Chapter 1

A survival kit of linear algebra

In this chapter we recall some elementary facts of linear algebra, which are needed throughout the course, in particular to set up notation.

1 Vector spaces and subspaces

Reminder 1.1 (Vector space)
Let $\mathbb{F}$ be a field. An $\mathbb{F}$-vector space V is a set with two operations

$$\begin{array}{ccc} V \times V & \to & V \\ (v, w) & \mapsto & v + w \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathbb{F} \times V & \to & V \\ (\lambda, w) & \mapsto & \lambda \cdot w \end{array}$$

called **addition** and **multiplication by scalars** with the usual axioms (see MT3501 for details).

Example 1.2 (Complex row space)
The set of row vectors of length n containing complex numbers is denoted by

$$\mathbb{C}^{1\times n} := \{[\alpha_1, \ldots, \alpha_n] \mid \alpha_1, \ldots, \alpha_n \in \mathbb{C}\}.$$

It is a $\mathbb{C}$-vector space, we add vectors and multiply them by scalars as exhibited in the following examples:

$$[1, -2, 3] + [4, 5, 6] = [1 + 4, -2 + 5, 3 + 6] = [5, 3, 9]$$

and

$$(-3) \cdot [4, 1/2, -2] = [-3 \cdot 4, (-3) \cdot (1/2), (-2) \cdot (-3)] = [-12, -3/2, 6] = (-1)[12, 3/2, 6].$$

Remarks: Multiplication by -1 is additive inversion, we often leave out the dot $\cdot$ for multiplication.

Reminder 1.3 (Linear combinations, span, spanning set)
If V is a $\mathbb{C}$-vector space, $v_1, \ldots, v_k \in V$ and $\lambda_1, \ldots, \lambda_k \in \mathbb{C}$, then

$$w := \sum_{i=1}^{k} \lambda_i v_i = \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_k v_k$$

is called a **linear combination** of the v_i, we say that "$w \in V$ **is a linear combination** of the v_i", the **coefficients** $\lambda_1, \ldots, \lambda_k$ are **not necessarily uniquely defined**!
The **set of linear combinations**

$$\mathrm{Span}(v_1, \ldots, v_k) := \{w \in V \mid w \text{ is a linear combination of the } v_i\}$$

is called the **span** of the vectors $v_1, \dots, v_k$.
If $M \subseteq V$ is a (possibly infinite) subset, then its span is the **union**

$$\operatorname{Span}(M) := \bigcup_{n\in\mathbb{N}} \left(\bigcup_{v_1,\dots,v_n \in M} \operatorname{Span}(v_1, \dots, v_n) \right),$$

of all spans of all finite sequences of vectors in M.
Remember: Linear combinations are **always finite sums**.

Reminder 1.4 (Subspace)
Let V be a $\mathbb{C}$-vector space. A non-empty subset $W \subseteq V$ is called a **subspace**, if

$$u + v \in W \qquad \text{and} \qquad \lambda u \in W \qquad \text{for all } u, v \in W \text{ and all } \lambda \in \mathbb{C}.$$

In particular, a subspace is itself a $\mathbb{C}$-vector space. In fact, every subspace W is the span of some vectors $v_1, \dots, v_k$ for some k, and every such span is a subspace.

Example 1.5 (Sub-row space)
The following is a subspace of $\mathbb{C}^{1\times 3}$:

$$\operatorname{Span}([1, 0, -1], [0, 2, 1], [1, 2, 0]) = \left\{ [x, y, z] \in \mathbb{C}^{1\times 3} \middle| z = y/2 - x \right\}.$$

Exercise: Prove this equality (consider an arbitrary linear combination of the three row vectors)!

