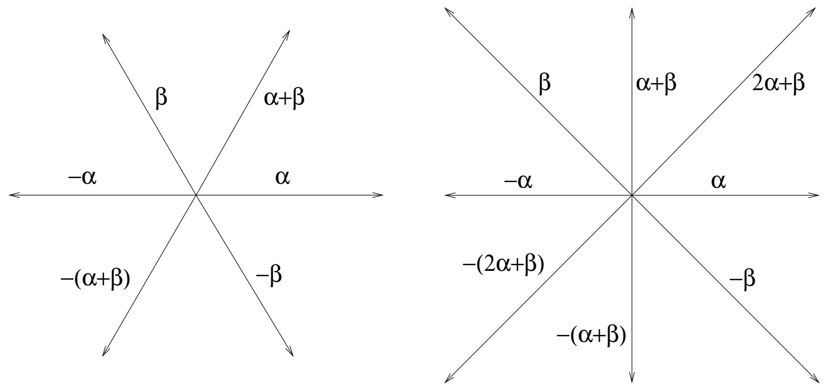
We immediately conclude that there are only very few possibilities for $\langle\alpha|\beta\rangle$, $\langle\beta|\alpha\rangle$, the angle θ and the ratio $(\beta|\beta)/(\alpha|\alpha)$ (without loss of generality we assume $(\beta|\beta) \geq (\alpha|\alpha)$):

$\langle\alpha\|\beta\rangle$	$\langle\beta\|\alpha\rangle$	θ	$\frac{(\beta\|\beta)}{(\alpha\|\alpha)}$
0	0	$\pi/2$	$-$
1	1	$\pi/3$	1
-1	-1	$2\pi/3$	1
1	2	$\pi/4$	2
-1	-2	$3\pi/4$	2
1	3	$\pi/6$	3
-1	-3	$5\pi/6$	3

Lemma 16.2
Let R be a root system with E as in Lemma 16.1 and let $\alpha, \beta \in R$ with $(\alpha|\alpha) \leq (\beta|\beta)$. If the angle between α is strictly obtuse, then $\alpha + \beta \in R$. If the angle between α and β is strictly acute, then $\alpha - \beta \in R$.

Proof. Use (R3) saying that $\alpha s_\beta = \alpha - \langle\alpha|\beta\rangle\, \beta \in R$ together with the above table. ■

Example 16.3 (Examples of root systems)
The following are two different root systems in $\mathbb{R}^2$:



Check the axioms yourself. You find examples for most but not all cases in the above table.

Definition 16.4 (Bases for root systems)
Let R be a root system in a real vector space E. A subset $\mathcal{B} \subseteq R$ is called a **base** of R, if

(B1) $\mathcal{B}$ is a vector space basis of E, and

(B2) every $\alpha \in R$ can be written as $\alpha = \sum_{\beta\in\mathcal{B}} k_\beta \beta$ with $k_\beta \in \mathbb{Z}$, such that all the non-zero coefficients k_β are either all positive or all negative.

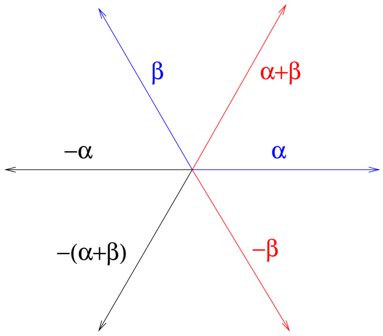
For a fixed base $\mathcal{B}$, we call α **positive** if all its non-zero coefficients with respect to $\mathcal{B}$ are positive and **negative** otherwise. We denote the subset of R of positive roots by R^+ and the subset of negative roots R^-.

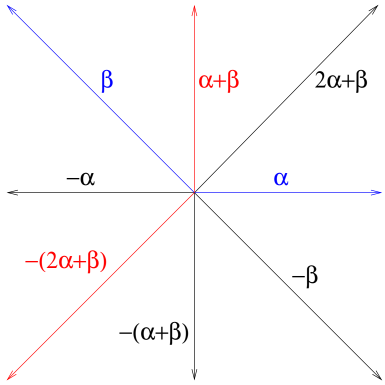
Note that some coefficients can be equal to zero, only the non-zero ones need to have the same sign! Note furthermore that the definition of R^+ and R^- actually depends on $\mathcal{B}$ and that there are different choices for $\mathcal{B}$ possible! For example, for any base $\mathcal{B}$, the set $-\mathcal{B}$ is also a base!

Theorem 16.5 (Existence of bases for root systems)
Let R be a root system in the real vector space E. Then R has a base $\mathcal{B}$.

Proof. Omitted here. ∎

Example 16.6 (Example of a root system)
In the following two diagrams we have coloured a base of the root system in blue and one in red:





So in the first diagram, both (α, β) and $(\alpha + \beta, -\beta)$ are bases. In the second diagram, both (β, α) and $(\alpha + \beta, -(2\alpha + \beta))$ are bases. These are not all possible choices!

Definition 16.7 (Isomorphism of root systems)
Let $R_1 \subseteq E_1$ and $R_2 \subseteq E_2$ be two root systems. An isomorphism between the two root systems R_1 and R_2 is a bijective $\mathbb{R}$-linear map $\psi : E_1 \to E_2$ such that

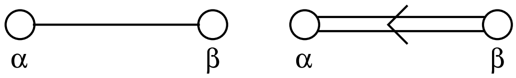
(i) $R_1\psi = R_2$, and

(ii) for any $\alpha, \beta \in R_1$ we have $\langle \alpha | \beta \rangle = \langle \alpha\psi | \beta\psi \rangle$.

Note that condition (ii) basically ensures that the angle θ between $\alpha\psi$ and $\beta\psi$ is the same as the angle between α and β since $4\cos^2\theta = \langle \alpha | \beta \rangle \cdot \langle \beta | \alpha \rangle$.
We can now come up with a graphical way to describe root systems. At first however, it seems that we describe a basis of a root system!

Definition 16.8 (Coxeter graphs and Dynkin diagrams)
Let R be a root system in a real vector space E and let $\mathcal{B} = (b_1, \ldots, b_n)$ be a base of R. The **Coxeter graph** of $\mathcal{B}$ is an undirected graph with n vertices, one for every element b_i and with $\langle b_i | b_j \rangle \cdot \langle b_j | b_i \rangle$ edges between vertex b_i and b_j for all $1 \leq i < j \leq n$. In the **Dynkin diagram**, we add for any pair of vertices $b_i \neq b_j$ for which $(b_i | b_i) \neq (b_j | b_j)$ (which are then necessarily connected) an arrow from the vertex corresponding to the longer root to the one corresponding to the longer root.

Example 16.9 (Dynkin diagrams)
Here are the two Dynkin diagrams for the base (α, β) in each of the two root systems in Example 16.6:



Surprisingly, the information in the Dynkin diagram is sufficient to describe the isomorphism type of the root system:

Proposition 16.10 (Dynkin diagram decides isomorphism type)
Let $R_1 \subseteq E_1$ and $R_2 \subseteq E_2$ be two root systems and let $\mathcal{B}_1$ be a base of R_1 and $\mathcal{B}_2$ one of R_2. If there is a bijection $\psi : \mathcal{B}_1 \to \mathcal{B}_2$ such that ψ maps the Dynkin diagram of $\mathcal{B}_1$ to the one of $\mathcal{B}_2$, then R_1 and R_2 are isomorphic in the sense of Definition 16.7.

More formally, if

$$\langle\alpha|\beta\rangle\cdot\langle\beta|\alpha\rangle=\langle\alpha\psi|\beta\psi\rangle\cdot\langle\beta\psi|\alpha\psi\rangle$$

and $(\alpha|\alpha) < (\beta|\beta)$ if and only if $(\alpha\psi|\alpha\psi) < (\beta\psi|\beta\psi)$ for all $\alpha, \beta \in \mathcal{B}_1$, then the $\mathbb{R}$-linear extension of ψ to an $\mathbb{R}$-linear map from $E_1 \to E_2$ is an isomorphism between the root systems R_1 and R_2.

Proof. Omitted. ∎

Proposition 16.11 (Dynkin diagram is property of isomorphism type)
If two root systems are isomorphic then they have the same Dynkin diagram. In particular, the Dynkin diagram does not depend on the choice of base.