2 Bases, dimension and linear maps

Definition 2.1 (Linear independence)
A tuple $(v_1, \dots, v_k)$ of vectors in a $\mathbb{C}$-vector space V are called **linearly independent**, if one of the following equivalent statements is true:

(i) For arbitrary numbers $\lambda_1, \dots, \lambda_k \in \mathbb{C}$ the following implication holds:

$$\sum_{i=1}^{k} \lambda_k v_k = 0 \qquad \Longrightarrow \qquad \lambda_1 = \lambda_2 = \cdots = \lambda_k = 0.$$

(ii) Every vector in $\operatorname{Span}(v_1, \dots, v_k)$ can be expressed as a linear combination of the vectors $v_1, \dots, v_k$ **in a unique way**.

(iii) No vector v_i is contained in the span of the others:

$$v_i \notin \operatorname{Span}(v_j \mid 1 \le j \le k, j \ne i) \qquad \text{for all } i.$$

Otherwise the tuple is called **linearly dependent**. Linear dependence is a property of **the tuple** and **not** of **the individual vectors**.

Example 2.2 (Linear independent vectors)
The tuple of vectors

$$([5, 0, 2], [2, 3, 0], [-1, 0, 0])$$

is linearly independent.

Definition 2.3 (Basis of a vector space)
Let V be a $\mathbb{C}$-vector space. A tuple $(v_1, \dots, v_n)$ of vectors in V is called a **basis of** V, if

- $V = \mathrm{Span}(v_1, \ldots, v_n)$ and
- $(v_1, \ldots, v_n)$ is linearly independent.

Theorem 2.4 (Dimension)
In a $\mathbb{C}$-vector space V any two bases have the same number of elements. The number of elements in an arbitrary basis of V is called the **dimension** of V.

Note: In this course, we only deal with **finite-dimensional** vector spaces.

Example 2.5
The $\mathbb{C}$-vector space $\mathbb{C}^{1\times n}$ is n-dimensional because

$$([1, 0, \ldots, 0], [0, 1, 0, \ldots, 0], \ldots, [0, \ldots, 0, 1])$$

is a basis of length n, it is called the **standard basis**.

Reminder 2.6 (Linear maps)
Let V and W be $\mathbb{C}$-vector spaces. A map $\varphi : V \to W$ is called **$\mathbb{C}$-linear**, if

$$(u + v)\varphi = u\varphi + v\varphi \quad \text{and} \quad (\lambda v)\varphi = \lambda(v\varphi)$$

for all $u, v \in V$ and all $\lambda \in \mathbb{C}$. We write all maps on the right hand side.

Theorem 2.7 (Linear map determined by values on a basis)
Let $(v_1, \ldots, v_n)$ be a basis of a $\mathbb{C}$-vector space V and let W be a $\mathbb{C}$-vector space. Then for every tuple $(w_1, \ldots, w_n)$ of vectors in W, there is a unique linear map $\varphi : V \to W$ with $v_i\varphi = w_i$ for $1 \leq i \leq n$, it maps

$$\left(\sum_{i=1}^{n} \lambda_i v_i\right) \qquad \text{to} \qquad \left(\sum_{i=1}^{n} \lambda_i v_i\right)\varphi := \sum_{i=1}^{n} \lambda_i w_i \in W.$$

Example 2.8 (Example for a linear map)
The map

$$\mathbb{C}^{1\times 3} \to \mathbb{C}^{1\times 3}, \quad [x, y, z] \mapsto [2x - y + 3z, x + z, -x + 7z + 6y]$$

is $\mathbb{C}$-linear. It is uniquely defined by doing

$$[1, 0, 0] \mapsto [2, 1, -1] \quad \text{and} \quad [0, 1, 0] \mapsto [-1, 0, 6] \quad \text{and} \quad [0, 0, 1] \mapsto [3, 1, 7].$$