Proof. Omitted. ∎

So Dynkin diagrams are the same as isomorphism types of root systems.

17 How everything fits together

Definition 17.1 (Irreducible root systems)
A root system R is called **irreducible**, if it cannot be written as the disjoint union $R_1 \cup R_2$ such that $(\alpha|\beta) = 0$ whenever $\alpha \in R_1$ and $\beta \in R_2$.

Lemma 17.2 (Root systems can be decomposed into irreducible ones)
Let R be a root system in the real vector space E. Then R is the disjoint union $R = R_1 \cup \cdots \cup R_k$ of subsets $R_1, \ldots, R_k$ where each R_i is an irreducible root system in $E_i := \mathrm{Span}(R_i)$ and E is an orthogonal direct sum of the subspaces $E_1, \ldots, E_k$.

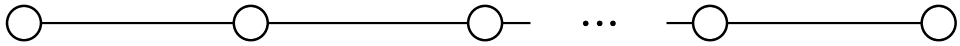
Proof. Omitted here. ∎

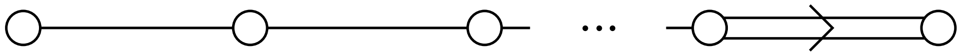
Note that both root systems in Example 16.3 are irreducible.

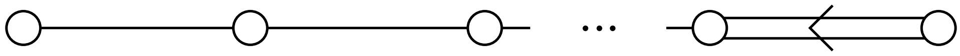
Proposition 17.3 (Irreducibility in the Dynkin diagram)
A root system is irreducible if and only if its Dynkin diagram is connected.

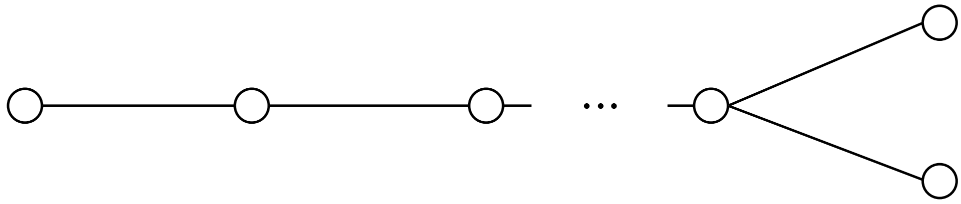
Proof. Follows immediately from the definitions of "irreducible" for root systems and of Dynkin diagrams. ∎

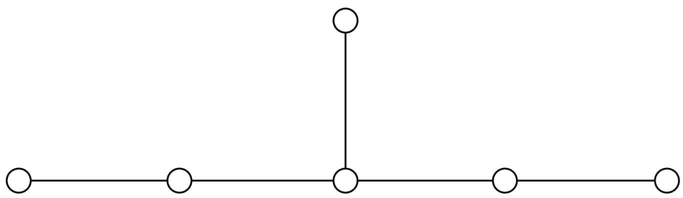
Theorem 17.4 (Classification of irreducible root systems)
Every irreducible root system has one of the following Dynkin diagrams and these diagrams all occur as Dynkin diagrams of a root system:

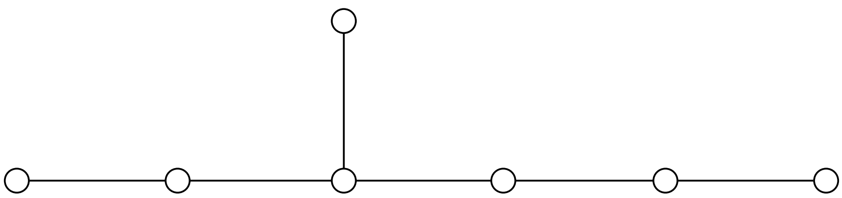
$A_n (n \geq 1)$: 

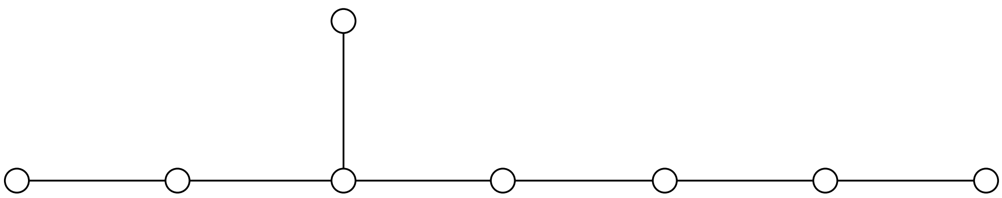
$B_n (n \geq 2)$: 

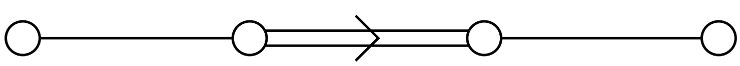
$C_n (n \geq 3)$: 

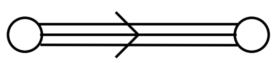
$D_n (n \geq 4)$: 

E_6: 

E_7: 

E_8: 

F_4: 

G_2: 

The first four types A_n to D_n cover each infinitely many cases. Each diagram has n vertices. The restrictions on n are there to avoid duplicates.

Proof. Very nice, but omitted here, unfortunately. ■

We have now done the following:

$$\begin{tikzcd}
\boxed{\text{F.d. semisimple Lie algebra } L \text{ over } \mathbb{C}} \arrow[d, "\text{we choose}"] \\
\boxed{\text{Cartan subalgebra } H} \arrow[d, "\text{which gives}"] \\
\boxed{\text{Root space decomposition } L = H \oplus \bigoplus_{\alpha\in\Phi} L_\alpha} \arrow[d, "\text{which defines}"] \\
\boxed{\text{Root system } \Phi \subseteq H^*} \arrow[d, "\text{we choose}"] \\
\boxed{\text{Base } \mathcal{B} \text{ of root system } \Phi} \arrow[d, "\text{which defines}"] \\
\boxed{\text{Dynkin diagram}}
\end{tikzcd}$$

The big plan is:

- We know all resulting diagrams that can possibly occur.
- The result does not depend on our choices (we need to show this!).
- Two isomorphic Lie algebras give the same Dynkin diagram.
- Two non-isomorphic Lie algebras give different Dynkin diagrams.
- All Dynkin diagrams actually occur.
- L is simple if and only if the Dynkin diagram is irreducible.

To this end, we would need to prove the following results:

Theorem 17.5
Let L be a finite dimensional semisimple Lie algebra over $\mathbb{C}$ and let H_1 and H_2 be two Cartan subalgebras with associated root systems Φ_1 and Φ_2. Then Φ_1 and Φ_2 are isomorphic as root systems.

Theorem 17.6 (Serre)
Let Φ be an irreducible root system with n vertices and base $\mathcal{B} = (b_1, \ldots, b_n)$ and let $c_{i,j} := \langle b_i | b_j \rangle$ for $1 \leq i, j \leq n$ (the so-called **Cartan matrix**).
Let L be the Lie algebra over $\mathbb{C}$ generated by generators e_i, f_i and h_i for $1 \leq i \leq n$ subject to the relations

(S1) $[h_i, h_j] = 0$ for all $1 \leq i, j \leq n$,
(S2) $[e_i, h_j] = c_{i,j} e_i$ and $[f_i, h_j] = -c_{i,j} f_i$,
(S3) $[e_i, f_i] = h_i$ for all $1 \leq i \leq n$ and $[e_i, f_j] = 0$ for all $i \neq j$,
(S4) $(e_i)(e_j^{\mathrm{ad}})^{1-c_{i,j}} = 0$ and $(f_i)(f_j^{\mathrm{ad}})^{1-c_{i,j}} = 0$ if $i \neq j$.

Then L is finite dimensional and semisimple, $H := \mathrm{Span}(h_1, \ldots, h_n)$ is a Cartan subalgebra and its root system is isomorphic to Φ.

Theorem 17.7
Let L be a finite dimensional simple Lie algebra over $\mathbb{C}$ with root system Φ. Then Φ is irreducible.

Bibliography

[1] Karin Erdmann and Mark J. Wildon, Introduction to Lie Algebras. *Springer Undergraduate Mathematics Series.* 2006.

[2] James E. Humphreys, Introduction to Lie Algebras and Representation Theory. *Springer Graduate Texts in Mathematics*, **9**, 1972.