Theorem 2.9 (Matrix of a linear map)
Let V and W be $\mathbb{C}$-vector spaces, and $(v_1, \ldots, v_m)$ and $(w_1, \ldots, w_n)$ be bases of V and W respectively. Then there is a $\mathbb{C}$-linear bijection between the set of $\mathbb{C}$-linear maps from V to W and the set $\mathbb{C}^{m\times n}$ of $m \times n$-matrices with entries in $\mathbb{C}$, given by

$$\varphi \mapsto [a_{i,j}]_{1\leq i\leq m, 1\leq j\leq n} \qquad \text{where } v_i\varphi = \sum_{j=1}^{n} a_{i,j} w_j \quad \text{for all } i.$$

Note that this convention might be different from what you know, it comes from the fact that we write mappings on the right hand side and use row vectors.
For three spaces, the composition $\varphi \cdot \psi$ (do φ first, then ψ) is mapped to the **matrix product** of the matrices corresponding to φ and ψ respectively, if the same basis is chosen in the range of φ and the source of ψ.

Example 2.10 (Continuation of Example 2.8)
The matrix of the linear map in Example 2.8 with respect to the standard basis (in both domain and range) is

$$\begin{bmatrix} 2 & 1 & -1 \\ -1 & 0 & 6 \\ 3 & 1 & 7 \end{bmatrix}$$

since $[1, 0, 0]\varphi = [2, 1, -1] = 2 \cdot [1, 0, 0] + 1 \cdot [0, 1, 0] + (-1) \cdot [0, 0, 1]$.

Definition 2.11 (Endomorphisms)
For a $\mathbb{C}$-vector space V we denote the set of $\mathbb{C}$-linear maps from V to V by $\mathrm{End}(V)$ and call them **linear endomorphisms**. The subset (in fact, subgroup) of **invertible** endomorphisms is denoted by $\mathrm{GL}(V)$. We call an endomorphism $\varphi \in \mathrm{End}(V)$ **nilpotent**, if there is an $n \in \mathbb{N}$ with $\varphi^n = 0$.

3 Direct sums

Definition 3.1 (Direct sum)
The $\mathbb{C}$-vector space V is said to be the **direct sum** $U \oplus W$ of two subspaces U and W of V, if one and thus both of the following equivalent conditions holds:

- $V = U + W := \{u + w \mid u \in U, w \in W\}$ and $U \cap W = \{0\}$,
- every vector $v \in V$ can be written as a sum $u + w$ of a vector $u \in U$ and a vector $w \in W$ **in a unique way**.

Both statements generalise more than two subspaces: The $\mathbb{C}$-vector space V is said to be the **direct sum** $U_1 \oplus \cdots \oplus U_k$ of k subspaces $U_1, \ldots, U_k$ if one and thus both of the following equivalent conditions holds:

- $V = U_1 + \cdots + U_k := \{u_1 + u_2 + \cdots + u_k \mid u_i \in U_i\}$ and
  $U_i \cap (U_1 + \cdots + U_{i-1} + U_{i+1} + \cdots + U_k) = \{0\}$ for $1 \le i \le k$.
- Every vector $v \in V$ can be written as a sum $u_1 + \cdots + u_k$ of vectors $u_i \in U_i$ for $1 \le i \le k$ **in a unique way**.

Theorem 3.2 (Basis of a direct sum)
If $V = U \oplus W$ and $(u_1, \ldots, u_m)$ is a basis of U and $(w_1, \ldots, w_n)$ is a basis of W, then

$$(u_1, \ldots, u_m, w_1, \ldots, w_n)$$

is a basis of V and we have $\dim(V) = \dim(U) + \dim(W)$.

Example 3.3 (Direct sum decomposition)
We have

$$\mathbb{C}^{1\times 3} = \mathrm{Span}([1, 2, 3]) \oplus \left\{[x, y, z] \in \mathbb{C}^{1\times 3} \mid z = x - y\right\}.$$

Exercise: Prove this statement.

Remark 3.4 (Complements)
Note that for every subspace U of a $\mathbb{C}$-vector space V there is a (not necessarily unique) subspace W of V such that $V = U \oplus W$.